Some q-exponential formulas involving the double lowering operator ψ for a tridiagonal pair

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Definition of a tridiagonal pair

Let V denote a finite-dimensional vector space over a field $\mathbb{K}$.

Definition

By a **tridiagonal pair** (or TD pair) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ satisfying:

1. Each of A, A^* is diagonalizable.
2. There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that
$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$
where $V_{-1} = 0$ and $V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$
where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
4. There does not exist a subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

Example: Q-polynomial distance-regular graph

- Let $\Gamma = \Gamma(X, E)$ denote a Q-polynomial distance-regular graph.
- Let A denote the adjacency matrix of Γ.
- Fix $x \in X$. Let $A^* = A^*(x)$ denote the dual adjacency matrix of Γ with respect to x.
- Let W denote an irreducible (A, A^*)-submodule of $\mathbb{C}^{|X|}$.
- Then A, A^* form a TD pair on W.

Tridiagonal system

By a **tridiagonal system** (or TD system) on V, we mean a sequence

$$\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$$

that satisfies (1)–(3) below.

1. A, A^* is a tridiagonal pair on V.
2. $\{V_i\}_{i=0}^d$ is an ordering of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d).$$

3. $\{V_i^*\}_{i=0}^d$ is an ordering of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq d).$$

Relatives of a TD system

A given TD system can be modified in a number of ways to get a new TD system.

$$(A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) \qquad (A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$$

$$(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) \qquad (A^*; \{V_{d-i}^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$$

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These eight TD systems are said to be **relatives** of one another.

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Big Goal: Better understand the relationship between these relatives!

Relatives of a TD system

A given TD system can be modified in a number of ways to get a new TD system.

$$
\begin{array}{ll}
\longrightarrow (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) & (A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d) \\
\longrightarrow (A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) & (A^*; \{V_{d-i}^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d) \\
(A; \{V_i\}_{i=0}^d; A^*; \{V_{d-i}^*\}_{i=0}^d) & (A^*; \{V_i^*\}_{i=0}^d; A; \{V_{d-i}\}_{i=0}^d) \\
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\end{array}
$$

These eight TD systems are said to be **relatives** of one another.

Big Goal: Better understand the relationship between these relatives!

Smaller Goal: Better understand the relationship between these 2 relatives.

Assumptions/Notation

- Fix a TD system $\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ on V.

  Let $\Phi^{\Downarrow} = (A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ denote the second inversion of Φ.

- For $0 \le i \le d$, we let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) corresponding to the eigenspace V_i (resp. V_i^*).

q-Racah case

Definition

We say that the TD system Φ has q-**Racah type** whenever there exist nonzero scalars $q, a, b \in \overline{\mathbb{K}}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d},$$
$$\theta^*_i = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$.

Assumption

Throughout this talk, we assume that Φ has q-Racah type. For simplicity, we also assume that $\mathbb{K}$ is algebraically closed.

The split decompositions of V

Definition
For $0 \leq i \leq d$, define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d),$$

$$U_i^{\Downarrow} = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

We refer to $\{U_i\}_{i=0}^d$ as the **first split decomposition** of V.

We refer to $\{U_i^{\Downarrow}\}_{i=0}^d$ as the **second split decomposition** of V.

The maps K, B

Definition

Let $K : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i}. That is,

$$(K - q^{d-2i} I) U_i = 0$$

for $0 \leq i \leq d$.

Definition

Let $B : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, $U_i^{\Downarrow}$ is an eigenspace of B with eigenvalue q^{d-2i}. That is,

$$(B - q^{d-2i} I) U_i^{\Downarrow} = 0$$

for $0 \leq i \leq d$.

The linear transformation ψ

There is a linear transformation $\psi : V \to V$ associated with the TD system Φ. The exact definition is somewhat technical. One key feature of Ψ is given below.

Lemma (B. 2012)
For $0 \leq i \leq d$, both

$$\psi U_i \subseteq U_{i-1},$$
$$\psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}.$$

Moreover, $\psi^{d+1} = 0$.

In light of the above result, we refer to ψ as the **double lowering operator.**

We see that both $K\psi = q^2\psi K$ and $B\psi = q^2\psi B$.

The linear transformation Δ

We now introduce a linear transformation $\Delta : V \to V$ which sends the first split decomposition to the second split decomposition.

Lemma (B. 2012)
There exists a unique linear transformation $\Delta : V \to V$ which satisfies

$$\Delta(U_i) \subseteq U_i^{\Downarrow},$$
$$(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1},$$

for $0 \leq i \leq d$.

Δ as a polynomial in ψ

Theorem (B. 2014)

Both

$$\Delta = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{aq^{j-1} - a^{-1}q^{1-j}}{q^j - q^{-j}} \right) \psi^i,$$

$$\Delta^{-1} = \sum_{i=0}^{d} \left(\prod_{j=1}^{i} \frac{a^{-1}q^{j-1} - aq^{1-j}}{q^j - q^{-j}} \right) \psi^i.$$

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Question

Does this polynomial factor nicely?
If it does, what does that factorization mean?

The linear transformation $\mathcal{M}$

Definition

Define a linear transformation $\mathcal{M} : V \to V$ by

$$\mathcal{M} = \frac{aK - a^{-1}B}{a - a^{-1}}.$$

We will use this map $\mathcal{M}$ to find a factorization of Δ.

The q-exponential function

We now recall the q-exponential function. For nilpotent $T \in \mathrm{End}(V)$,

$$exp_q(T) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q^!} T^n.$$

Here

$$[n]_q^! = [n]_q[n-1]_q \cdots [1]_q$$

and

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Recall that the map $\exp_q(T)$ is invertible and its inverse is given by

$$\exp_{q^{-1}}(-T) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}}}{[n]_q^!} T^n.$$

Lemma

Both

$$K \exp_q \left(\frac{a^{-1}}{q - q^{-1}} \psi \right) = \exp_q \left(\frac{a^{-1}}{q - q^{-1}} \psi \right) \mathcal{M},$$

$$B \exp_q \left(\frac{a}{q - q^{-1}} \psi \right) = \exp_q \left(\frac{a}{q - q^{-1}} \psi \right) \mathcal{M}.$$

These results turns out to be the key to being able to factor the polynomial in ψ for Δ.

Δ as a product of q-exponentials

Theorem

Both

$$\begin{aligned} \Delta &= \exp_q\left(\frac{a}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right), \\ \Delta^{-1} &= \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right)\exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right). \end{aligned}$$

If we multiply out the right-hand side of the above product and use the q-binomial theorem to simplify the coefficients, we will obtain the polynomial for Δ given earlier in the talk.

Δ as a transition matrix

We view Δ as a transition matrix from the first split decomposition of V to the second. Consequently, we view

$$\exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right)$$

as a transition matrix from the first split decomposition to a decomposition of V which we interpret as a kind of half-way point.

We will describe this new decomposition of V using the linear transformation $\mathcal{M}$.

The eigenspaces of $\mathcal{M}$

Lemma

The map $\mathcal{M}$ is diagonalizable with eigenvalues $q^d, q^{d-2}, q^{d-4}, \ldots, q^{-d}$.

Definition

For $0 \leq i \leq d$ let W_i denote the eigenspace of $\mathcal{M}$ corresponding to the eigenvalue q^{d-2i}. Note that $\{W_i\}_{i=0}^d$ is a decomposition of V.

The eigenspaces of $\mathcal{M}$ as a half-way point

Lemma

For $0 \leq i \leq d$,

$$U_i = \exp_q\left(\frac{a^{-1}}{q-q^{-1}}\psi\right) W_i, \qquad W_i = \exp_{q^{-1}}\left(-\frac{a^{-1}}{q-q^{-1}}\psi\right) U_i,$$

$$U_i^{\Downarrow} = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right) W_i, \qquad W_i = \exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right) U_i^{\Downarrow}.$$

The eigenspaces of $\mathcal{M}$ as a half-way point

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For $0 \le i \le d$,

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$$U_i^{\Downarrow} = \exp_q\left(\frac{a}{q-q^{-1}}\psi\right) W_i, \qquad W_i = \exp_{q^{-1}}\left(-\frac{a}{q-q^{-1}}\psi\right) U_i^{\Downarrow}.$$

We see that W_i is the image of U_i under our q^{-1}-exponential in ψ!

We now know that the decomposition that we have regarded as a half-way point between the two split decompositions is the eigenspace decomposition for $\mathcal{M}$.

The actions of various linear transformations on W_i

Now that we know how to describe this half-way point, we can investigate the actions of our other linear transformations on this decomposition.

The actions of ψ, K, and B

Lemma

For $0 \le i \le d$,

$$\psi W_i \subseteq W_{i-1}.$$

Lemma

For $0 \le i \le d$,

$$(K - q^{d-2i} I) W_i \subseteq W_{i-1},$$
$$(B - q^{d-2i} I) W_i \subseteq W_{i-1}.$$

The action of Δ

Lemma

For $0 \leq i \leq d$,

$$(\Delta - I)W_i \subseteq W_0 + W_1 + \cdots + W_{i-1},$$
$$(\Delta^{-1} - I)W_i \subseteq W_0 + W_1 + \cdots + W_{i-1}.$$

The End

Thank you for your attention!