Surface and Interfacial Waves over Currents and Point-Vortices

Chris Curtis

Department of Mathematics and Statistics
San Diego State University

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Collaborators

- Henrik Kalisch (U. of Bergen)
- Katie Oliveras (Seattle U.)
- Theresa Morrison (SDSU)

Waves over Depth-Varying Currents and Vortices

- Currents are ubiquitous in fluids. How do they interact with free surfaces and interfaces?
- Eddies are ubiquitous as well. Same question.
- How much of the above questions can we answer with simple vorticity models?

Depth-Varying Currents

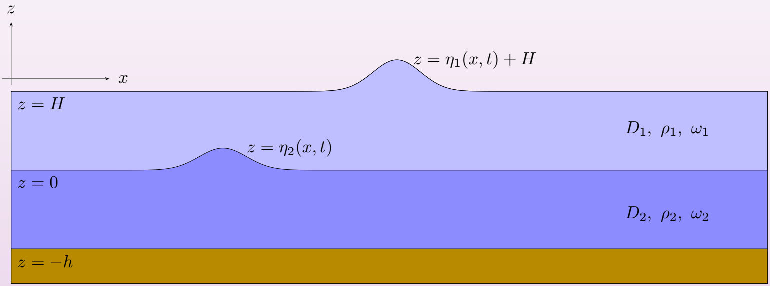


Figure: Density stratified fluid with piecewise constant vorticity.

Depth Varying Currents

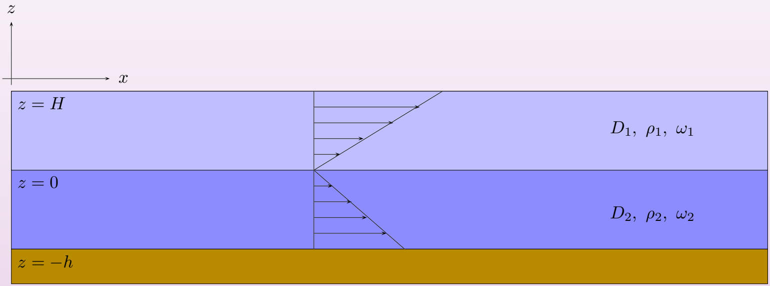
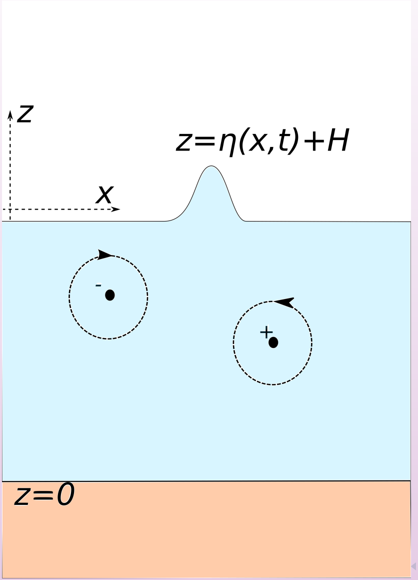


Figure: Discontinuous Linear Shear Profiles.

Irrotational Point Vortices



The DNO and AFM Methods

- To close our system in terms of surface height η and surface velocity potential q, we introduce the DNO $G(\eta)$ (see the ouvre of Craig and collaborators) so that

$$\eta_t = G(\eta)q.$$

- A complimentary point of view to doing this is the method of Ablowitz, Fokas, and Mussilimani (AFM). One can readily show the AFM and DNO approaches are formally equivalent.
- AFM leverages solving $\Delta\phi = 0$ so that you can turn the boundary equation into the equivalent set of integral equations

$$\int e^{-ikx}\left(\cosh(k(\eta+h))\eta_t + iq_x \sinh(k(\eta+h))\right) dx = 0, \; k \neq 0.$$

Outline

- **Part I**: We use the AFM approach to deal with the depth-varying currents, derive shallow-water approximations, and present numerical results of dynamics.
- **Part II**: We use a modified version of AFM to rewrite the problem in terms of surface variables and vortex positions alone. We then use the DNO to build numerical schemes.

Part I: Waves over Currents

- The constant vorticity and shear flow literature is far too vast to summarize.
- Civil engineer, Dalrymple, began looking at layer models to study more general density and shear profiles.
- Maslow and Redekopp also study arbitrary incompressible density/shear profiles varying in vertical.
- Generally, one surface, or rigid lid/internal layer is studied.

Problem Formulation

- Restricting ourselves to a linear shear, or constant vorticity, background flow within each layer of the fluid ($j = 1, 2$), Euler's equations of motion for $(x, z) \in D_j$ become

$$\nabla \cdot \mathbf{u}_j = 0, \tag{1}$$

$$\nabla \times \mathbf{u}_j = \omega_j \hat{\mathbf{y}} \tag{2}$$

$$\partial_t \mathbf{u}_j + \mathbf{u}_j \cdot \nabla \mathbf{u}_j = -\frac{1}{\rho_j} \nabla p_j - g\hat{\mathbf{z}}, \tag{3}$$

where $\mathbf{u}_j = \begin{bmatrix} u_j & v_j \end{bmatrix}^T$ represents the fluid velocities in D_j.

Problem Formulation

- At the various interfaces, we enforce the kinematic boundary conditions

$$
\begin{aligned}
v_2 &= 0, && z = -h, && (4)\\
\partial_t \eta_1 &= v_1 - u_1 \partial_x \eta_1, && z = \eta_1(x,t) + H, && (5)\\
\partial_t \eta_2 &= v_1 - u_1 \partial_x \eta_2, && z = \eta_2^+(x,t) && (6)\\
\partial_t \eta_2 &= v_2 - u_2 \partial_x \eta_2, && z = \eta_2^-(x,t) && (7)
\end{aligned}
$$

as well as the pressure relations given by

$$
\begin{aligned}
p_1 &= p_c, && z = \eta_1(x,t) + H, && (8)\\
p_1 &= p_2, && z = \eta_2(x,t). && (9)
\end{aligned}
$$

By η_2^+ and η_2^- we mean the regions just above and just below the surface $z = \eta_2(x,t)$ respectively.

Problem Formulation

- Using $\mathbf{u}_j = \omega_j z\hat{\mathbf{z}} + \nabla\phi$ coupled with AFM, we can find a closed system describing the evolution of the surface and internal layer.
- We rescale the bistratified, bilinear shear system via the following non-dimensional parameters

$$\tilde{x} = x/L,\ \tilde{z} = z/H,\ \tilde{t} = \frac{\sqrt{gH}}{L}t,\ \tilde{k} = Lk,$$

$$\eta_j = a\tilde{\eta}_j,\ Q = \frac{agL}{\sqrt{gH}}\tilde{Q},\ q_j = \frac{agL}{\sqrt{gH}}\tilde{q}_j.$$

- Take the balances

$$\frac{H}{h} = \frac{1}{\bar{d}},\ \frac{a}{H} = \epsilon,\ \frac{H}{L} = \gamma,\ \omega_j\sqrt{\frac{H}{g}} = \tilde{\omega}_j,\ \frac{\rho_2}{\rho_1} = \tilde{\rho},$$

Problem Formulation

- While complicated, AFM formulation readily allows for asymptotically reduced models to be derived.
- However, just putting interfaces next to one another is not always physically sensible i.e. what about Kelvin–Helmholtz instabilities?

Kelvin–Helmholtz Instabilities

- Using $e^{-ikx+i\Omega t}$, we find the dispersion relationship

$$
\begin{aligned}
\Pi = & k^4 \Big((\rho + \bar{d}s^2 \varphi(s)\varphi(\bar{d}s))\tilde{\Omega}^4 + \Big(\rho(\omega_1 \varphi(s) + \omega_2 \bar{d}\varphi(\bar{d}s)) - 2\rho\omega_1 \\
& + \Big(\rho\omega_1^2 - \rho(1+\omega_1^2)\varphi(s) + (\omega_1^2 - 2\omega_1\omega_2\rho - \rho)\bar{d}\varphi(\bar{d}s) + \omega_1(\omega_1 \\
& + \bar{d}\varphi(\bar{d}s) \Big(2\omega_1(\rho - 1) + \omega_1^2(\omega_2\rho - \omega_1) + (2\omega_1 - \rho(\omega_1 + \omega_2 + \\
& + (\rho - 1)\bar{d}\varphi(\bar{d}s)(-\omega_1^2 + \varphi(s)(1+\omega_1^2)) \Big)
\end{aligned}
$$

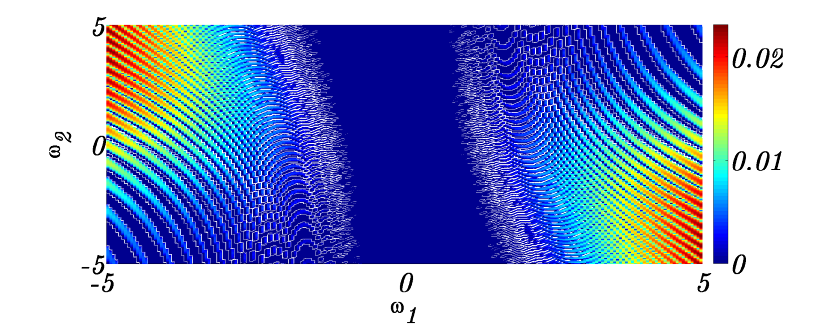
where $s = \gamma k$, $\tilde{\Omega} = \Omega/k$, and

$$\varphi(s) = \frac{\tanh(s)}{s}.$$

- I know I left a formula trailing off the edge of the slide. But that kind of makes my point.

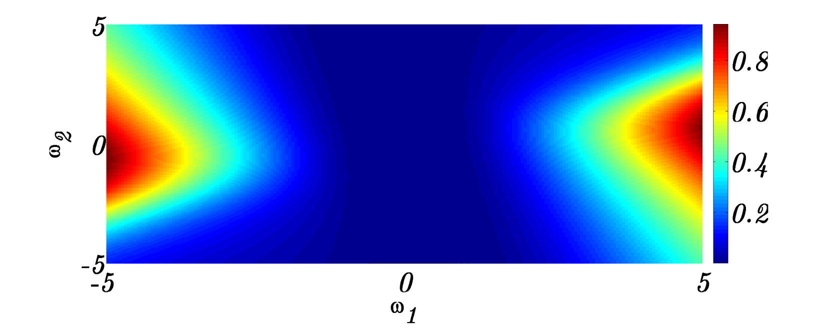
Kelvin–Helmholtz Instabilities

- Taking $\bar{d} \ll 1$, we can use asymptotic arguments to show there are shear/density configurations for which only real spectra can be found for all wave numbers.
- Letting $\bar{d} = .25$, $\rho = 820$, we get the following plot of the maximum imaginary part of the spectrum for $0 \leq s \leq 300$.



Kelvin–Helmholtz Instabilities

- Letting $\bar{d} = 4$, $\rho = 1.5$, we get the following plot of the maximum imaginary part of the spectrum for $0 \leq s \leq 300$.



Reduced Model/System of KdV equations

- To capture nonlinear effects, we take the balance $\epsilon = \gamma^2$, and then expand in ϵ.
- From $\epsilon = 0$ problem, we have four wave speeds λ_j as $\epsilon \to 0$.
- This ultimately leads to four KdV equations of the form

$$c_{nl}(\lambda_j) w_j^{(0)} \partial_{\xi_j} w_j^{(0)} + c_d(\lambda_j) \partial_{\xi_j}^3 w_j^{(0)} + c_t(\lambda_j) \partial_\tau w_j^{(0)} = 0,$$

- This allows us to recreate the surface profiles via the equations

$$\eta_1(x,t) \sim \sum_{j=1}^{4} (1 - \rho + \rho(\bar{d}\omega_2\lambda_j + \lambda_j^2)/\bar{d}) w_j^{(0)}(x - \lambda_j t, \epsilon t),$$

$$\eta_2(x,t) \sim \sum_{j=1}^{4} w_j^{(0)}(x - \lambda_j t, \epsilon t).$$

Finding Nonlinear Response: Strong Stratification

- Define

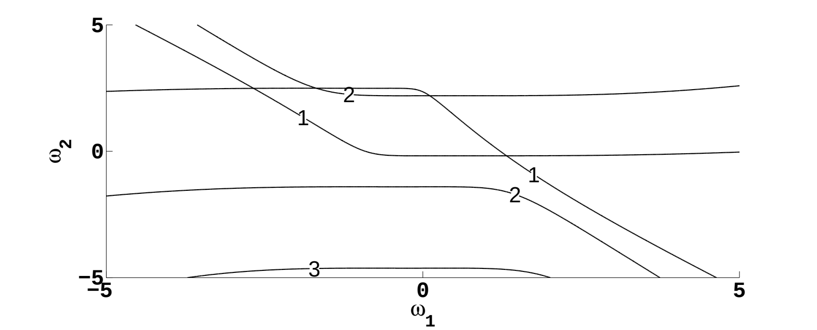
$$c_b(\lambda) = \frac{c_{nl}(\lambda)}{6c_d(\lambda)c_t(\lambda)}\left(\lambda + \left(\lambda^2 - \omega_1\lambda - 1\right)\left(\frac{1-\rho}{\lambda} + \rho\right) + 1\right),$$

- Looking at $c_b(\lambda_j)$, we can determine when we expect large initial conditions in the rescaled KdV equation

$$\tilde{w}_\tau + \partial_\xi^3 \tilde{w} + \tilde{w}\partial_\xi \tilde{w} = 0.$$

- Roughly, the larger the initial condition that goes into this rescaled equation, the more nonlinear phenomena we expect to see.

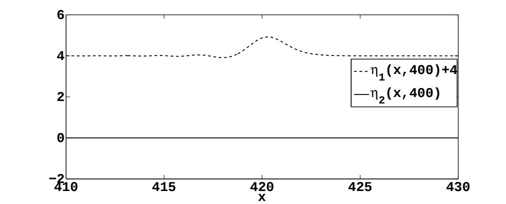
Finding Nonlinear Response: Strong Stratification



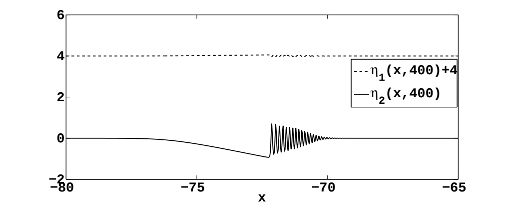
$\log_{10}(|c_b(\lambda_2)|)$, $\log_{10}(|c_b(\lambda_3)|)$

Figure: Contour plots of $c_b(\lambda_j)$ for $\rho = 820$ and $\bar{d} = .25$.

Finding Nonlinear Response: Strong Stratification



(a) Surface wave for $410 \leq x \leq 430$.



(b) Internal wave for $-80 \leq x \leq -65$.

Figure: $\omega_1 = .1$, $\omega_2 = -4.62$, $\rho = 820$, $\bar{d} = .25$, and $\epsilon = .0025$.

Conclusion for Part One

- Shear and density variation can result in significant nonlinear wave response, especially along internal layers.
- Hints at a wide variety of phenomena in the presence of more layers.
- Also calls for higher order, or fully nonlinear solves.
- Traveling waves? Almost done with K. Oliveras... fully nonlinear, and stability.

Part II: Point Vortices under Waves

- In 2D, for vorticity $\omega(\mathbf{x}, t)$ we have

$$\omega_t + \mathbf{u} \cdot \nabla\omega = 0.$$

- Close the system via Biot-Savart and harmonic potential ϕ so that

$$\mathbf{u} = \int \mathbf{K}(x-y)\omega(y,t)dy + \nabla\phi.$$

- Let

$$\omega(\mathbf{x}, t) = \frac{1}{2\pi}\sum_{j=1}^{N}\Gamma_j\delta(\mathbf{x} - \mathbf{x}_j(t)),$$

  and so we can approximate arbitrary vorticity profiles (technical details: see Cottet et al., Krasny, and several others) via irrotational point vortices.

Irrotational Vortices under Waves

- The fluid velocity **u** is given by the gradient of a potential, say ϕ where

$$\phi(x,z,t) = \phi_v(x,z,t) + \tilde{\phi}(x,z,t).$$

- ϕ_v is defined to be

$$\phi_v(x,z,t) = \frac{1}{2\pi}\sum_{j=1}^{N} \Gamma_j \phi_{v,j}(x,z,t),$$

with Γ_j denoting the circulation strength of the vortices and

$$\phi_{v,j}(x,z,t) = \Phi_p(x - x_j(t), z - z_j(t)) - \Phi_p(x - x_j(t), z + z_j(t)),$$

where

$$\Phi_p(x,z) = \sum_{m=-\infty}^{\infty} \tan^{-1}\left(\frac{z}{x - 2mL}\right).$$

Irrotational Vortices under Waves

- This modifies the boundary equations so that

$$\eta_t = -\eta_x \tilde{\phi}_x + \tilde{\phi}_z + P_v(x_1, z_1, \cdots, x_N, z_N),$$

and

$$\tilde{\phi}_t + \frac{1}{2}\left|\nabla\tilde{\phi}\right|^2 + g\eta = -\nabla\phi_v \cdot \nabla\phi + E_v(x_1, z_1, \cdots, x_N, z_N)$$

- Apologies, but P_v and E_v are not very nice to look at.

Irrotational Vortices under Waves

- Likewise we now tack on the ODE's

$$\dot{x}_j = \frac{1}{4L}\left(\Gamma_j \mathrm{cotanh}\left(\frac{\pi z_j}{L}\right) + 2\sum_{l\neq j}\Gamma_l v_{jl}^{(h)}\right) + \tilde{\phi}_x(x_j, z_j, t),$$

$$\dot{z}_j = \frac{1}{2L}\sinh\left(\frac{\pi}{L} z_j\right)\sum_{l\neq j}\Gamma_l v_{jl}^{(v)} + \tilde{\phi}_z(x_j, z_j, t),$$

- $v_{jl}^{(h)}$ and $v_{jl}^{(v)}$ are even worse to look at.

Irrotational Vortices under Waves

- So AFM lets us write for $\tilde{\eta}_t = \eta_t - P_v$

$$\int_{-L}^{L} e^{-i\pi kx/L}\left(\cosh(k(\eta+H))\tilde{\eta}_t + i\tilde{q}_x \sinh(k(\eta+H))\right) dx = 0$$

- But now we need $\tilde{\phi}_x(x_j, z_j, t)$ and $\tilde{\phi}_z(x_j, z_j, t)$ in terms of surface variables and vortex positions alone to close the system.
- Modify original AFM argument by introducing fundamental solutions

$$\psi_j(x,z,t) = -\frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\left(\ln\left(\tilde{x}_{j,m}^2 + \tilde{z}_{j,-}^2\right) + \ln\left(\tilde{x}_{j,m}^2 + \tilde{z}_{j,+}^2\right)\right)$$

Irrotational Vortices under Waves

- Use Green's third identity and some tricks, we end up with

$$\tilde{\phi}_x(x_j, z_j, t) = -\int_{-L}^{L} \left((\eta_t - P_v)\,\partial_x \psi_j + \tilde{q}_x \partial_z \psi_j\right)\Big|_{z=\eta+H} dx$$

- Likewise we can show

$$\tilde{\phi}_z(x_j, z_j, t) = -\int_{-L}^{L} \left((\eta_t - P_v)\,\partial_z \tilde{\psi}_j - \tilde{q}_x \partial_x \tilde{\psi}_j\right)\Big|_{z=\eta+H} dx$$

- So we have now closed the system in terms of surface variables and vortex positions alone.

Shallow-Water Scalings

- We now choose the following non-dimesionalizations

$$\tilde{x} = \frac{x}{L},\ \tilde{z} = \frac{z}{H},\ \tilde{t} = \frac{\sqrt{gH}}{L}t,\ \eta = d\tilde{\eta},\ \tilde{\phi} = \mu L\sqrt{gH}\tilde{\tilde{\phi}},$$

where we define the non-dimensional parameters

$$\mu = \frac{d}{H},\ \gamma = \frac{H}{L},$$

and where we define the Froude number F to be

$$F = \frac{\Gamma}{\mu L\sqrt{gH}}.$$

Shallow-Water Scalings

- From now on, the Froude number determines the strength of vortex interactions.
- Can now readily find the DNO expansion

$$\eta_t - \frac{1}{\gamma} P_v(x, 1 + \mu\eta, t) = \left(G_0 + \mu G_1 + \mu^2 G_2 + \cdots \right) Q$$

  where $Q = \tilde{q}_x$.
- Numerically simple and fast to implement. Extends to 3D.
- But David Ambrose has a point.

Numerics

- Pseudo-spectral in space, 4th order Runge-Kutta in time.
- If we run to time t_f, truncate DNO expansion when

$$\frac{||G_{\tilde{N}}Q||_2}{||Q||_2} \leq \text{mach. prec.}$$

- The range here is $18 \leq \tilde{N} \leq 35$.
- Orszag's $2/3$-rule is used for filtering.
- Quiescent fluid: we track energy input into the surface via

$$E(t) = \frac{1}{2}\int_{-1}^{1} qG(\eta)Q\,dx + \frac{1}{2}\int_{-1}^{1} \eta^2\,dx.$$

Numerics: Cnoidal Waves

- Can show as $F \to 0$, we can derive KdV equation

$$2Q_\tau + 3QQ_\xi + \frac{1}{3}Q_{\xi\xi\xi} = 0$$

- Thus we can look at how vortices modify propagation of 'cnoidal' profiles where

$$Q(x,t) \sim \frac{2}{3}q_0 + \frac{4}{3}\tilde{m}^2\mathcal{K}^2(\tilde{m})\mathrm{cn}^2\left(\mathcal{K}(\tilde{m})\left(x - (1+\mu\tilde{c})\,t\right);\tilde{m}\right)$$

$$\eta(x,t) \sim \left(1 + \frac{2}{3}\mu\mathcal{K}^2(\tilde{m})(2\tilde{m}^2 - 1)\right)Q(x,t)$$

$$\tilde{c} = \frac{2}{3}\mathcal{K}^2(\tilde{m})(2\tilde{m}^2 - 1) + q_0$$

Numerics: Cnoidal Waves

- Place two vortices of opposite signs at

$$x_1(0) = -\mu\gamma, \; x_2(0) = \mu\gamma, \; z_1(0) = z_2(0) = .25.$$

- Then we get

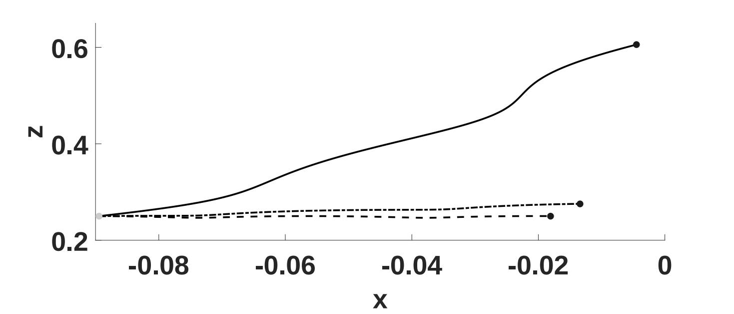


Figure: Paths of the left-side vortex moving under a cnoidal wave for $0 \leq t \leq 5$ for Froude numbers $F = 0$ (dashed line), .02 (dashed/dotted line), and .2 (solid line).

Numerics: Cnoidal Waves

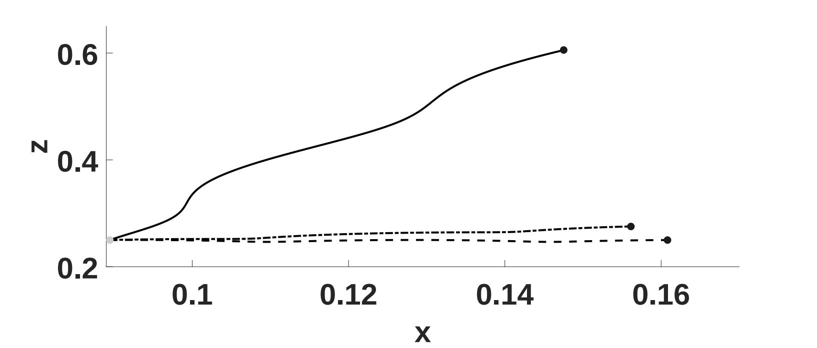


Figure: Paths of the right-side vortex moving under a cnoidal wave for $0 \leq t \leq 5$ for Froude numbers $F = 0$ (dashed line), .02 (dashed/dotted line), and .2 (solid line).

Numerics: Cnoidal Waves

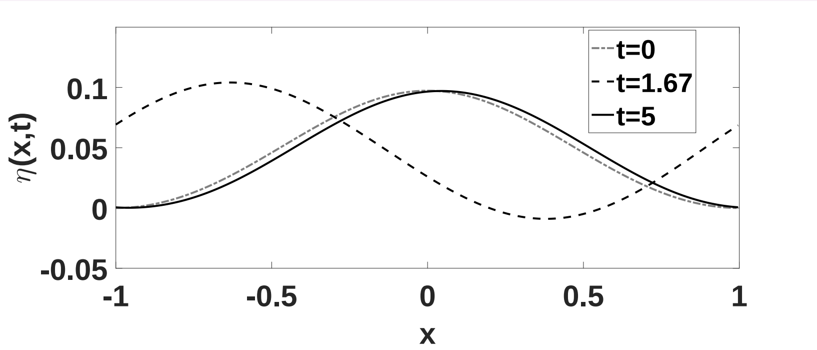


Figure: Surface response $\eta(x, 5)$ elliptic modulus $\tilde{m} = .2$ and Froude number $F = .02$.

Numerics: Cnoidal Waves

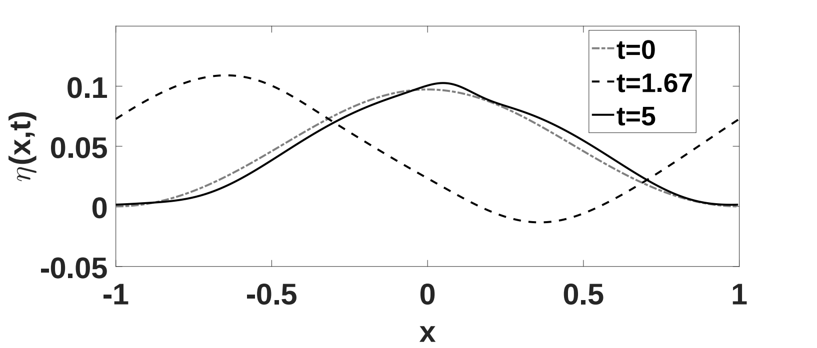


Figure: Surface response $\eta(x, 5)$ with elliptic modulus $\tilde{m} = .2$ and Froude number $F = .2$.

Two Counter Propagating Vortices

- Start from a quiescent surface.

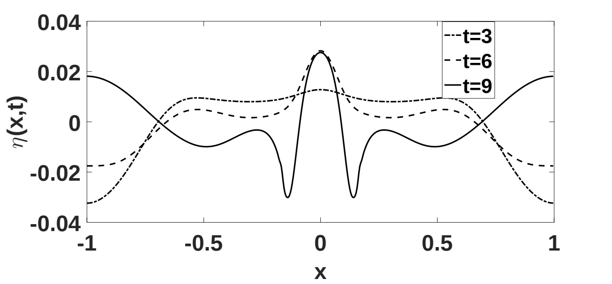


Figure: Surface response $\eta(x,t)$ at $t=3$, $t=6$, and $t=9$ over two counter-propagating vortices. $F=.2$

Two Counter Propagating Vortices

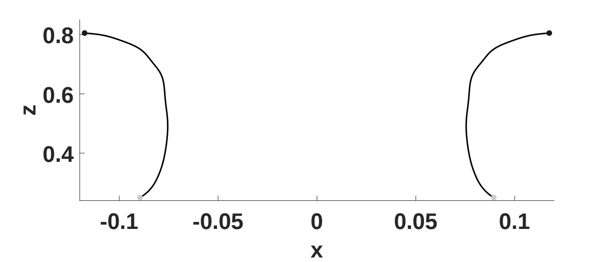


Figure: Motion of the two counter-propagating vortices for $0 \leq t \leq 9$. The light grey dots indicate where the vortices begin and the black dots indicate their positions at $t = 9$. $F = .2$

Two Counter Propagating Vortices

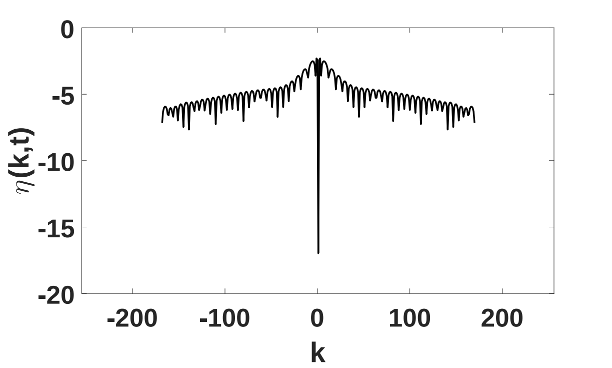


Figure: Log plot of the power spectrum at $t = 9$ over wave numbers $-256 \leq k \leq 256$. As can be seen, the rising vortices pump more energy into higher wavenumbers in the surface profile.

Two Counter Propagating Vortices

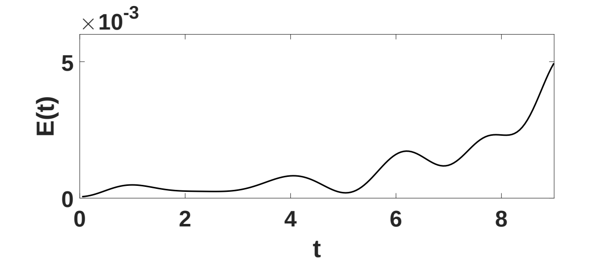


Figure: Surface energy profile $E(t)$ for a surface over two vortices for $0 \leq t \leq 9$.

Four Counter Propagating Vortices

- Place four vorices at

$$x_1(0) = -2\mu\gamma - \mu\gamma/2,\ x_2(0) = -2\mu\gamma + \mu\gamma/2,$$
$$x_3(0) = 2\mu\gamma - \mu\gamma/2,\ x_4(0) = 2\mu\gamma + \mu\gamma/2,$$

$z_j(0) = .25$ in $+/+, -/-$ configuration. Take $F = .2$.

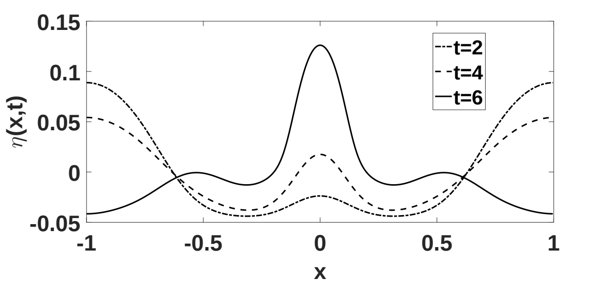


Figure: Surface response $\eta(x,t)$ for $t = 2$, 4, and 6 over four vortices in the Plus/Plus, Minus/Minus configuration.

Four Counter Propagating Vortices

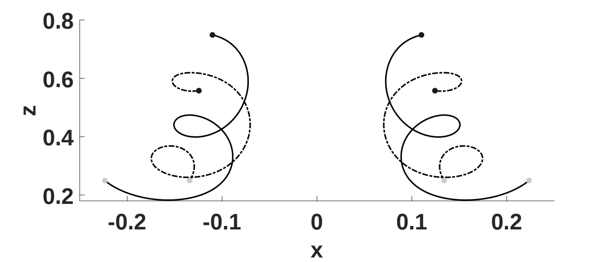


Figure: Motion of four vortices in the Plus/Plus, Minus/Minus configuration for $0 \leq t \leq 6$.

Four Counter Propagating Vortices

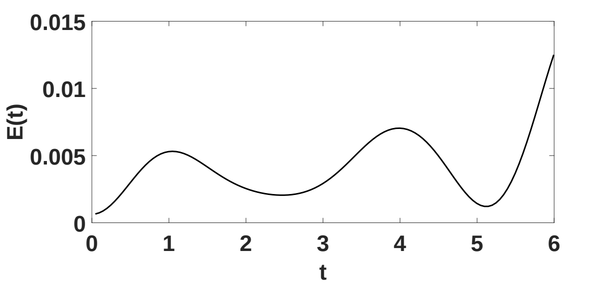


Figure: Surface energy profile $E(t)$ in response to the motion of four vortices in the Plus/Plus, Minus/Minus configuration for $0 \leq t \leq 6$.