Bott-Samelson Algebras and Junzo's Bold Conjecture

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(joint work with Larry Smith)

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Preliminaries

$$A = \bigoplus_{i=0}^{d} A^i = \mathbb{F}[x_1, \ldots, x_n] / \langle f_1, \ldots, f_n \rangle$$

- complete intersection means a graded Artinian complete intersection algebra with the standard grading
- The formal dimension of A is the maximum d for which $A^d \neq 0$.
- A has the sLp if $\exists \ell \in A^1$ such that $\times \ell^{d-2i} \colon A^i \to A^{d-i}$ are isomorphisms for $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.
- An embedding is an injective $\mathbb{F}$ algebra homomorphism $\phi \colon A \to A'$ between two rings of the same formal dimension.
- A inherits the sLp from A' if there is a simultaneous Lefschetz element for both A and A'.

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Every complete intersection can be embedded in a quadratic complete intersection.

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Examples

- (monomial complete intersections)

$$\mathbb{F}[x]\Big/\langle x^{m+1}\rangle \longrightarrow \mathbb{F}[X_1,\ldots,X_m]\Big/\langle X_1^2,\ldots,X_m^2\rangle$$

$$x \longmapsto (X_1+\cdots+X_m)$$

- ("split" complete intersections)

$$\mathbb{F}[x,y]\Big/\left\langle x\prod_{i=0}^{m-1}(x-\lambda_i y), y\prod_{i=0}^{n-1}(y-\mu_i x)\right\rangle$$

- (certain coinvariant rings)

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- (certain coinvariant rings)

Coinvariant Rings

- $\mathbb{F}$ any field.
- $V = (\mathbb{F}^n)$ a finite dimensional vector space over $\mathbb{F}$.
- $R = \mathbb{F}[V] = \mathrm{Sym}(V^*)$ the ring of polynomial functions on V.
- $W \subset \mathrm{GL}(V)$ a finite group so that W acts on R.
- $R^W \subset R$ the subring of W-invariant polynomials.
- $R_W := R \Big/ \left\langle \left(R^W\right)^+ \right\rangle$ = the **coinvariant ring** of W.

Theorem (Shephard-Todd '54, Chevalley '55)

Assuming that $|W| \in \mathbb{F}^\times$, R_W is a complete intersection if and only if W is generated by pseudo-reflections.

Definition

- a pseudo-reflection is an element $s \in \mathrm{GL}(V)$ with $|s| < \infty$ that fixed a hyperplane V_s pointwise
- W is field friendly (or f.f.) if $|W| \in \mathbb{F}^\times$

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The Main Result

Theorem (Smith-M.- '15)

If W is a finite f.f. pseudo-reflection group, and is generated by reflection of order two, then there is a quadratic complete intersection Q and an embedding

$$\phi \colon R_W \to Q.$$

Bott-Samelson Bimodules

- W any finite f.f. pseudo-reflection group
- $s \in W$ any pseudo-reflection
- $R^s \subset R$ the s-invariant subring

Construction

Given any sequence of pseudo-reflections $\underline{w} = s_1, \ldots, s_k$ define the ***Bott-Samelson bimodule*** *for $\underline{w}$ by*

$$BS(\underline{w}) := R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_k}} R$$

and define the ***Bott-Samelson algebra*** *by*

$$\overline{BS}(\underline{w}) := \mathbb{F} \otimes_R BS(\underline{w}) \cong \mathbb{F} \otimes_{R^{s_1}} \cdots \otimes_{R^{s_k}} R$$

e.g. $k = 1$:

- $BS(s) := R \otimes_{R^s} R$
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Bott-Samelson Algebras

Fact

For any sequence of pseudo-reflections $s_1, \ldots, s_k$ we have

$$\overline{BS}(s_1, \ldots, s_k) \cong \overline{BS}(s_1, \ldots, s_{k-1})[X] \big/ \langle X^{|s_k|} - B \rangle$$

Fact

For any sequence of pseudo-reflections $\underline{w} = s_1, \ldots, s_k$,

1. *the Bott-Samelson algebra $\overline{BS}(\underline{w})$ is a complete intersection of formal dimension k.*
2. *Moreover, if $|s_1| = 2, \ldots, |s_k| = 2$ then $\overline{BS}(\underline{w})$ is a quadratic complete intersection.*

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The Bott-Samelson Map

Fact

There is a well defined map of $\mathbb{F}$ algebras

$$\iota_{(s_1,\ldots,s_k)} : R_W \longrightarrow \overline{BS}(s_1,\ldots,s_k)$$
$$\bar{r} \longmapsto (\bar{1} \otimes 1 \otimes \cdots \otimes 1) \cdot r$$
$$= \bar{1} \otimes 1 \otimes \cdots \otimes r$$

called the ***Bott-Samelson map****.*

Problem

Find a sequence of pseudo-reflections $\underline{w}$ for which $\iota_{\underline{w}}$ is an embedding.

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Demazure Operators

- $s \in W$ any pseudo-reflection
- $\ell_s \in V^*$ any s-anti invariant
- $\Delta_s : R \to R(-1)$ the Demazure operator for s

$$\Delta_s(f) = \frac{f - s(f)}{\ell_s}$$

- $\underline{w} = s_1, \ldots, s_k$ any sequence of pseudo-reflections
- $\Delta_{\underline{w}} : R \to R(-k)$ the composition

$$\Delta_{\underline{w}} = \Delta_{s_1} \circ \cdots \circ \Delta_{s_k}$$

Key Lemma (Neumann-Neusel-Smith '96)

Let W be finite f.f. pseudo-reflection group, and let $u \in (R_W)^N$ be a socle generator in the coinvariant ring. Then there is some sequence of pseudo-reflections $\underline{w_0} := s_1, \ldots, s_N$ for which

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Extended Demzure Operators and the Bott-Samelson Map

Fix any sequence of pseudo-reflections $\underline{w} = s_1, \ldots, s_k$ and assume that $|s_1| = 2, \ldots, |s_k| = 2$.

Fact 1

There is a well defined "extended Demazure composition"

$$\hat{\Delta}_{\underline{w}} \colon \overline{BS}(\underline{w}) \to \overline{BS}(\underline{w})(-k).$$

Fact 2

The (extended) Demazure composition commutes with the Bott-Samelson map, i.e.

$$\hat{\Delta}_{\underline{w}}\left(\iota_{\underline{w}}(f)\right) = \iota_{\underline{w}}\left(\Delta_{\underline{w}}(f)\right), \quad \forall f \in R.$$

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The Main Result Again

Theorem (Smith-M.- '15)

Let W be a finite f.f. pseudo-reflection group generated by reflections of order 2, and let $\underline{w_0} := s_1, \ldots, s_N$ be a sequence of pseudo-reflections for which $\Delta_{\underline{w_0}}(u) \neq 0$ for some socle generator $u \in (R_W)^N$. Then

1. *the Bott-Samelson algebra $\overline{BS}(\underline{w_0})$ is a quadratic complete intersection, and*
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Proof.

$$\begin{aligned} \iota_{\underline{w_0}}\left(\Delta_{\underline{w_0}}(u)\right) &= \Delta_{\underline{w_0}}(u)(\bar{1} \otimes 1 \otimes \cdots \otimes 1) && \neq 0 \\ &= \hat{\Delta}_{\underline{w_0}}\left(\iota_{\underline{w_0}}(u)\right) && \neq 0 \\ &\Rightarrow \iota_{\underline{w_0}}(u) && \neq 0 \end{aligned}$$

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Chris McDaniel (joint work with Larry Smith)

Lefschetz Properties

Fact

If $\mathbb{F}$ has characteristic 0 then the Bott-Samelson algebra $\overline{BS}(s_1, \ldots, s_k)$ has the strong Lefschetz property.

Question

Does R_W inherit the sLp from the Bott-Samelson it's embedded in?

Recall: The Bott-Samelson map is an R module map:

$$\iota_{\underline{w_0}} : R_W \longrightarrow \overline{BS}(\underline{w_0})$$

$$\bar{r} \longmapsto (\bar{1} \otimes 1 \cdots \otimes 1) \cdot r$$

Fact

R_W inherits the sLp from $\overline{BS}(\underline{w_0}) \Leftrightarrow \overline{BS}(\underline{w_0})$ is a Lefschetz R module.

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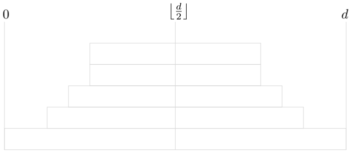
Lefschetz R Modules

- A Lefschetz R Module: is a graded finite dimensional R module that has a Lefschetz element in R, i.e. $M = \bigoplus_{i=0}^{d} M^i$ such that $\exists$ a linear form $\ell \in R$ such that

$$\times \ell^{d-2i} : M^i \xrightarrow{\cong} M^{d-i} \qquad 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$$

- Lefschetz R Module $\Rightarrow$ Centered Decomposition:

$$M = \bigoplus_{j \in \mathbb{Z}} \bigoplus_{a_j \in \mathbb{Z}} M_j(a_j)$$



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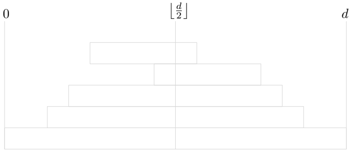

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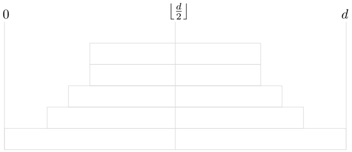

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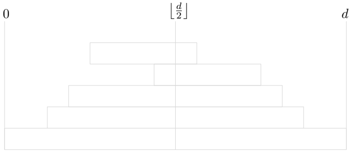

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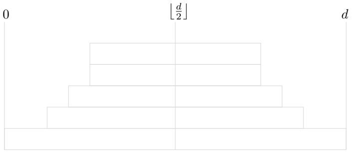

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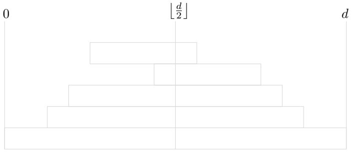

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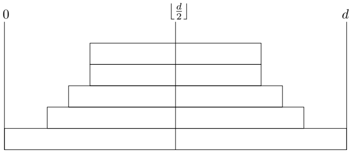

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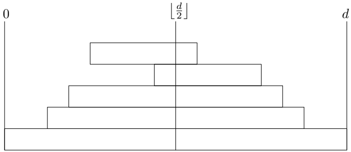

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Decompositions of Bott-Samelson Algebras

Observation

If any summand in an R module decomposition of $\overline{BS}(\underline{w_0})$ is OFF CENTER, then R_W CANNOT inherit the sLp.

Question

How to find R module decompositions of $\overline{BS}(\underline{w_0})$???

Enter representation theory:

Amazing Fact

For Coxeter groups, the R module decompositions of Bott-Samelson algebras can be read off from the multiplicative structure of the Hecke algebra!!

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$$\mathcal{H}_W := \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}] \cdot H_w \quad \leftarrow \left(\text{standard basis}\right)$$

$$H_w \cdot H_s = \begin{cases} H_{ws} & \text{if } w < ws \\ (v^{-1} - v) \cdot H_w + H_{ws} & \text{if } ws < w \end{cases}$$

- $\exists$ an involution $\iota : \mathcal{H}_W \to \mathcal{H}_W$ s.t. $\begin{cases} \iota(P(v)) = & P(v^{-1}) \\ \iota(H_x) = & H_{x^{-1}}^{-1} \end{cases}$
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- Let $R = \mathbb{F}[V]$ have *doubled degrees*, i.e. $\deg(V^*) = 2$.
- $\mathcal{R}$ = the category of R bimodules
- For any simple reflection $s \in S$ define the simple bimodule

$$B_s := R \otimes_{R^s} R(1)$$

- Soergel's Bimodule Category: $\mathcal{B}_S$ = the full Karoubian monoidal subcategory of $\mathcal{R}$ generated by the simple bimodules B_s.

Fact

The Bott-Samelson bimodule $BS(\underline{w})$ is a Soergel bimodule for every sequence of simple reflections $\underline{w} = s_1, \ldots, s_k$.

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$$\begin{aligned} BS(\underline{w})(k) &\cong R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_k}} R(k) \\ &\cong (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \end{aligned}$$

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Soergel Bimodules for $W = (W, S)$

- Let $R = \mathbb{F}[V]$ have *doubled degrees*, i.e. $\deg(V^*) = 2$.
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Soergel's Categorification Theorem

Chris McDaniel (joint work with Larry Smith)

- $\mathcal{H}_W$ Hecke algebra
- $\mathcal{B}_S$ = category Soergel bimodules
- Define the split Grothendieck group of $\mathcal{B}_S$:

$$[\mathcal{B}_S] := \bigoplus_{B \in \mathcal{B}_S} \mathbb{Z} \cdot [B] \Big/ \langle [B_1 \oplus B_2] - [B_1] - [B_2] \rangle$$

  - a ring via tensor product, i.e. $[B_1] \cdot [B_2] = [B_1 \otimes_R B_2]$
  - a $\mathbb{Z}[v, v^{-1}]$ algebra, i.e. $v^j \cdot [B] = [B(j)]$

Theorem (Soergel '07)

There is a $\mathbb{Z}[v, v^{-1}]$ algebra isomorphism

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defined by $\mathcal{E}(v) = [R(1)]$ and $\mathcal{E}(C'_s) = [B_s]$.

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The Character Map and Soergel's Conjecture

Theorem (Soergel '07)

The (left) inverse of the categorification map is the character map $\mathcal{E}^{-1}\colon\ [\mathcal{B}_S] \to \mathcal{H}_W$ *defined by*

$$\mathcal{E}^{-1}([B]) = \sum_{x \in W} \mathrm{Poin}\left(\overline{\mathrm{Hom}(B, D_x)}, v\right) \cdot H_x.$$

for some "standard bimodules" $D_w \in \mathcal{R}$*.*

$$\text{Recall:} \quad C'_w = \sum_{x \in W} h_{x,w}(v) \cdot H_x.$$

Conjecture (Soergel '07)

There are bimodules $B_w \in \mathcal{B}_S$ $(w \in W)$ *such that*

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Indecomposable Soergel Bimodules

Theorem (Soergel '07)

There is a one-to-one correspondence between (isomorphism classes of) indecomposable R bimodules in $\mathcal{B}_S$ and $W \times \mathbb{Z}$,

$$(w, j) \mapsto B_w(j).$$

where $\overline{B_w} = \bigoplus_{i=-\ell(w)}^{\ell(w)} \left(\overline{B_w}\right)^i$ lives in degrees "centered around zero".

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Soergel's Conjecture is True!!

Theorem (Elias-Williamson '14)

1. *Soergel's Conjecture is true, and*
2. $\overline{B_w}$ *is a Lefschetz* R *module, i.e. there is* $\ell \in V^*(= R^2)$ *such that*

$$\times \ell^i : \left(\overline{B_w}\right)^{-} i \to \left(\overline{B_w}\right)^{i}$$

is an isomorphism for each $0 \leq i \leq \ell(w)$.

Proof.
By induction on the Bruhat ordering of W. □

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Decomposing Bott-Samelson Bimodules

For a sequence of (simple) reflections $s_1, \ldots, s_k$, we have

$$\begin{array}{ccccc}
\mathcal{H}_W \ni & C'_{s_1} \cdots C'_{s_k} & = & \sum_{w \in W} P_{w,(s_1,\ldots,s_k)}(v) C'_w \\
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Conclusion:

$$BS(s_1, \ldots, s_k)(k) \cong \bigoplus_{w \in W} P_{w,(s_1,\ldots,s_k)}(v) B_w.$$

Fact

$\overline{BS}(s_1, \ldots, s_k)$ is a Lefschetz R module (if and) only if the polynomials $P_{w,(s_1,\ldots,s_k)}(v)$ are constant.

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Example

$W = A_3 = \mathfrak{S}_4, \qquad S = s - t - u, \qquad \underline{w_0} = s, u, t, u, s, t$

- The Bott-Samelson map is an embedding

$$\iota_{\underline{w_0}} : R_W \xrightarrow{\text{(embedding)}} \overline{BS}(\underline{w_0})$$

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Observation

The embedded coinvariant ring $\iota_{\underline{w_0}}(R_W) \subset \overline{BS}(\underline{w_0})$ does NOT inherit the sLp from $\overline{BS}(\underline{w_0})$.

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3. Prove Junzo's Bold Conjecture for all complete intersections!!

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Chris McDaniel (joint work with Larry Smith)

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