Envelope:
Localization for the Spectrum of a Matrix

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Bendixson

Hermitian part of $A \in \mathbb{C}^{n\times n}$:

$$H(A) = \frac{A + A^*}{2}$$

Eigenvalues of $H(A)$:

$$\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$$

For every eigenvalue $\lambda \in \sigma(A)$,

$$\delta_n(A) \leq \mathrm{Re}\lambda \leq \delta_1(A)$$

Numerical range

- Thus $\sigma(A)$ lies in

$$\{(s + \mathrm{i}\, t) : \; s, t \in \mathbb{R} \;\text{ with }\; s \leq \delta_1(A)\}$$

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$$\left\{ e^{-\mathrm{i}\,0}(s+\mathrm{i}\,t) : \ s,t \in \mathbb{R} \ \text{ with } \ s \le \delta_1(e^{\mathrm{i}\,0}A) \right\}$$

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- Thus $\sigma(A)$ lies in $(\forall\theta \in [0, 2\pi))$

$$\left\{ e^{-\mathrm{i}\,\theta}(s + \mathrm{i}\,t) : \; s, t \in \mathbb{R} \;\text{ with }\; s \le \delta_1(e^{\mathrm{i}\,\theta} A) \right\}$$

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- This infinite intersection of half-planes coincides with the **numerical range** (field of values) of A:

$$F(A) \;=\; \{ v^* A v \in \mathbb{C} : \; v \in \mathbb{C}^n \;\text{ with }\; v^* v = 1 \}$$

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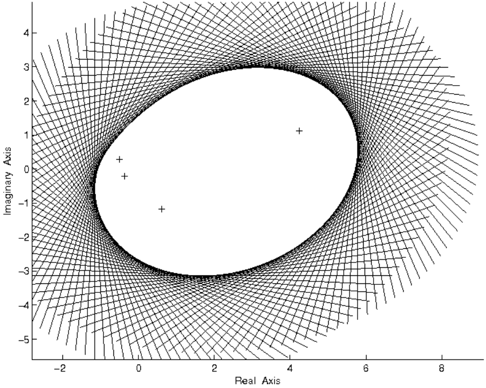
- In fact, this is Johnson's algorithm for computing and plotting the boundary points of $F(A)$.

The numerical range of a 4×4 Toeplitz matrix.



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- Replace infinite intersection of half-planes by an infinite intersection of regions in the complex plane (defined by the above cubic curves).
- The outcome is a localization region for the spectrum called the **envelope** of A.
- The envelope is contained in the numerical range and can be quite smaller.

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- $0 \le \mathrm{v}(A) - \mathrm{u}(A)^2$ is a measure of how close $\delta_1(A) + i\mathrm{u}(A)$ is to being a normal eigenvalue of A.

The inequality

Theorem [Adam and T.]
Every eigenvalue λ of $A \in \mathbb{C}^{n\times n}$ satisfies

$$(\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Im}\lambda - \mathrm{u}(A))^2 \le$$

$$(\delta_1(A) - \mathrm{Re}\lambda)[\mathrm{v}(A) - \mathrm{u}(A)^2 + (\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Re}\lambda - \delta_1(A))]$$

In and Out Regions

- Theorem gives rise to a cubic algebraic curve $\Gamma(A)$:

$$\left\{ s + \mathrm{i}\, t : \ s, t \in \mathbb{R}, \ \delta_2(A) - s + \frac{(\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2)}{(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2} = 0 \right\}$$

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- $\Gamma_{out}(A) \ =$

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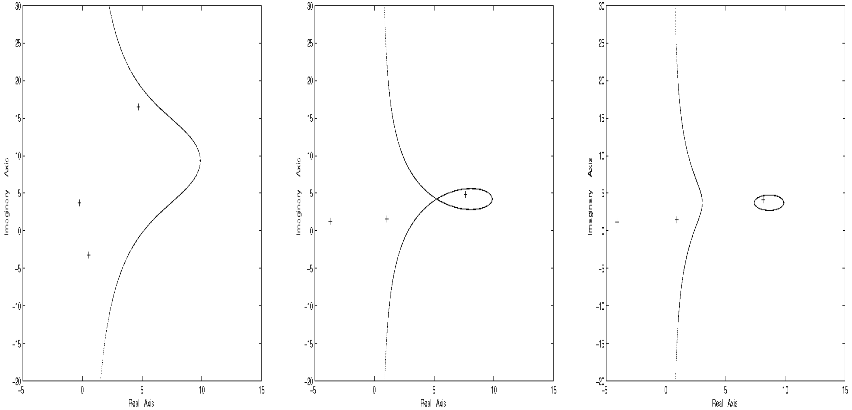
Possible configurations of $\Gamma(A)$



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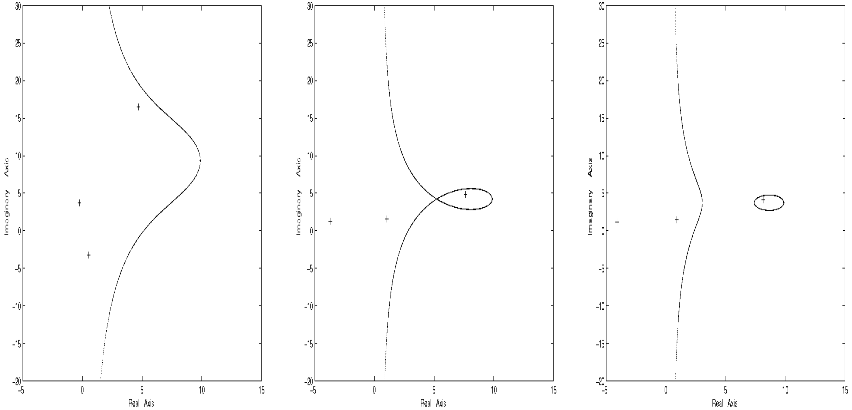
- $\Gamma(A)$ is asymptotic to the vertical line $\{z \in \mathbb{C} : \ \operatorname{Re} z = \delta_2(A)\}$
- $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a right most point of the numerical range.

See properties of $\Gamma(A)$



The envelope of A

Play the spinning game again: The **envelope** of A is

$$\mathcal{E}(A) = \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A)$$

Theorem [Psarrakos and T.] For any matrix $A \in \mathbb{C}^{n\times n}$,

$$\sigma(A) \subseteq \mathcal{E}(A) \subseteq F(A)$$

Proof

$$\mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A) \;=\; \{e^{-\mathrm{i}\,\theta}(s+\mathrm{i}\,t):\; s,t\in\mathbb{R}\;\text{ with }\; s\leq\delta_1(e^{\mathrm{i}\,\theta}A)\},$$

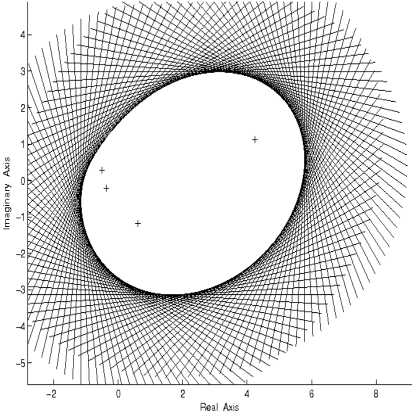
$$F(A) \;=\; \bigcap_{\theta\in[0,2\pi]} \mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A)$$

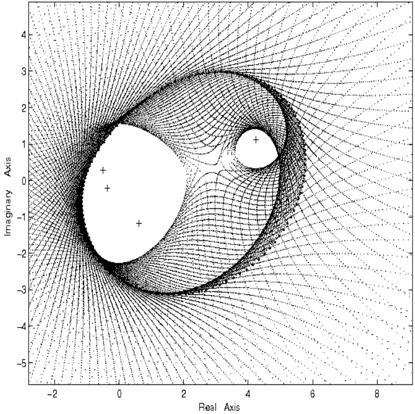
$$\sigma(A) \;=\; e^{-\mathrm{i}\,\theta}\sigma(e^{\mathrm{i}\,\theta}A) \;\subseteq\; e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;\subseteq\; \mathcal{H}_{in}(e^{\mathrm{i}\,\theta}A)$$

Hence

$$\sigma(A) \;\subseteq\; \mathcal{E}(A) \;=\; \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A) \;\subseteq\; F(A)$$

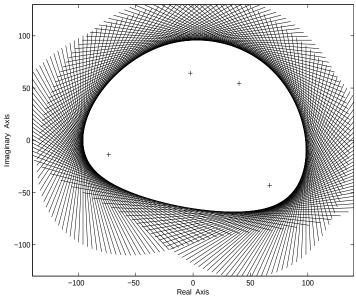
Numerical range and Envelope of a Toeplitz matrix

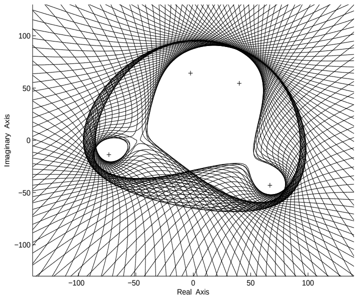




A complex matrix (and a better drawing method)

$$A = \begin{bmatrix} 14+i\,19 & -4-i & -55-i\,13 & -32+i\,13 \\ 27+i\,2 & 14-i\,25 & 64 & 72 \\ 54+i & 47-i\,3 & 14+i\,44 & -32-i\,42 \\ 76 & 73 & 4-i\,2 & -11+i\,24 \end{bmatrix}$$





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- For every real $r > 0$ and $a \in \mathbb{C}$, $\Gamma(rA) = r\,\Gamma(A)$ and $\mathcal{E}(aA) = a\,\mathcal{E}(A)$

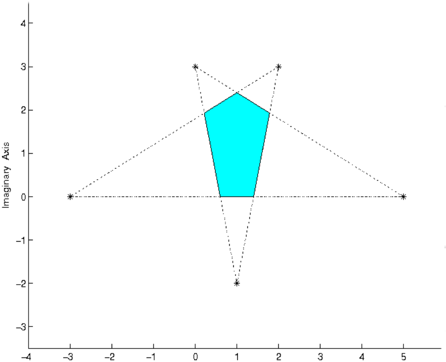
Interesting cases/behavior

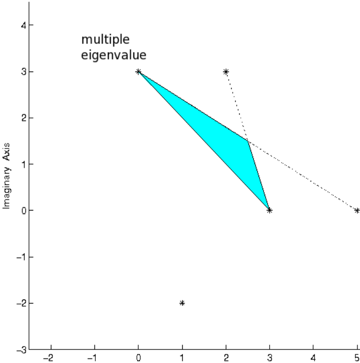
Proposition Let λ_0 be a **simple** eigenvalue of A on the **boundary** of $F(A)$. If λ_0 does **not** lie on a flat portion of $\partial F(A)$, or if it is a non-differentiable point of $\partial F(A)$, then λ_0 is an **isolated point** of the envelope $\mathcal{E}(A)$.

Normal matrices

$$D_1 = \mathrm{diag}\{\mathrm{i}\,3, 5, 2+\mathrm{i}\,3, 1-\mathrm{i}\,2, -3\},$$
$$D_2 = \mathrm{diag}\{\mathrm{i}\,3, \mathrm{i}\,3, 5, 2+\mathrm{i}\,3, 1-\mathrm{i}\,2, 3\}$$

$\mathcal{E}(D_1)$ and $\mathcal{E}(D_2)$ are the shaded regions **union** the isolated points.





Envelope of a normal matrix

- $\lambda_1, \lambda_2, \ldots, \lambda_k$ the simple **extremal** eigenvalues of normal A

(i.e., vertices of $\mathrm{Co}(\sigma(A))$ which must be isolated points of $\mathcal{E}(A)$)

- $\mathcal{C}(A) := \mathcal{E}(A) \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$

Proposition

(i) If all the eigenvalues of A are simple and extremal, then $\mathcal{E}(A) = \sigma(A)$.

(ii) If all the extremal eigenvalues of A are multiple, then $\mathcal{E}(A) = \mathcal{C}(A) = \mathrm{Co}(\sigma(A)) = F(A)$.

(iii) If $n = 2$ or 3, then $\mathcal{E}(A) = \sigma(A)$.

(iv) Let $n = 4$. If all the eigenvalues of A are extremal, and A does not have two double eigenvalues (for the case of two double eigenvalues, see (ii) above), then $\mathcal{E}(A) = \sigma(A)$.

Envelope of a hermitian matrix

Corollary Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix with eigenvalues $\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$. Then,

$$\mathcal{E}(A) = \{\delta_n(A)\} \cup [\delta_{n-1}(A), \delta_2(A)] \cup \{\delta_1(A)\} \subseteq [\delta_n(A), \delta_1(A)] = F(A)$$

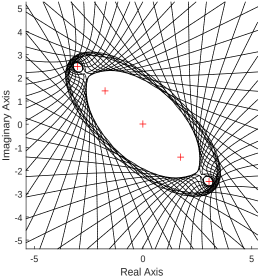
Tridiagonal Toeplitz matrices

$$T_n(c,a,b) = \begin{bmatrix} a & b & \cdots & 0 \\ c & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ 0 & \cdots & c & a \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad bc \neq 0.$$

- Numerical range of $T_n(c,a,b)$ is an elliptical disc.
- Envelope of $T_n(c,a,b)$, $bc \neq 0$, is symmetric with respect to a.
- Envelope of $T_n(c,a,b)$, $bc \neq 0$, is symmetric with respect to the line

$$\{a + \gamma e^{\mathrm{i}\frac{\arg(b)+\arg(c)}{2}} \,:\, \gamma \in \mathbb{R}\}.$$

Envelopes of a tridiagonal Toeplitz matrices



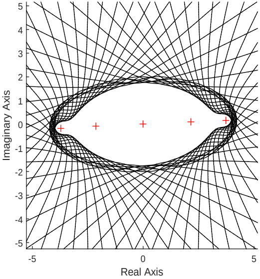


Figure: $\mathcal{E}(T_5(2+3i,0,-1-i))$ (left) and $\mathcal{E}(T_5(2+3i,0,0.8-i))$ (right).

Block-shift matrices

$$A = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_m \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{n\times n},$$

with $m > 1$ and square zero blocks along the main diagonal.
Theorem $\mathcal{E}(A)$ coincides with the circular disc $\mathcal{D}(0, R)$ centered at the origin, with radius

$$R = \left(\delta_1^2(A) - \left(\sqrt{2\delta_1(A)(\delta_1(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)} \right)^2 \right)^{1/2}.$$

Envelope of a Block-shift matrix

The numerical range of a block-shift matrix is also a circular disc.
The numerical radius of a block-shift matrix A is $r(A) = \delta_1(A)$.
Thus

$$r(A)^2 - R^2 = \left(\sqrt{2r(A)(r(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)}\right)^2.$$

2x2 matrices

Theorem Let A be a 2×2 complex matrix. Then $\mathcal{E}(A) = \sigma(A)$.

Similarities

Well-known result of Givens for the numerical range:

$$\bigcap \left\{ F(R^{-1}AR) : \; R \in \mathbb{C}^{n\times n}, \, \det(R) \neq 0 \right\} \; = \; conv\{\sigma(A)\}$$

An analogous result holds for the envelope (long proof if A is not diagonalizable):

$$\bigcap \left\{ \mathcal{E}(R^{-1}AR) : \; R \in \mathbb{C}^{n\times n}, \, \det(R) \neq 0 \right\} \; \subseteq \; \mathcal{E}(D(A)),$$

where $D(A)$ is the diagonal matrix whose diagonal entries are the eigenvalues of A.

Connection to k-rank Numerical Range

$$\begin{aligned}\Lambda_k(A) &= \left\{\mu \in \mathbb{C} : \; PAP = \mu P \text{ for some rank-}k \text{ orthog. proj. } P \in \mathbb{C}^{n\times n}\right\} \\ &= \left\{\mu \in \mathbb{C} : \; X^*AX = \mu I_k \text{ for some } X \in \mathbb{C}^{n\times k} \text{ such that } X^*X = I_k\right\}\end{aligned}$$

• Connected to the construction of quantum error correction codes for noisy quantum channels...

• Does not necessarily contain all of the eigenvalues of A.

Theorem $\Lambda_{n-1}(A) \subseteq \ldots \subseteq \Lambda_2(A) \subseteq \mathcal{E}(A) \subseteq F(A) = \Lambda_1(A)$

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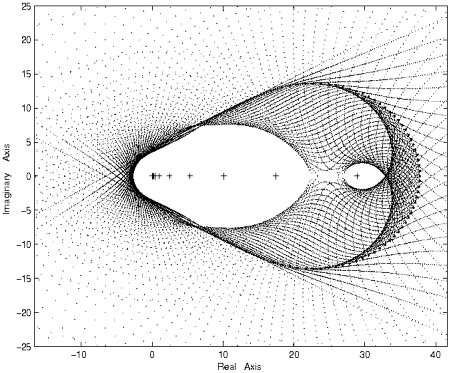
- The improvement expected depends on the **geometry of the eigenvalues**.
- Technique can potentially be generalized to utilize more eigenvalues of $H(A)$.

One last example

The envelope of a Frank matrix (11×11 highly ill-conditioned)



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