A Novel Method for Determining the Rank of a Matrix

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Outline

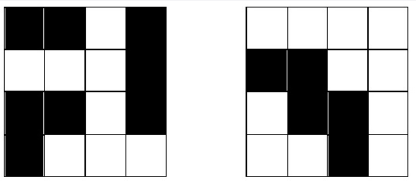
- Cauchon diagrams, Cauchon matrices, and the Cauchon algorithm;
- lacunary sequences and rank determination;
- the Cauchon algorithm, descending rank conditions, and bidiagonal factorization.

Cauchon Diagrams and Cauchon Matrices

Cauchon Diagram

An n-by-m *Cauchon diagram* C is an n-by-m grid consisting of $n \cdot m$ squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.

Examples of a Cauchon and a non-Cauchon diagrams



We denote by $\mathcal{C}_{n,m}$ the set of the n-by-m Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For $C \in \mathcal{C}_{n,m}$ and $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, $(i, j) \in C$ if the square in row i and column j is black.

Cauchon Matrix

Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and let $C \in \mathcal{C}_{n,m}$. We say that A is a *Cauchon matrix associated with the Cauchon diagram C* if for all (i, j), $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, we have $a_{ij} = 0$ if and only if $(i, j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a *Cauchon matrix*.

[Goodearl, Launois, and Lenagan, 2011]

There is a parametrization of the totally nonnegative cells by using the Cauchon diagrams. In fact, there is a one to one correspondence between these diagrams and the totally nonnegative cells.

The Cauchon Algorithm

Also called *deleting derivations algorithm*, *Cauchon reduction algorithm*.
The Cauchon algorithm was originally developed by G. Cauchon, while studying quantum matrices [Cauchon, 2003].

Notations

We denote by $\leq$ the lexicographic, on $\{1,\ldots,n\} \times \{1,\ldots,m\}$, i.e.,

$$(g,h) \leq (i,j) :\Leftrightarrow\ (g < i) \text{ or } (g = i \text{ and } h \leq j),$$

Set $E^\circ := \{1,\ldots,n\} \times \{1,\ldots,m\} \setminus \{(1,1)\}$, $E := E^\circ \cup \{(n+1,2)\}$.
Let $(s,t) \in E^\circ$. Then
$(s,t)^+ := \min\{(i,j) \in E \mid (s,t) \leq (i,j),\ (s,t) \neq (i,j)\}$.

The Cauchon Algorithm, [Goodearl, Launois, and Lenagan, 2011]

Let $A \in \mathbb{R}^{n,m}$. As r runs in decreasing order over the set E with respect to the lexicographical order, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,m}$ as follows.

1. Set $A^{(n+1,2)} := A$.
2. For $r = (s,t) \in E^{\circ}$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows.
   (a) If $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.
   (b) If $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \dfrac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$; $\tilde{A}$ is called *the matrix obtained from A (by the Cauchon algorithm)*.

$$A^{(r+)} = \begin{array}{cc} & \begin{array}{cc} j & t \end{array} \\ \begin{array}{c} i \\ s \end{array} & \begin{pmatrix} a_{ij}^{(r+)} & a_{it}^{(r+)} \\ a_{sj}^{(r+)} & a_{st}^{(r+)} \end{pmatrix} \end{array}$$

Example 1

Let

$$A = \begin{bmatrix} 6 & 3 & 3 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then by application of the Cauchon algorithm to A we obtain

$$A^{(4,4)} = \begin{bmatrix} 5 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(4,3)} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A^{(4,2)} = A^{(4,1)}$$

Example 1 Cont.

$$A^{(3,4)} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(3,3)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A^{(3,2)} = A^{(3,1)},$$

$$A^{(3,1)} = A^{(2,4)} = A^{(2,3)} = A^{(2,2)} = A^{(2,1)} = A^{(1,4)} = A^{(1,3)} = A^{(1,2)} = \tilde{A}.$$

Condensed Form of the Cauchon Algorithm

The condensed form of the Cauchon algorithm reduces the number of required arithmetic operations from $O(n^4)$ to $O(n^3)$.
This is accomplished by relating the entries of $A^{(k,2)}$ to the entries of $A^{(k+1,2)}$, $k = 2, \ldots, n$.

Condensed form of Cauchon Algrithm, [Adm and Garloff, 2014]

Let $A \in \mathbb{R}^{n,m}$. Set $A^{(n)} := A$.
For $k = n-1, \ldots, 1$ define $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{n,m}$ as follows:
For $i = 1, \ldots, k$,
for $j = 1, \ldots, m-1$
set $u_j := \min \left\{ h \in \{j+1, \ldots, m\} \,|\, a_{kh}^{(k+1)} \neq 0 \right\}$ (we set $u_j := \infty$ if this set is empty)

$$a_{ij}^{(k)} := \begin{cases} a_{ij}^{(k+1)} - \dfrac{a_{k+1,j}^{(k+1)} a_{iu_j}^{(k+1)}}{a_{k+1,u_j}^{(k+1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1)} & \text{if } u_j = \infty, \end{cases}$$

for $i = k+1, \ldots, n$, $j = 1, \ldots, m$, and $i = 1, \ldots, k$, $j = m$ $a_{ij}^{(k)} := a_{ij}^{(k+1)}$. Put $\hat{A} := A^{(1)}$.

$$A^{(k+1)} = \begin{array}{c c} & \begin{array}{ccccc} j & & & & u_j \end{array} \\ \begin{array}{c} i \\ \\ k+1 \end{array} & \left(\begin{array}{ccccc} a_{ij}^{(k+1)} & & \cdots & & a_{iu_j}^{(k+1)} \\ & & & & \\ a_{k+1,j}^{(k+1)} & 0 & \cdots & 0 & a_{k+1,u_j}^{(k+1)} \neq 0 \end{array} \right) \end{array}$$

Example 2

Example 2

Let

$$A = \begin{bmatrix} 6 & 3 & 3 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then by application of the condensed form of the Cauchon algorithm to A we obtain

$$A^{(3)} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A^{(1)} = \tilde{A}.$$

Lacunary sequence, [Launois and Lenagan, 2014]

Let $C \in \mathcal{C}_{n,m}$. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0, 1, \ldots, p)$$

which is strictly increasing in both arguments is a *lacunary sequence with respect to C* if the following conditions hold:

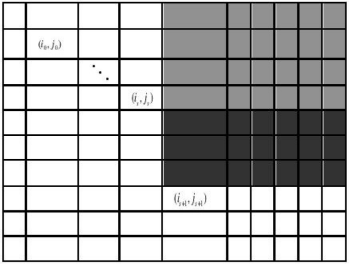
1. $(i_k, j_k) \notin C$, $k = 1, \ldots, p$;
2. $(i, j) \in C$ for $i_p < i \leq n$ and $j_p < j \leq m$.
3. Let $s \in \{0, \ldots, p-1\}$. Then $(i, j) \in C$ if

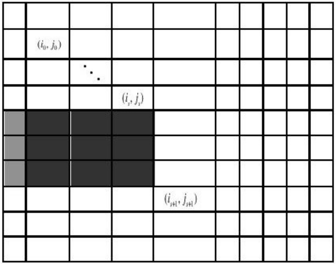
   either for all (i, j), $i_s < i < i_{s+1}$ and $j_s < j$,
   or for all (i, j), $i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$

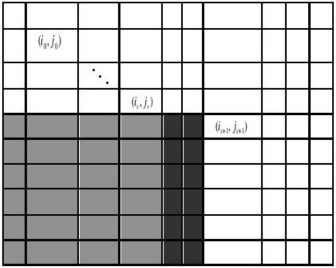
   and

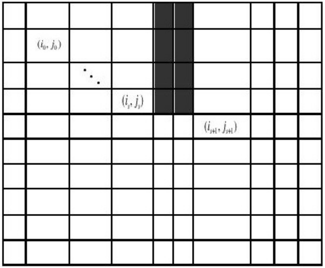
   either for all (i, j), $i_s < i$ and $j_s < j < j_{s+1}$
   or for all (i, j), $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

Condition 3 in the definition of the lacunary sequence






(i_0, j_0)
(i_s, j_s)
(i_{s+1}, j_{s+1})


(i_0, j_0)
(i_s, j_s)
(i_{s+1}, j_{s+1})

[Launois and Lenagan, 2014, Proposition 4.1], [Adm and Garloff, 2016, Proposition 4.11]

Let $A \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. For each position in C fix a lacunary sequence $\gamma = ((i_0, j_0), (i_1, j_1), \ldots, (i_t, j_t))$ with respect to C starting at this position. Assume that for all (i_0, j_0), we have

$$0 = \det A[i_0, i_1, \ldots, i_t | j_0, j_1, \ldots, j_t] \text{ if and only if } (i_0, j_0) \in C.$$

Then

$$\det A[i_0, i_1, \ldots, i_t | j_0, j_1, \ldots, j_t] = \tilde{a}_{i_0,j_0} \cdot \tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_t,j_t} \tag{1}$$

holds for all lacunary sequences γ.

Proposition, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,m}$ be such that $\tilde{A}$ is a Cauchon matrix and let γ be a lacunary sequence. Then γ allows the representation (1).

Rank Determination

Procedure, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,m}$ be a Cauchon matrix. Construct the sequence

$$\gamma = ((i_p, j_p), \ldots, (i_0, j_0)) \tag{2}$$

as follows:

- Put $(i_{-1}, j_{-1}) := (n+1, m+1)$.
- For $k = 0, 1, \ldots$, define

$$M := \{(i,j) \mid 1 \le i < i_{k-1},\ \ 1 \le j < j_{k-1},\ a_{ij} \ne 0\}.$$

If $M = \phi$, put $p := k - 1$. Otherwise, put $(i_k, j_k) := \max M$, where the maximum is taken with respect to the lexicographical order.

Lemma, [AAAFG, 2017]

The sequence that is obtained by the Procedure is a lacunary sequence with respect to C_A.

Theorem, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,m}$ be such that $\tilde{A}$ is a Cauchon matrix. Then $\mathrm{rank} A = p + 1$, where p is the length of the sequence which is obtained by application of the Procedure to $\tilde{A}$.

Theorem, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,m}$ be such that $\tilde{A}$ is a Cauchon matrix. Then for $i = 1, \ldots, n$ and $0 \leq l \leq n - i$, the rows $i, i+1, \ldots, i+l$ of A are linearly independent if and only if application of the Procedure to $\tilde{A}[i, \ldots, i+l|1, \ldots, m]$ results in a sequence of length l.

Corollary, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,n}$ be such that $\tilde{A}$ is a Cauchon matrix. Then A is nonsingular if and only if $\tilde{a}_{ii} \neq 0$ for all $i = 1, 2, \ldots, n$.

Example 3

Let

$$A = \begin{bmatrix} 6 & 3 & 3 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then by application of the condensed form of the Cauchon algorithm to A we obtain

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence $\mathrm{rank} A = 3$ and the rows 2 and 3 are linearly dependent while 3 and 4 are linearly independent.

SEB Factorization

Definition

Let $A \in \mathbb{R}^{n,n}$. Then we say that A has a *successively ordered elementary factorization* (SEB) if A can be written as

$$A = \left(\prod_{k=1}^{n-1} \prod_{j=n}^{k+1} L_j(l_{jk}) \right) D \left(\prod_{k=n-1}^{1} \prod_{j=k+1}^{n} U_j(u_{kj}) \right), \tag{3}$$

where $L_i(s) = I + sE_{i,i-1}$, $U_j(t) = I + tE_{j-1,j}$, $2 \leq i, j \leq n$, and D is a diagonal matrix.

Descending Rank Conditions

Definition, [Johnson, Olesky, and van den Driessche]

Let $A \in \mathbb{R}^{n,n}$. Then A satisfies the *column descending rank condition* if for all l with $1 \leq l \leq n-1$, for all z with $0 \leq z \leq l-1$, and for all p satisfying $l-z \leq p \leq n-z-1$,

$$\mathrm{rank} A[p+1,\ldots,p+z+1|1,\ldots,l] \leq \mathrm{rank} A[p,\ldots,p+z|1,\ldots,l].$$

Similarly, A satisfies the *row descending rank condition* if with the indices as above

$$\mathrm{rank} A[1,\ldots,l|p+1,\ldots,p+z+1] \leq \mathrm{rank} A[1,\ldots,l|p,\ldots,p+z].$$

A satisfies the descending rank conditions if A satisfies both the row and column descending rank conditions.

Theorem, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,n}$ and $B := PAP$. If A satisfies the descending rank conditions, then the following statements hold:

(i) If $\tilde{b}_{ij} = 0$ for some $i \geq j$, then $\tilde{b}_{it} = 0$ for all $t < j$;

(ii) if $\tilde{b}_{ij} = 0$ for some $i \leq j$, then $\tilde{b}_{tj} = 0$ for all $t < i$;

(iii) $\tilde{B}$ is a Cauchon matrix.

Theorem, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,n}$ and $B := PAP$. If B satisfies (i) and (ii) in the above Theorem, then A satisfies the descending rank conditions.

Theorem, [AAAFG, 2017]

Let $A \in \mathbb{R}^{n,n}$ and $B := PA^TP$. Then the following statements are equivalent:

(a) A satisfies the descending rank conditions.

(b) B satisfies (i) and (ii) in the above Theorem.

(c) A has an SEB factorization and l_{jk} and u_{kj}, $k = 1, \ldots, n-1$, $j = k+1, \ldots, n$, and d_{ii}, $i = 1, \ldots, n$ are given by

  (a) $l_{n1} = \frac{\tilde{b}_{n1}}{\tilde{b}_{n2}}$, $l_{n-1,1} = \frac{\tilde{b}_{n2}}{\tilde{b}_{n3}}$, …, $l_{21} = \frac{\tilde{b}_{n,n-1}}{\tilde{b}_{nn}}$,
  $l_{n2} = \frac{\tilde{b}_{n-1,1}}{\tilde{b}_{n-1,2}}$, $l_{n-1,2} = \frac{\tilde{b}_{n-1,2}}{\tilde{b}_{n-1,3}}$, …, $l_{3,2} = \frac{\tilde{b}_{n-1,n-2}}{\tilde{b}_{n-1,n-1}}$, …,
  $l_{n,n} = \frac{\tilde{b}_{21}}{\tilde{b}_{22}}$;

  (b) $d_{ii} = \tilde{b}_{n-i,n-i}$, $i = 1, \ldots, n$;

  (c) $u_{n-1,n} = \frac{\tilde{b}_{12}}{\tilde{b}_{22}}$,
  $u_{n-2,n-1} = \frac{\tilde{b}_{13}}{\tilde{b}_{23}}$, $u_{n-2,n} = \frac{\tilde{b}_{23}}{\tilde{b}_{33}}$, …,
  $u_{12} = \frac{\tilde{b}_{1,n}}{\tilde{b}_{2,n}}$, $u_{13} = \frac{\tilde{b}_{2,n}}{\tilde{b}_{3,n}}$, …, $u_{1n} = \frac{\tilde{b}_{n-1,n}}{\tilde{b}_{n,n}}$,

with the convention $\frac{0}{0} := 0$.

The End

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