A higher index theorem for proper cocompact actions

Xiang Tang

Washington University in St. Louis

In this talk, I will discuss some results about index theory for invariant elliptic operators. The recent developments in non-commutative geometry and differential geometry led to some interesting new progress.

This is joint work with Markus Pflaum and Hessel Posthuma.

Part I: Elliptic operators and the Atiyah-Singer index theorem

We briefly explain the concept of elliptic differential operator, and the Atiyah-Singer index theorem.

Differential operators

Let M be a smooth compact manifold of dimension n.

In a coordinate chart, a different operator D on M has the form

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where

$$\alpha = (\alpha_1, \cdots, \alpha_n), \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If D is a system (i.e. acts between sections of two different vector bundles E and F over M), then the coefficients $a_\alpha(x)$ lie in $\mathrm{Hom}(E_x, F_x)$.

Principal Symbol and elliptic operator

The principal symbol of D is a function on T^*M, which on a coordinate chart is defined to be

$$\sigma(D)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}.$$

A differential operator D on M is elliptic if $\sigma(D)(x,\xi)$ is invertible for all $\xi \neq 0$.

Theorem: If D is an elliptic operator of order m acting between sections of two vector bundles E and F on a closed manifold M, then

$$D : L^2(M;E) \longrightarrow L^2(M;F)$$

is a Fredholm operator.

de Rham operator

Let M be a closed riemannian manifold. Consider the de Rham differential

$$C^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \cdots ,$$

and its adjoint

$$\Omega^n(M) \xrightarrow{d^*} \Omega^{n-1}(M) \xrightarrow{d^*} \Omega^{n-2}(M) \cdots .$$

Then

$$D_{\text{de Rham}} := d + d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$$

is an elliptic differential operator.

The index of $D_{\text{de Rham}}$ is the Euler characteristic $\chi(M)$ of M.

Dolbeault operator

Let M be a closed complex manifold. Consider the d-bar operator

$$\Omega^{(0,0)}(M) \xrightarrow{\bar{\partial}} \Omega^{(0,1)}(M) \xrightarrow{\bar{\partial}} \Omega^{(0,2)}(M) \cdots ,$$

and its adjoint

$$\Omega^{(0,n)}(M) \xrightarrow{\bar{\partial}^*} \Omega^{(0,n-1)}(M) \xrightarrow{\bar{\partial}^*} \Omega^{(0,n-2)}(M) \cdots .$$

Then

$$D := \bar{\partial} + \bar{\partial}^* : \Omega^{(0,\text{even})}(M) \longrightarrow \Omega^{(0,\text{odd})}(M).$$

is an elliptic differential operator. The Dolbeault operator naturally extends to holomophic vector bundles.

The index of $D_{\text{Dolbeault}}$ is the holomorphic Euler characteristic $\chi(M)$ of M.

Atiyah-Singer Theorem

Theorem: (Atiyah-Singer) Let D be an elliptic operator on a closed, oriented, and n-dimensional manifold M. Then

$$\mathrm{ind}_a(D) = \mathrm{ind}_t(D) = \int_{T^*M} \mathrm{ch}(\sigma(D))\hat{A}(T^*M).$$

- The symbol $\sigma(D)$ defines an element in $K^0(T^*M)$.

- $\mathrm{ch} : K^0(T^*M) \to H^{\mathrm{even}}(T^*M)$ is the Chern character.

- $\hat{A}(T^*M) \in H^{\mathrm{even}}(T^*M)$ is the A-hat genus of T^*M.

Examples

1. (Hirzebruch-Riemann-Roch theorem) Let V be a holomorphic vector bundle on a closed complex manifold M.

$$\chi(X,V) = \mathrm{ind}_a(D^V_{\mathrm{Dolbeault}}) = \int_M \mathrm{ch}(V)Td(M).$$

2. Let M be a spin manifold, and $\partial\!\!\!/ : \mathcal{S}^+ \to \mathcal{S}^-$ be the Dirac operator.

$$\mathrm{ind}_t(\partial\!\!\!/) = \int_M \widehat{A}(M).$$

(Lichnerowicz) If the scalar curvature of M is positive, then $\mathrm{ind}_a(\partial\!\!\!/) = 0$, and therefore $\int_M \widehat{A}(M) = 0$.

Part II: Equivariant index theory

The study of equivariant index theory was started when Atiyah and Singer proved their seminal index theorem. We review some of the developments related to operator algebras and noncommutative geometry.

Invariant elliptic operators

Let G be a Lie group acting properly on a manifold M, i.e. $G \times M \to M \times M$ is a proper map.

Assume that the action is cocompact, i.e. the quotient space M/G is compact.

Let D be a G-invariant elliptic operator M on G-equivariant vector bundles, i.e. $gD = Dg$.

Examples:

- The de Rham operator on M.

- The lift of an elliptic differential operator on X to the universal covering space $M = \widetilde{X}$ with the deck transformation of the fundamental group.

Lefschetz fixed point theorem

Let $f : M \to M$ be a diffeomorphism on a closed manifold M. $f^* : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$ descends to

$$f^* : H^\bullet(M) \to H^\bullet(M).$$

The Lefschetz number $L(f)$ of f is defined to be

$$\sum_i (-1)^i \operatorname{tr}(f^*|_{H^i(M)}).$$

Theorem: (Lefschetz) When f only has isolated nondegenerated fixed points, then

$$L(f) = \sum_{p=f(p)} \pm 1.$$

C^*-algebra

As the G action commutes with D, the (co)kernel of G is equipped with a unitary G representation.

In general, the (co)kernel of an elliptic operator D on a noncompact manifold M is not finite dimensional. Tools from operator algebras are needed to study D.

A C^*-algebra, A, is a Banach algebra over the field of complex numbers, together with an involution $^* : A \to A$ such that for all $x \in A$,

$$||x^*x|| = ||x|| ||x^*|| = ||x||^2.$$

Group C^*-algebra

Let G be a Lie group. Fix a Haar measure dg on G.

Let $f \in L^1(G)$. Define $\lambda(f) \in \mathcal{L}(L^2(G))$ by

$$\lambda(f)(\xi)(g) := \int f(g_1)\xi(g_1^{-1}g)dg_1.$$

Define $||f||_r = ||\lambda(f)||$. The completion of $(L^1(G), ||\cdot||_r)$ is called the reduced C^*-algebra $C_r^*(G)$ of the group G.

K-theory

Let A be a unital algebra.

Let Proj(A) be the set of isomorphism classes of projective modules of A.

Define an addition $+$ on Proj(A) by

$$M + N := M \oplus N.$$

(Proj$(A), +$) is a commutative monoid.

The Grothendieck group of (Proj$(A), +$) is defined to be $K_0(A)$.

For a compact manifold M, $K^0(M)$ is defined to be $K_0(C(M))$.

L^2-index

Assume G is unimodular. Let G act properly and cocompactly on a smooth manifold M. Let D be a G-invariant elliptic operator on M.

The kernel and cockerel of D are equipped with unitary G-representations. The formal difference

$$\text{ind}^G(D) := [\ker(D)] - [\text{coker}(D)]$$

defines an element in $K_0(C_r^*(G))$.

Define the L^2-index $\text{ind}_{L^2}(D)$ to be

$$\text{tr}_G(\text{ind}^G(D)) \in \mathbb{R}.$$

Covering space

Let X be a closed manifold, and D_X be an elliptic differential operator on X.

Let M be the universal covering of X, and G be the fundamental group of X. G acts on M properly, freely, and cocompactly.

The differential operator D_X lifts to a G-invariant elliptic differential operator D_M on M.

Theorem: (Atiyah)

$$\text{ind}_{L^2}(D_M) = \text{ind}_a(D_X).$$

Homogeneous spaces

Let G be a unimodular Lie group, and H be a compact subgroup of G. Consider the homogeneous space $M = G/H$. G acts properly and cocompactly on M from the left.

Let D be a G-invariant elliptic differential operator on M. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H.

Theorem: (Connes-Moscovici)

$$\mathrm{ind}_{L^2}(D) = \left\langle \widehat{A}(\mathfrak{g}, H) \wedge \mathrm{ch}(\sigma(D))_{\mathfrak{m}^*}, [V] \right\rangle,$$

where $\mathfrak{m}^* \subset \mathfrak{g}^*$ is the conormal space of $\mathfrak{h}$ in $\mathfrak{g}$, $[V]$ is the fundamental class of $\mathfrak{m}^*$.

This result was recently generalized to the L^2-index of a G-invariant elliptic operator on a manifold with a proper and cocompact action by Wang.

Example

Let Σ_g to be the closed Riemann surface of genus $g \geq 2$ with the $\bar{\partial}_{\Sigma_g}$ operator.

The universal covering space $\mathbb{H}$ is the upper half plane with the $\bar{\partial}_{\mathbb{H}}$ operator.

The index of $\bar{\partial}_{\Sigma_g}$ is $g-1 \geq 0$. The Atiyah covering index theorem implies that $\bar{\partial}_{\mathbb{H}}$ has nontrivial L^2 kernel.

The kernel of $\bar{\partial}_{\mathbb{H}}$ gives a Hilbert space representation of $SL(2,\mathbb{R})$, belonging to the discrete series representations.

Part III: Higher index theory

Motivated by application to Novikov conjecture, Connes-Moscovici proved a far reaching generalization of the Atiyah-Singer index theorem.

Higher index theorem

Theorem: (Connes-Moscovici) Let G be a countable discrete group acting properly and freely on a manifold M and D a G-invariant elliptic differential operator on M. For any $[c] \in H^{2k}(G, \mathbb{C})$,

$$\langle \mathrm{ind}^G(D), [c] \rangle = \frac{1}{(2\pi\sqrt{-1})^k (2k)!} \int_{T^*X} \mathrm{ch}(\sigma(D)) \widehat{A}(T^*X) \Psi^*([c]),$$

where $X = M/G$, $\Psi : X \to BG$ is the classifying map, and $\Psi^*([c]) \in H^{2k}(X, \mathbb{C})$ is the pull-back of the class $[c]$.

Signature operator

The signature operator:

$$D^{\text{sign}} = d + d^* : L^2\Omega^+(M) \longrightarrow L^2\Omega^-(M)$$

where $\Omega^{\pm}(M)$ is the $\pm$ eigenspace of an involution

$$\tau : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M), \quad \tau^2 = 1.$$

When $\dim(M) = 4k$, then $\text{ind}_a(D^{\text{sign}})$ is the signature of the bilinear form on $H^{2k}(M)$.

Let Γ be the fundamental group of M with $\mu : M \to B\Gamma$. For $\alpha \in H^\bullet(B\Gamma; \mathbb{Q})$, the Connes-Moscovici higher index theorem states

$$\langle \text{ind}^\Gamma(D^{\text{sign}}_{\widetilde{X}}), [c]\rangle = \int_M L(M) \wedge \mu^*(\alpha) \in \mathbb{Q}.$$

Differentiable group cohomology

Let G be a Lie group. Let $C^\infty(G^{\times k})$ be the space of smooth functions on

$$\underbrace{G \times \cdots \times G}_{k}.$$

Define a differential $\delta : C^\infty(G^{\times k}) \to C^\infty(G^{\times k})$ by

$$\begin{aligned} & \delta(\varphi)(g_1, \cdots, g_{k+1}) \\ = \ & \varphi(g_2, \cdots, g_k) \\ - \ & \varphi(g_1 g_2, \cdots, g_{k+1}) + \cdots + (-1)^k \varphi(g_1, \cdots, g_k g_{k+1}) \\ + \ & (-1)^{k+1} \varphi(g_1, \cdots, g_k). \end{aligned}$$

The differentiable group cohomology $H^\bullet_{\mathrm{diff}}(G)$ is defined to be the cohomology of $(C^\infty(G^{\times \bullet}), \delta)$.

Index pairing

Assume G to be unimodular. Fix a Haar measure on G.

There is a natural pairing between $C^\infty(G^{\times k})$ and $C_c^\infty(G)^{\hat{\otimes}(k+1)}$ by

$$< \hat{\varphi}, f_0 \otimes \cdots \otimes f_k > \; := \int f_0(g_k^{-1} \cdots g_1^{-1}) f_1(g_1) \cdots f_k(g_k) \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k$$

The above pairing descends to define a pairing between $H^\bullet_{\mathrm{diff}}(G)$ and $K_\bullet(C_c^\infty(G))$.

Let D be a G-invariant elliptic operator on M. For $[\varphi] \in H^\bullet_{\mathrm{diff}}(G)$, define $\mathrm{ind}_{[\varphi]}(D)$ to be

$$< [\varphi], \mathrm{ind}^G(D) > .$$

An index theorem

Theorem: (Pflaum-Posthuma-Tang) Let G be a Lie group acting properly and cocompactly on a manifold M. Suppose that D is an elliptic G-invariant differential operator on M, and $[\varphi] \in H^{2k}_{\text{diff}}(G; L)$. The index pairing evaluated on these elements is given by

$$\text{ind}_{[\varphi]}(D) = \frac{1}{(2\pi\sqrt{-1})^k (2k)!} \int_{T^*M} c\Phi([\varphi]) \wedge \hat{A}(T^*M) \wedge \text{ch}(\sigma(D)),$$

where

i) $c \in C^\infty_{\text{cpt}}(M)$ is a cut-off function, i.e.

$$\int_G c(g^{-1}x)dg = 1, \ \forall x \in M.$$

ii) Φ is a map from $H^\bullet_{\text{diff}}(G; L)$ to the de Rham cohomology of G-invariant differential forms on M with coefficient in L.

Examples:

1. When G is the fundamental group of X with $M = \widetilde{X}$, $H^{\bullet}_{\text{diff}}(G) = H^{\bullet}(G)$. The index formula is the Connes-Moscovici higher index theorem.

2. When G is a unimodular Lie group and H is a compact subgroup, choose $M = G/H$. The index formula for $\varphi = 1 \in H^0_{\text{diff}}(G)$ is the Connes-Moscovici index theorem for homogeneous spaces.

3. When $\mathsf{H}_{\mathcal{F}}$ is the holonomy groupoid of a regular foliation $\mathcal{F}$ on X, choose M to be $\mathsf{H}_{\mathcal{F}}$. Assume that $\mathsf{H}_{\mathcal{F}}$ is unimodular. The index formula for $[\varphi] = 1 \in H^0_{\text{diff}}(\mathsf{H}_{\mathcal{F}})$ is the Connes index theorem for measured foliations.

The $\bar{\partial}$-operator on $\mathbb{C}$ (after Connes and Moscovici)

Consider the abelian group $G = \mathbb{R}^2$ and the trivial subgroup H. The homogeneous space $M = G/H$ is $\mathbb{R}^2$ on which $\mathbb{R}^2$ acts by translation.

Consider the $\bar{\partial}$-operator on $M \cong \mathbb{C}$. $\bar{\partial} + \bar{\partial}^*$ is an $\mathbb{R}^2$-invariant elliptic operator on M. As there are no L^2 (anti)holomorphic functions on $\mathbb{C}$, the kernel $\ker(\bar{\partial})$ and cokernel $\mathrm{coker}(\bar{\partial})$ are trivial. Hence the L^2-index $\mathrm{ind}_{L^2}(\bar{\partial})$ of $\bar{\partial}$ is 0.

Higher index of the $\bar{\partial}$-operator

The Baum-Connes map defines a K-theory index Ind$(\bar{\partial})$ of $\bar{\partial}$ in $K_0(C^*(\mathbb{R}^2)) \cong K^0(\mathbb{R}^2)$.

On $\mathbb{R}^2$, consider the group 2-cocycle $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\alpha\Big((x,y),(x',y')\Big) = xy' - x'y.$$

A careful computation of the higher index associated to α in the main theorem gives

$$\mathrm{ind}_{[\alpha]}(\bar{\partial}) = 1.$$

Outlook

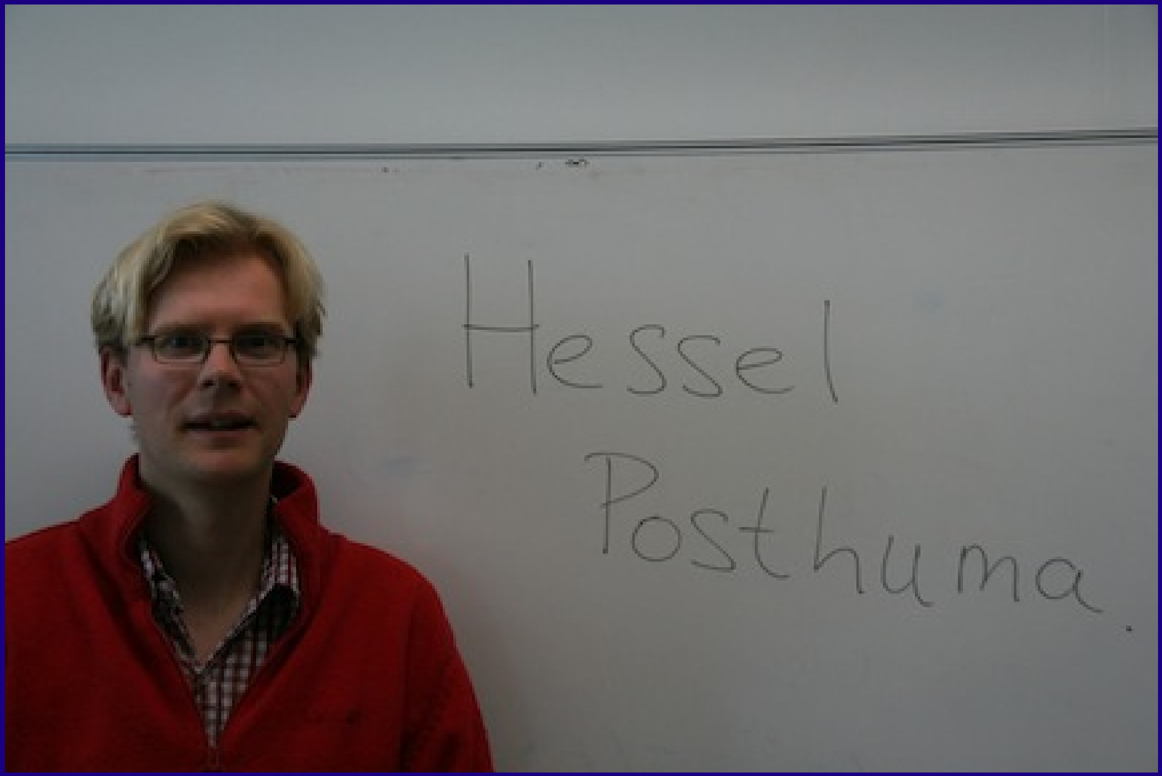
1. We are working on using the index theorem to extract more representation theory information.

2. We are applying the index theorem to study geometric properties of leaves of (singular) foliations.

3. The assumption about the ellipticity of D may be weakened, i.e. transversely elliptic to G-orbits.

Markus Pflaum

Hessel Posthuma



My friend

Thank you!