Theoretical and Computational Aspects of Nonlinear Surface Waves
2016 BIRS Workshop
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A Whitham-Boussinesq long-wave model for variable topography.

Rosa María Vargas Magaña
Joint work with: Panayotis Panayotaros

UNAM-IIMAS

1. Water waves problem with variable depth.
2. Long-wave approximation of the **Dirichlet-Neumann operator** in presence of non-trivial bottom topography.
3. A Whitham-Boussinesq model that involves a ***pseudo differential operator*** (PDO).
4. **Discretization of the PDO** associated with the bottom topography.
5. **Spectral analysis** of the linearized Whitham-Boussinesq model for different families of topographies.
6. Numerical integration of the **evolution of some initial wave-profiles** over different topographies.
7. **Work in progress. Project I. II. III. and IV**

Water waves problem in variable depth

Problem setting:

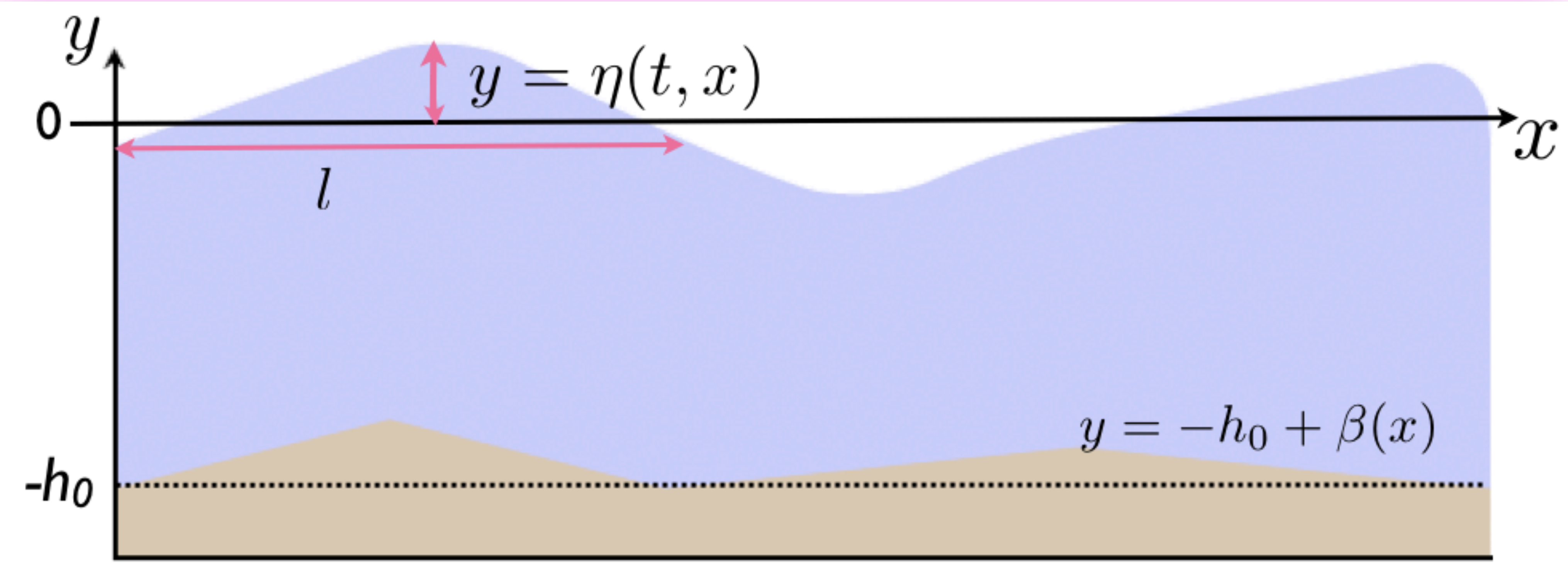


Figure 1. Cartoon of fluid domain

IDEAL FLUID:

- Perfect
- Incompressible
- Irrotational

FLUID DOMAIN:

- 2D
- Simple connected

EULER'S EQUATIONS

$\Delta\varphi = 0$ on the fluid domain

$N \cdot \nabla\varphi = 0$ on the variable bottom $y = -h_0 + \beta(x)$

Nonlinear boundary conditions on the free surface $y = \eta(x, t)$

Bernulli equation $\partial_t \varphi + \frac{1}{2}(\nabla\varphi)^2 + g\eta = 0$

Kinematic condition $\partial_t \eta + \partial_x \eta \cdot \partial_x \varphi - \partial_y \varphi = 0$

Hamilton equations with infinitely many degrees of freedom:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \eta} \\ \frac{\delta H}{\delta \xi} \end{pmatrix}$$

$$\xi(x,t) = \varphi(x, \eta(x,t), t)$$

ZAKHAROV CRAIG & SULEM EQUATIONS

with Hamiltonian: $H = \frac{1}{2} \int_{\mathbb{R}} (\xi G(\beta, \eta) \xi + g\eta^2) dx$

V.E. Zakharov, 1968, J.W. Miles, 1977, W. Craig and C. Sulem., 1992

G(β,η) is the DIRICHLET-NEUMANN operator for the fluid domain:

DEFINITION:
Let us consider the solution of the elliptic problem:

$$\Delta\varphi = 0$$

$$\varphi(x, \eta(x)) = \xi(x)$$

$$N \cdot \nabla\varphi = 0$$ On the variable bottom $y = -h_0 + \beta(x)$

$$[G(\beta, \eta)] : \quad \xi \longmapsto N \cdot \nabla\varphi(|\nabla_x \eta|^2)^{\frac{1}{2}}$$

Hamiltonian: $H = \frac{1}{2}\int_{\mathbb{R}} (\xi G(\beta,\eta)\xi + g\eta^2)dx$

In general there is not an explicit expression for it!

This case gives rise to an explicit expression to de DN operator:

$$[G(0,0)] : \quad \xi \longmapsto D\tanh(D)\xi$$

Where D is as usual the operator $D = -i\partial_x$

However this is not the case for the fluid domain which takes into account the variations in bottom topography as well as the deformations of the free surface from equlibrium.

Analitic Expansion of operator: $G(\beta,\eta)$

$$G(\beta,\eta) = G_0(\beta,\eta) + G_1(\beta,\eta) + G_2(\beta,\eta) +, \ldots$$

$G_j(\beta,\eta)$ are homogeneous of degree j in η.

$$\begin{aligned}
G_0(\beta,\eta) &= D\tanh(h_0 D) + DL(\beta), \\
G_1(\beta,\eta) &= D\eta D - G_0\eta G_0, \\
G_2(\beta,\eta) &= \frac{1}{2}(G_0 D\eta^2 D - D^2\eta^2 G_0 - 2G_0\eta G_1),
\end{aligned}$$

with $D = -i\partial_x$

and $L(\beta)$ Involve pseudo-differential operators

The bottom variation represented by β(x) are taken to be of order O(1), while the surface deformation η(x) will be small.

※ Proof for $G(0,\eta)$ Coifman and Meyer,1985 , ※ Craig, Sulem, 2005 ※ Craig, Guyenne Nicholls Sulem, 2005

※Proof for $G(\beta,\eta)$, Lannes, For $||\eta||$, for β smaller enough (in an appropriate norm)

The operator $L(\beta)$, can be written in the semi-explicit form:

$$L(\beta) = -B(\beta)A(\beta),$$

where

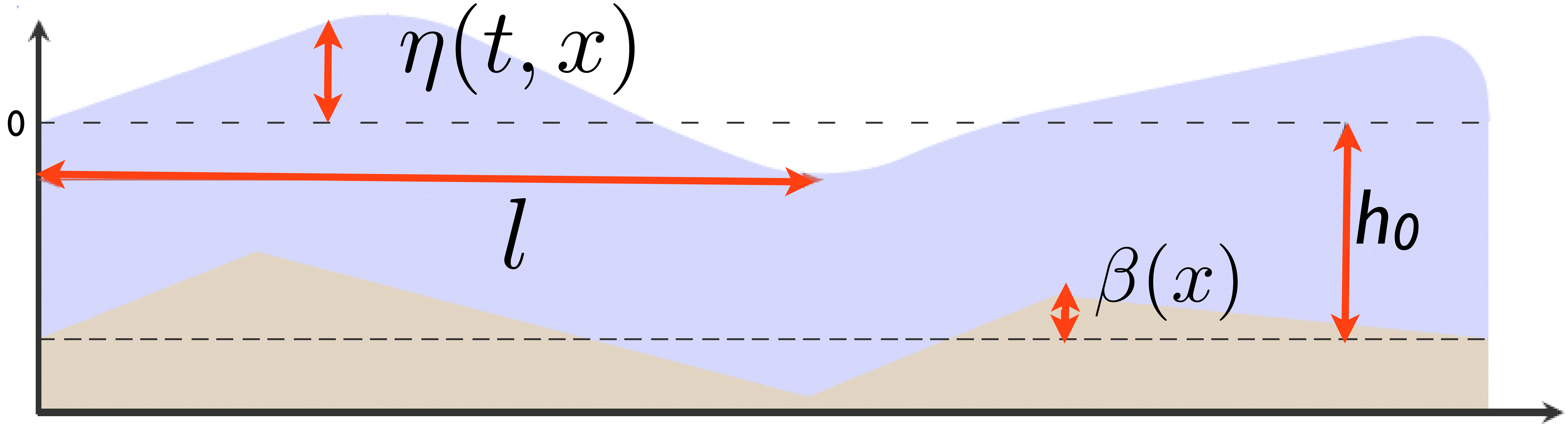
$$A(\beta)\xi = \int_{\mathbb{R}} e^{ikx} \sinh(\beta(x)k) \operatorname{sech}(hk)\hat{\xi}(k)dk,$$

$$C(\beta)\xi = \int_{\mathbb{R}} e^{ikx} \cosh((-h_0 + \beta(x))k)\hat{\xi}(k)dk,$$

and $B(\beta) = C(\beta)^{-1}.$

※ W.Craig, P. Guyenne, D. Nicholls & C. Sulem, 2005

ADIMENSIONAL PARAMETERS



$$\delta = \frac{h_0^2}{l^2} \qquad \epsilon = \frac{\eta}{h_0} \qquad \gamma = \frac{\beta}{h_0}$$

Shallow Water Regime (Long-Wave)

$$\delta = \frac{h_0^2}{l^2} \ll 1$$

Boussinesq Regime

$$\delta \sim \epsilon$$

Boussinesq Regime Variable Depth (Small Depth Variations)

$$\epsilon \sim \delta \sim \gamma$$

Aceves-Sánchez, Minzoni and Panayotaros

Numerical of a nonlocal Model for water waves with variable depth, 2013

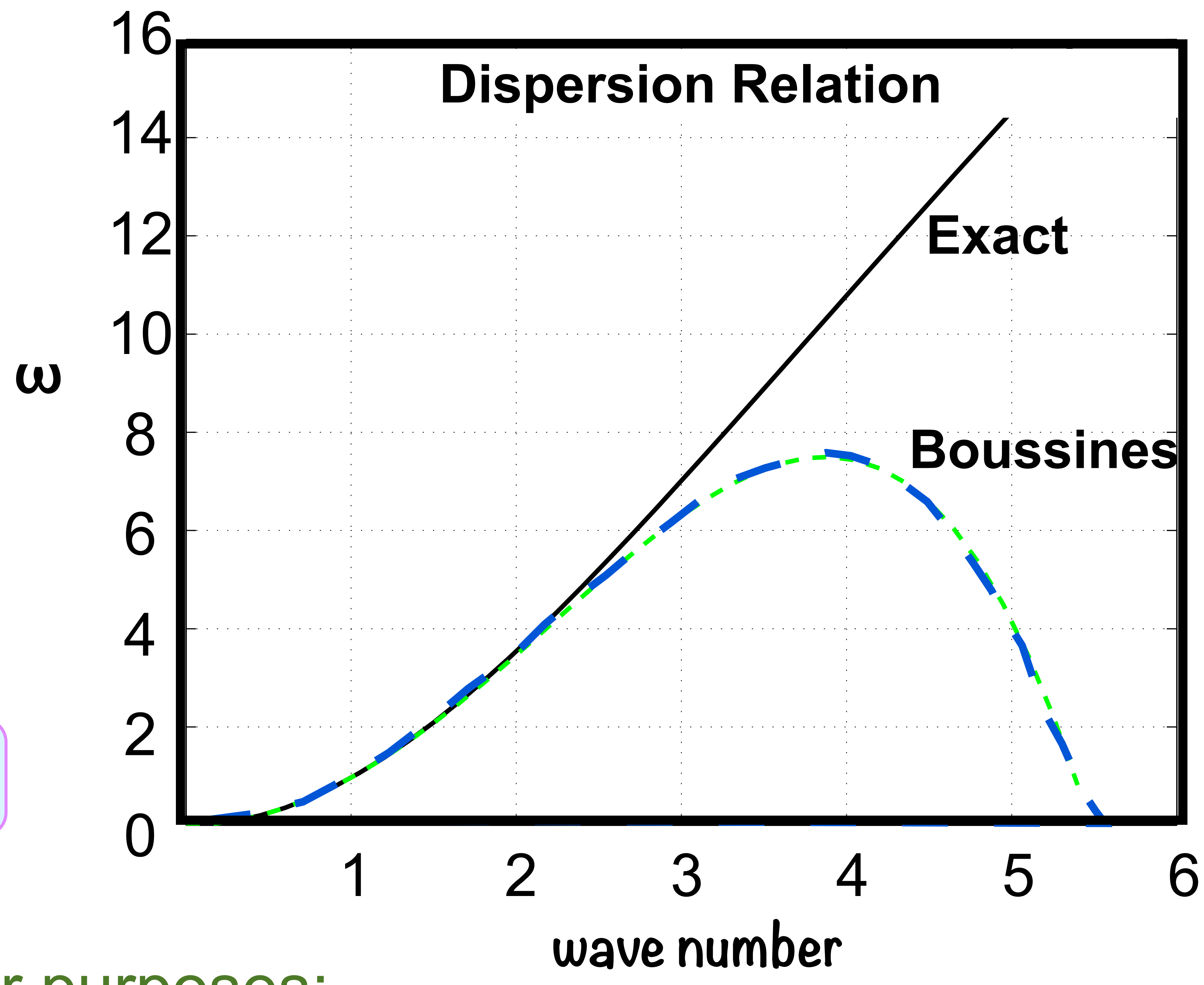
We want to capture bigger order depth variations!

Whitham's type equations

When the full water waves problem is linearized around the zero solution

$$\begin{cases} \eta_t = G(0,\eta)\xi \\ \xi_t = -g\eta. \end{cases}$$

one finds the classical DISPERSION RELATION: $\omega^2 = gk\tanh(h_0 k)$



Drawback on Boussinesq equations for our purposes:

-> Poor approximation to full dispersion relation for larger wave numbers.

-> Ill posed problem due to negative sign on larger wave numbers.

Whitham's- type equations offer the possibility of singularity formation at higher amplitudes and the existence of solitons with a cusped profile.

Keep term: $G_0(\beta,\eta) = D\tanh(h_0 D) + DL(\beta)$

D. Moldabayev, H. Kalisch and D.Dutykh The Whitham Equation as a Model for Surface Water Waves 2014

Long-wave approximation of $G(\beta, \eta)$ in presence of non-trivial bottom topography. Part I.

$$H = \frac{1}{2}\int_{\mathbb{R}} (\xi G(\beta,\eta)\xi + g\eta^2)dx$$

Constant depth. $\delta \sim \epsilon \ll 1$

$$G_{\mathcal{A}_0} = \frac{\tilde{D}}{\sqrt{\epsilon}}\tanh(h_0\sqrt{\epsilon}\tilde{D}) + \epsilon\tilde{D}\tilde{\eta}\tilde{D}$$

$\tilde{D} = \frac{1}{l}D \qquad \tilde{\eta} = \frac{\eta}{a}$

Smooth and small depth variations of order O(ε). $\delta \sim \epsilon \sim \gamma \ll 1$

$$G_0(\beta,\eta) = D\tanh(h_0 D) + DL(\beta)$$

$$G_{\mathcal{A}_1} = \frac{\tilde{D}}{\sqrt{\epsilon}}\tanh(h_0\sqrt{\epsilon}\tilde{D}) - h_0\tilde{D}\tilde{\beta}\tilde{D} + \epsilon\tilde{D}\tilde{\eta}\tilde{D}$$

Aceves-Sánchez, Minzoni and Panayotaros

Numerical of a nonlocal Model for water waves with variable depth, Wave Motion 2013

Long-wave approximation of $G(\beta,\eta)$ in presence of non-trivial bottom topography. Part II.

Depth variations of order **O(1)**

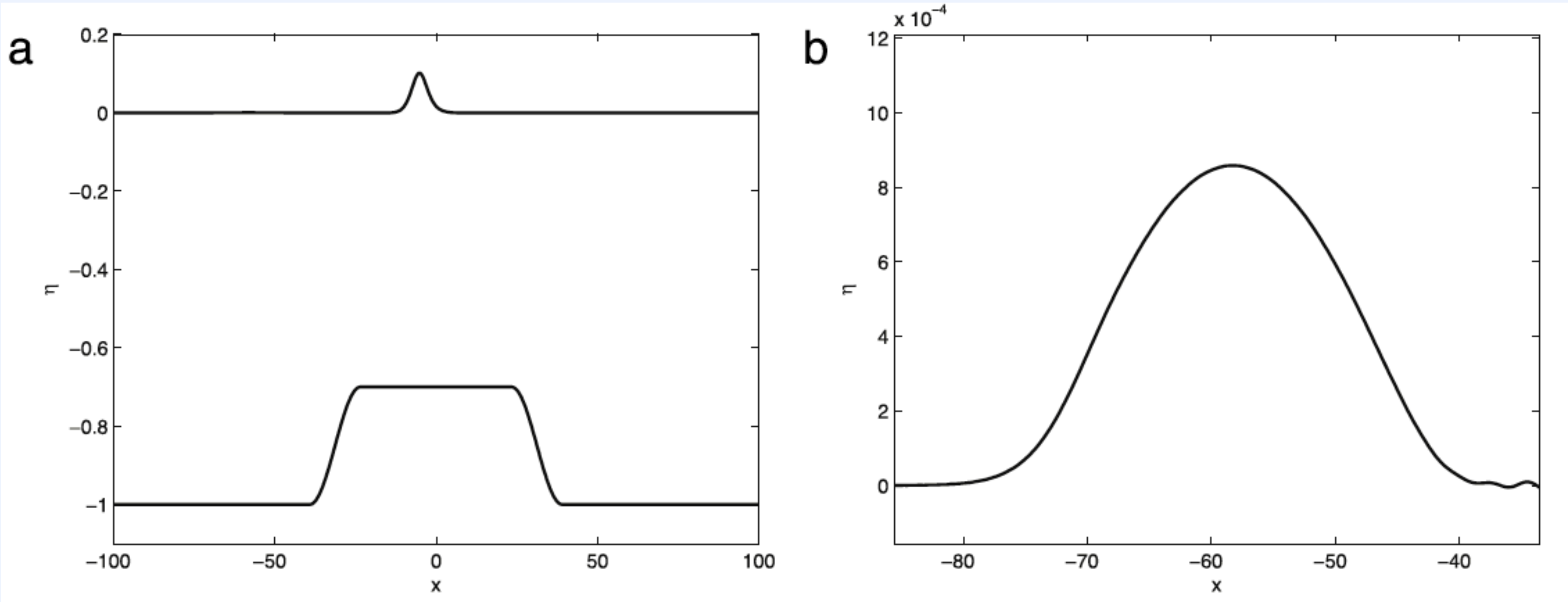
$$G_{\mathcal{A}_2} = Sym\left(\frac{\tilde{D}}{\sqrt{\epsilon}}\tanh(\sqrt{\epsilon}(-1+\tilde{\beta}(x))\tilde{D})\right) + \epsilon\tilde{D}\tilde{\eta}\tilde{D}$$

Vargas-Magaña, and Panayotaros

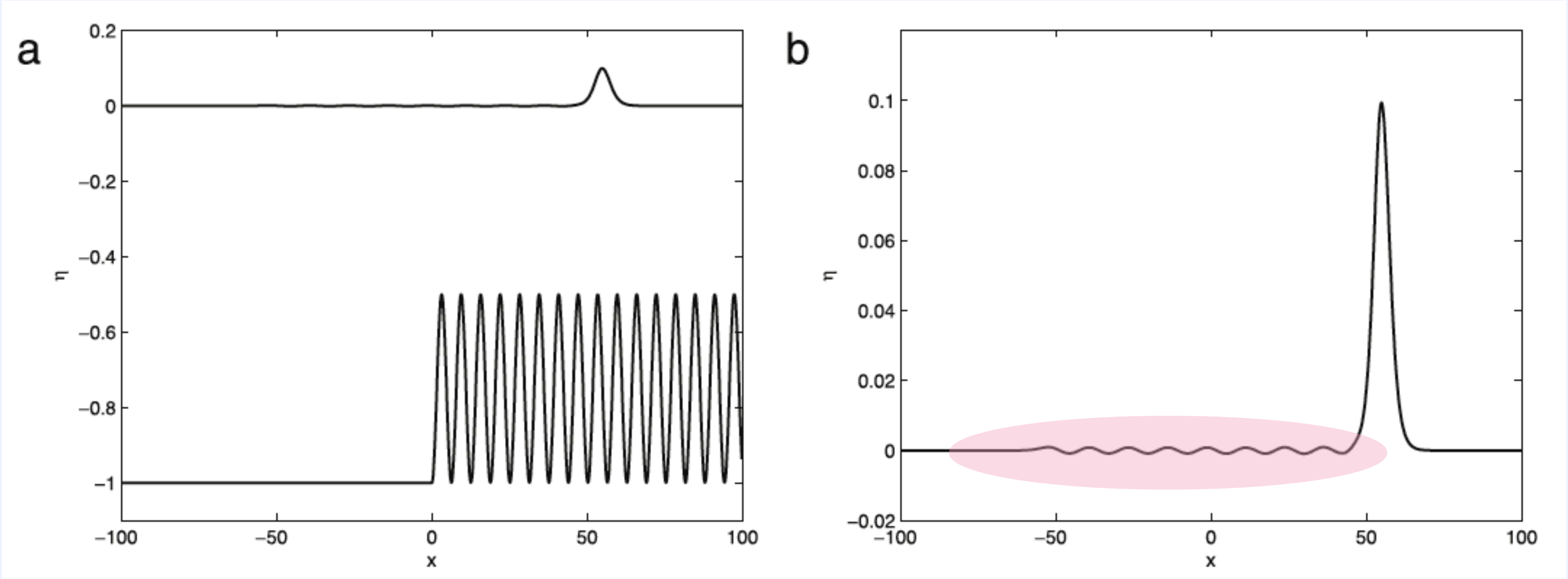
A Whitham-Boussinesq long-wave model for variable depth, Wave Motion 2016

Satisfies some structural properties of the exact linear DN operator:

- $G_{\mathcal{A}_2}[\xi](x)$ is real if ξ is real valued
- $G_{\mathcal{A}_2}$ is a symmetric operator
- Spectra of this operator has good asymptotic behavior as **κ** increase we approach the constant depth dispersion relation an the same condition apply to the eigenfunctions.
- $G_{\mathcal{A}_2}$ is a positive operator.



$$\eta(x) = u(x) = 0.1\,\mathrm{sech}^2(0.27(x+80))$$



$\varepsilon = 0.35$, $L = 200$, and $N = 1024$

$$\overline{\beta}(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{0.5}{2}\left(1 - \frac{1}{2}\cos(x)\right)dx = 0.25$$

Aceves-Sánchez, Minzoni y Panayotaros

Numerical of a nonlocal Model for water waves with variable depth, 2003

We show in the picture that the only visible effects detected with this model are very attenuated.

In 2007 **P.Guyenne and P. Nicholls** introduced an accurate numerical method for nonlinear surface water waves for variable bathymetry in 2D and 3D using Higher order expansions of the L(β) operator in powers of β(x),

P. Guyenne, D. Nicholls *Numerical simulations of solitary waves on plane slopes* 2007

P.Guyenne, D. Nicholls *A high-order spectral method for nonlinear wates waves over moving topography* 2007

A possible drawback in this formulation is the presence of higher derivatives in $\boldsymbol{L_j}$ as ***j*** increased and the authors also use high frequency truncations of the derivatives.

A Whitham-Boussinesq model that involves a *pseudo differential operator* (PDO).

$$H = \frac{1}{2}\int_{\mathbb{R}}(\xi G_{\mathcal{A}_2}(\beta,\eta)\xi + g\eta^2)dx$$

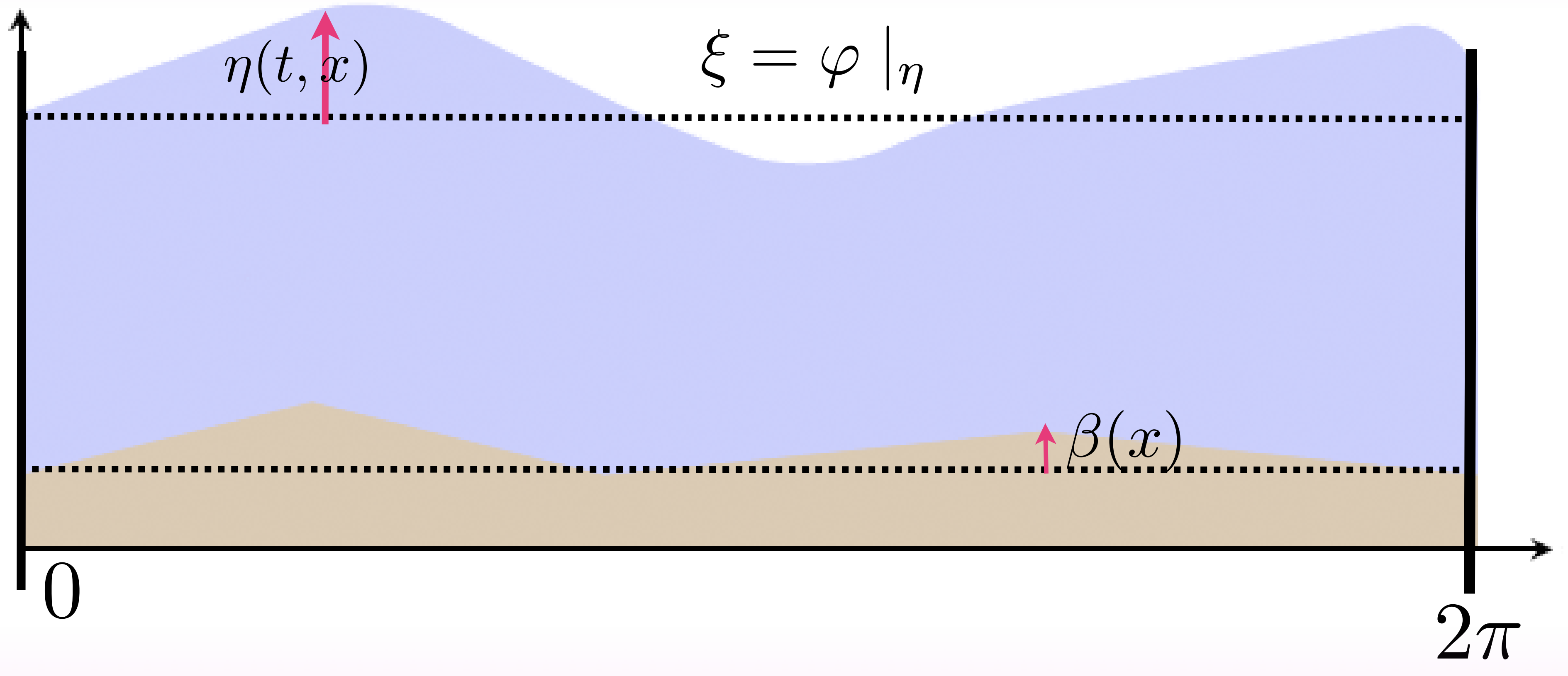
$$G_{\mathcal{A}_2} = Sym(\frac{D}{\sqrt{\epsilon}}\tanh(\sqrt{\epsilon}(-1+\beta(x))D)) + \epsilon D\eta D$$

Pseudo-differential operator of the form:

$$a(x,D)\xi(x) = \int_{\mathbb{R}} a(x,k)\hat{\xi}(k)e^{ikx}dk.$$

where the function $a(x,k)$ is the symbol of the operator.

Spectral representation in the 2π-periodic framework



PDO in the 2π-periodic frame

Let $a(x,k)$ be a function periodic in the variable x of period 2π:

$$a(x,D)[\xi](x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a(x,k)\hat{\xi}_k e^{ikx}.$$

with $$a(x,k) = \sum_{\lambda=-\infty}^{\infty} \hat{a}_\lambda(k) e^{i\lambda x}.$$

with $$\hat{a}_\lambda(k) = \int_0^{2\pi} a(x,k) e^{-i\lambda x} dx, \;\; \lambda \in \mathbb{Z}$$

The quadratic form in the Hamiltonian in the 2π-periodic setting:

$$K_{a(x,D)} = \frac{1}{2}\int_0^{2\pi} \xi a(x,D)\xi \, dx$$

$$K_{a(x,D)} = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} \hat{\xi}_k \hat{\xi}_\lambda \hat{a}_{-k-\lambda}(\lambda)$$

If ξ is real valued then $a(x,D)[\xi]$ is real valued then $K_{a(x,D)}$ is real valued too!

$$K_{Sym(a(x,D))} = \frac{1}{4\pi} \sum_{[k,\lambda]\in\mathbb{Z}^2} \hat{\xi}_k [\hat{a}_{\lambda-k}(k) + \overline{\hat{a}_{k-\lambda}(\lambda)}]\hat{\xi}_\lambda^*$$

Discretization of this PDO associated to the bottom topography.

Galerkin truncations of the quadratic form:

Let $\hat{\xi}^M = (\hat{\xi}_1, \ldots, \hat{\xi}_M)$

$$K^M_{a(x,D)} = \frac{1}{4\pi} \sum_{[k_1,k_2]\in J^2_M} \left[\hat{\xi}_{k_1}\hat{\xi}_{k_2}\hat{a}^*_{k_1+k_2}(k_2) + \hat{\xi}^*_{k_1}\hat{\xi}^*_{k_2}\hat{a}_{k_1+k_2}(k_2)\right]$$

$$+ \frac{1}{4\pi} \sum_{[k_1,k_2]\in J^2_M} \left[\hat{\xi}_{k_1}\hat{\xi}^*_{k_2}\hat{a}_{-k_1+k_2}(k_2) + \hat{\xi}^*_{k_1}\hat{\xi}_{k_2}\hat{a}^*_{-k_1+k_2}(k_2)\right]$$

$$= \frac{1}{4\pi}((\hat{\xi}^M)^T, ((\hat{\xi}^M)^*)^T) \begin{pmatrix} P^* & S \\ S^* & P \end{pmatrix} \begin{pmatrix} \hat{\xi}^M \\ (\hat{\xi}^M)^* \end{pmatrix},$$

- M is the truncation.
- Matrix sizes of P and S is MxM
- ξ^* is the conjugated of ξ
- $a_{-\lambda}(\kappa) = a^*_\lambda(\kappa)$

$$P = \begin{pmatrix} \hat{a}_{1+1}(1) & \hat{a}_{1+2}(2) & \cdots & \hat{a}_{1+(M-2)}(M-2) & \hat{a}_{1+(M-1)}(M-1) & 0 \\ \hat{a}_{2+1}(1) & \hat{a}_{2+2}(2) & \cdots & \hat{a}_{2+(M-2)}(M-2) & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{(M-2)+1}(1) & \hat{a}_{(M-2)+2}(2) & 0 & 0 & 0 & 0 \\ \hat{a}_{(M-1)+1}(1) & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad S = \begin{pmatrix} \hat{a}_0(1) & \hat{a}_1(2) & \cdots & \hat{a}_{M-2}(M-1) & \hat{a}_{M-1}(M) \\ \hat{a}^*_1(1) & \hat{a}_0(2) & \cdots & \hat{a}_{M-3}(M-1) & \hat{a}_{M-2}(M) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{a}^*_{M-1}(1) & \hat{a}^*_{M-2}(2) & \cdots & \hat{a}^*_1(M-1) & \hat{a}_0(M) \end{pmatrix}.$$

Letting $\hat{\xi}^M = \theta + i\zeta, \theta, \zeta \in \mathbb{R}^M$,

1 Splitting of ξ-fourier coefficients in real variables

$$K^M_{a(x,D)} = \frac{1}{4\pi}(\theta^T, \zeta^T)\begin{pmatrix} Re(P^* + S^* + P + S) & -Im(P^* - S^* + S - P) \\ -Im(P^* + S^* - S - P) & -Re(P^* - S^* - S + P) \end{pmatrix}\begin{pmatrix} \theta \\ \zeta \end{pmatrix}$$

Letting $\mathcal{M} = \frac{1}{4\pi}\begin{pmatrix} Re(P^* + S^* + P + S) & -Im(P^* - S^* + S - P) \\ -Im(P^* + S^* - S - P) & -Re(P^* - S^* - S + P) \end{pmatrix}$,

Matrix size of is 2Mx2M

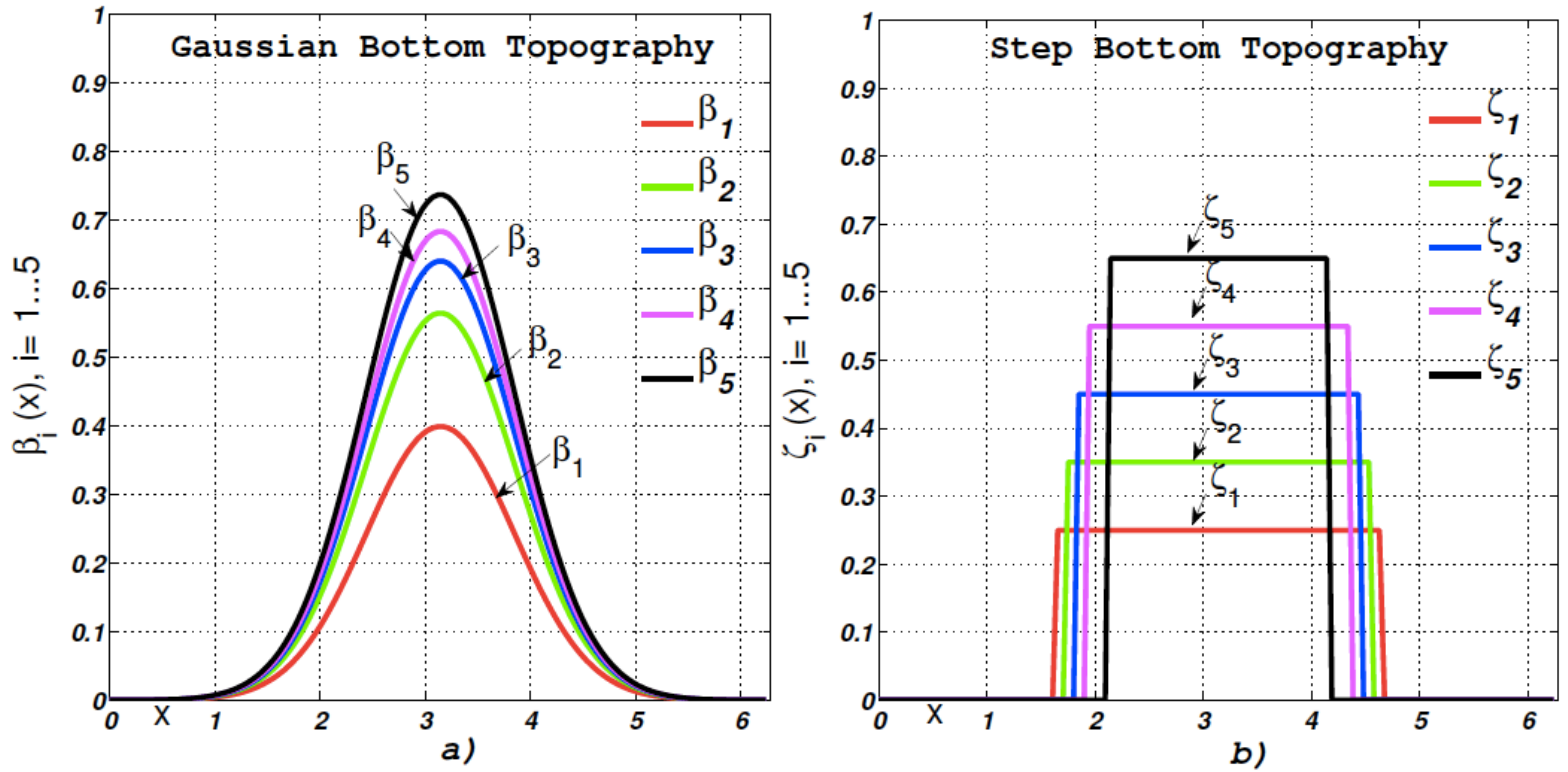
and $\mathcal{M}_{Sym} = \frac{1}{2}(\mathcal{M} + \mathcal{M}^T)$,

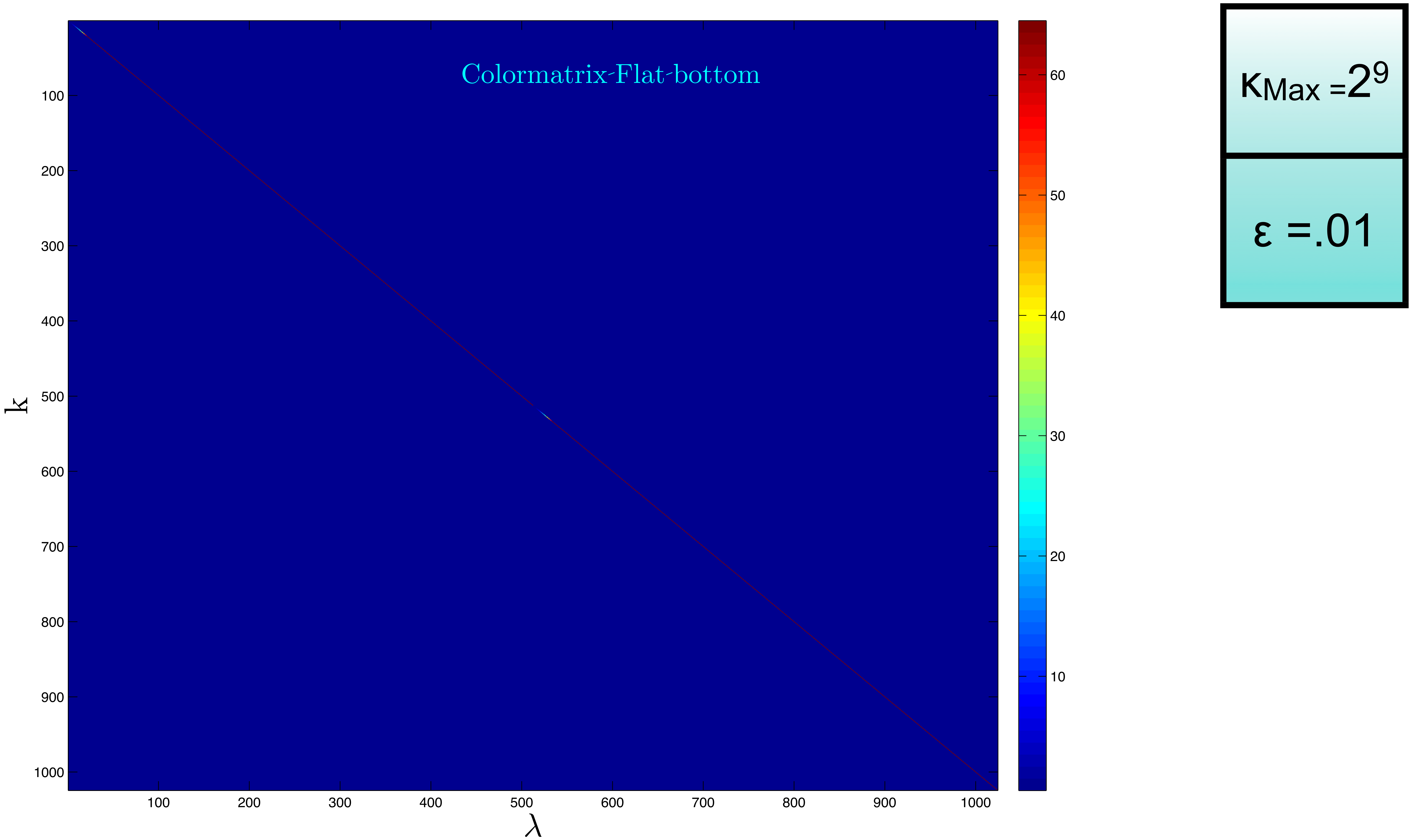
2 Symmetrization

$$K^M_{Sym(a(x,D))} = (\theta^T, \zeta^T)\, \mathcal{M}_{Sym} \begin{pmatrix} \theta \\ \zeta \end{pmatrix}.$$

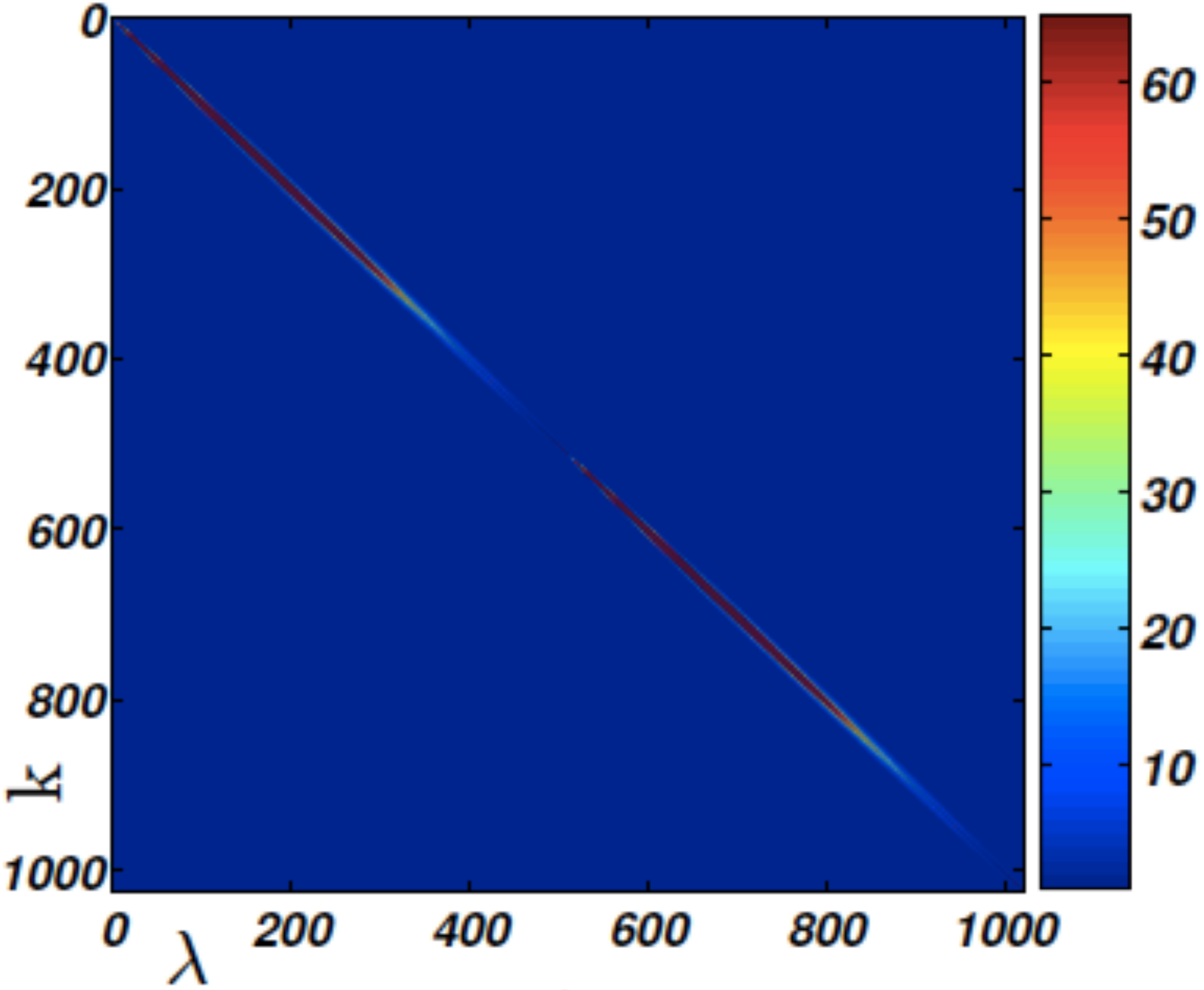
3 Matrix representation in Real variables

Bottom Topography

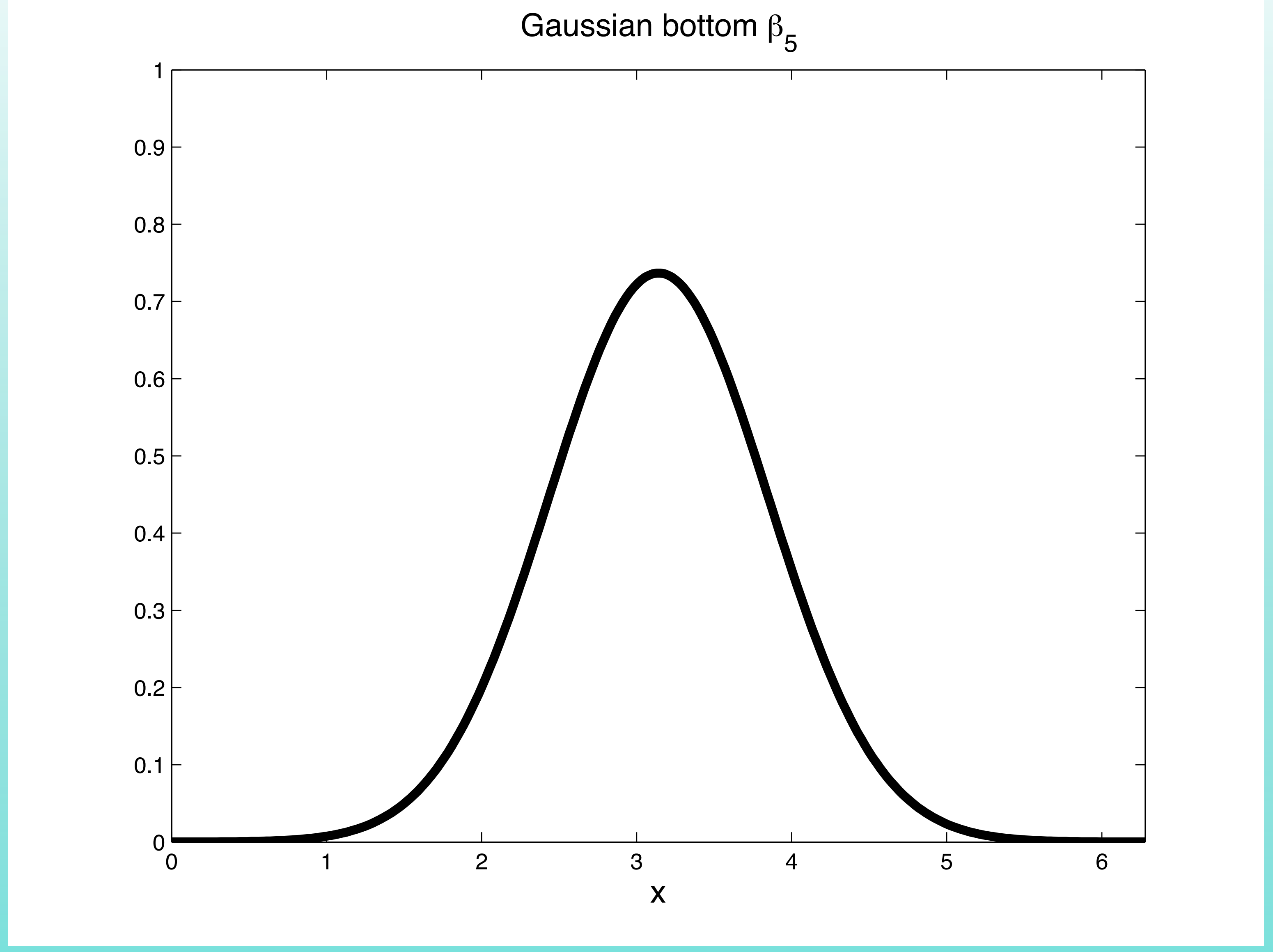




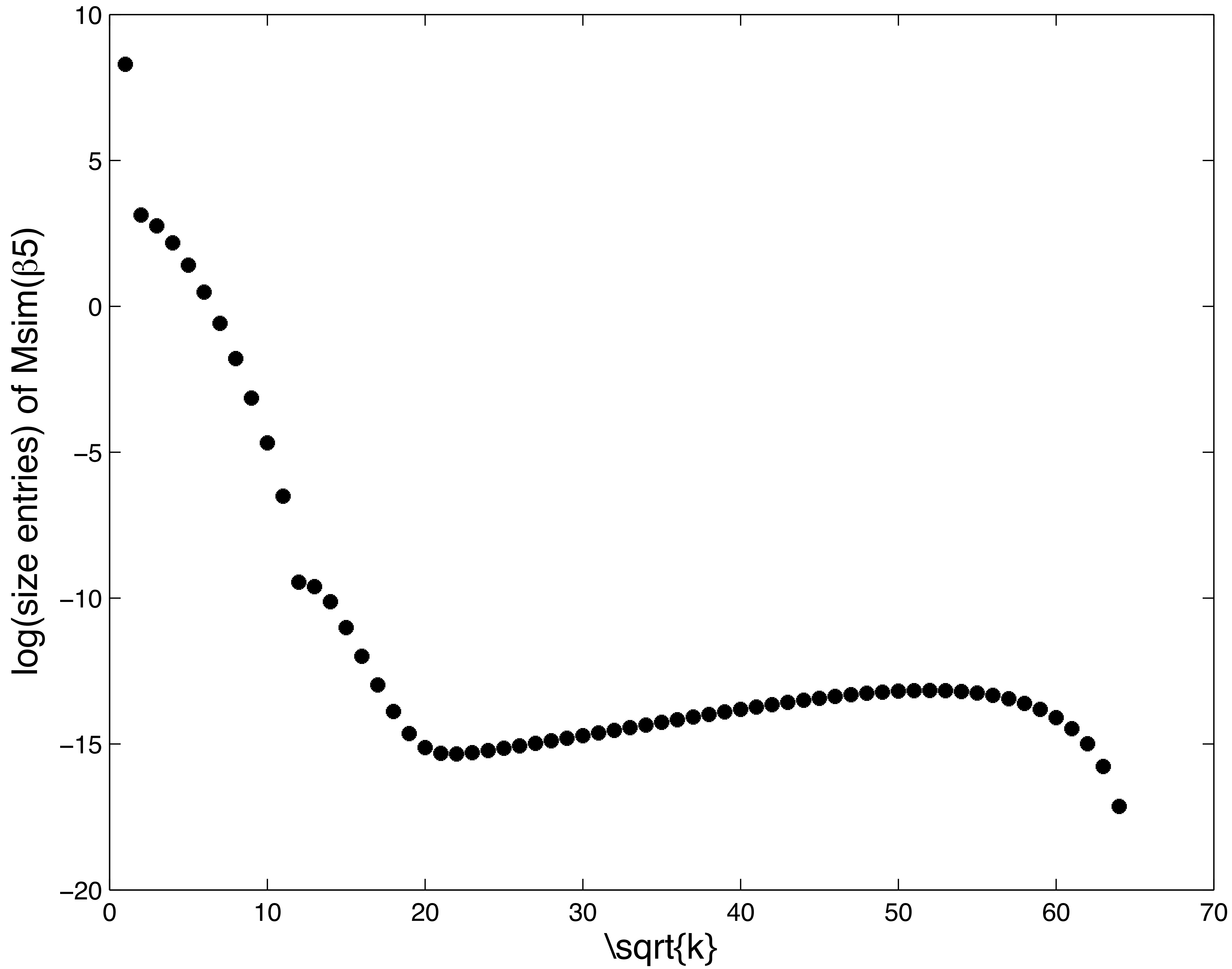
The matrix corresponding to **β=0** is a diagonal matrix

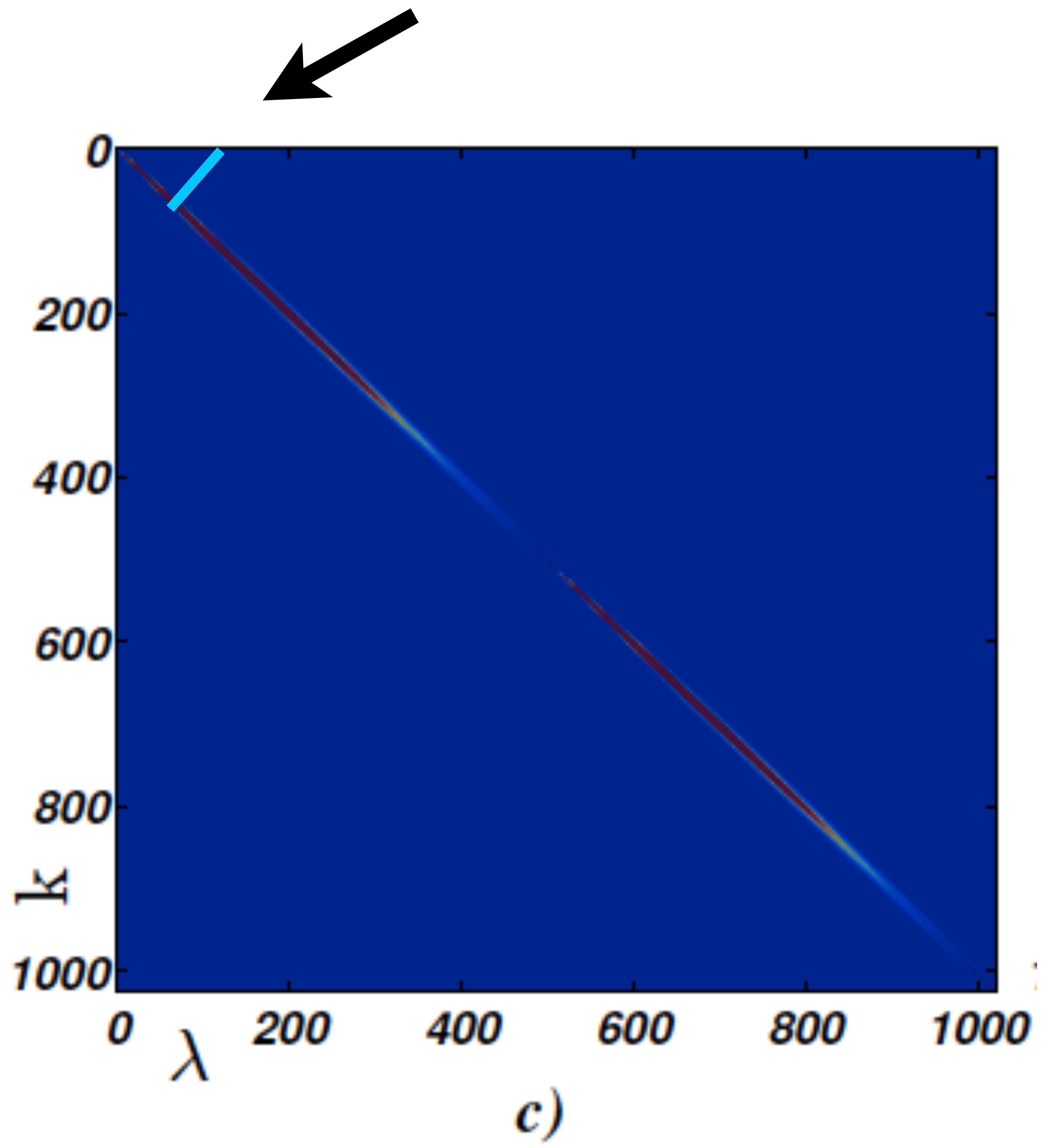


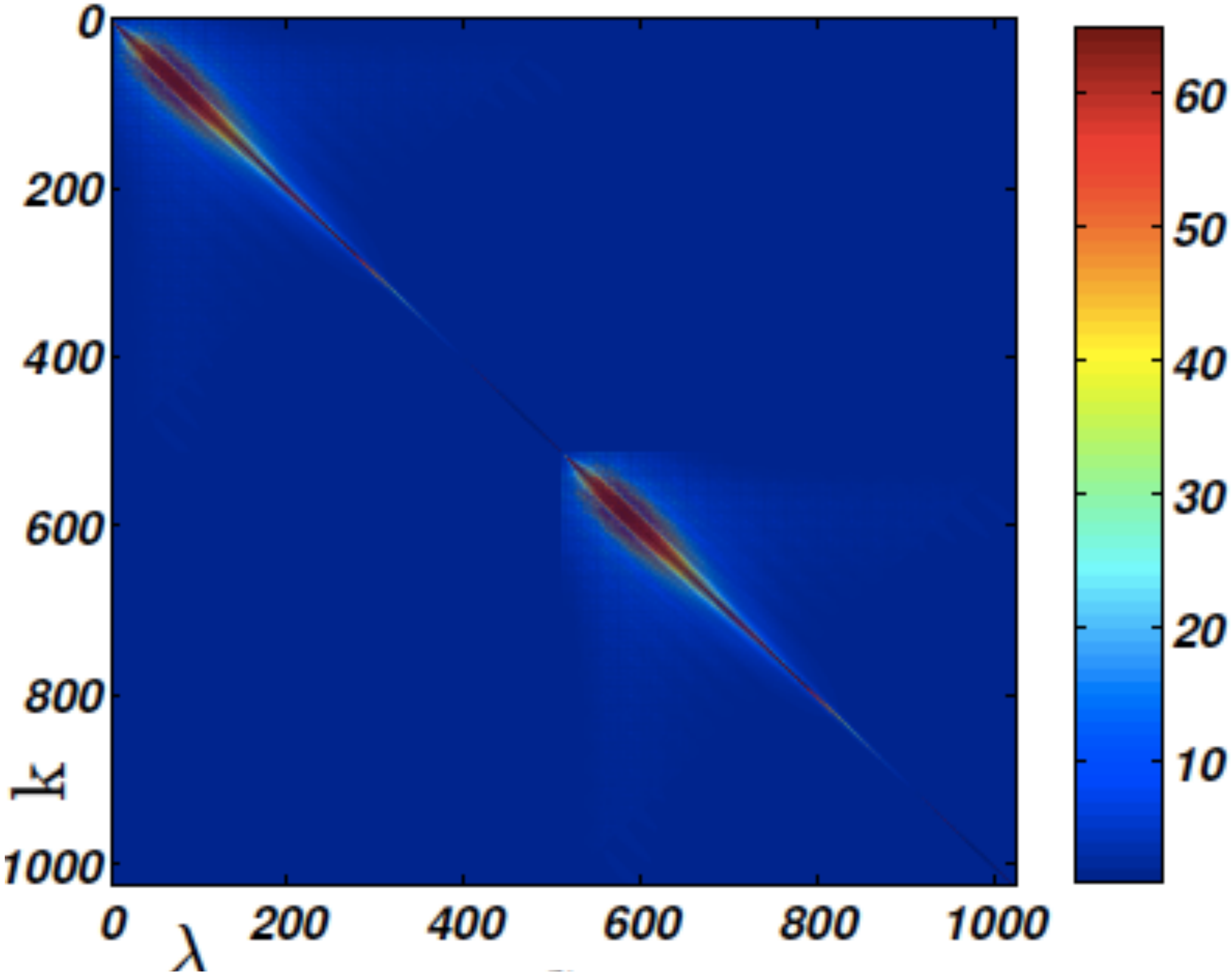
c)

κ_{Max} 2^9	Bottom profile
ε .01	

The larger entries are on the diagonal and the decay of the entries from the diagonal is related to the amplitud and smoothness of β(x).


log(size entries) of Msim(β5)
\sqrt{k}
10
5
0
−5
−10
−15
−20
0
10
20
30
40
50
60
70

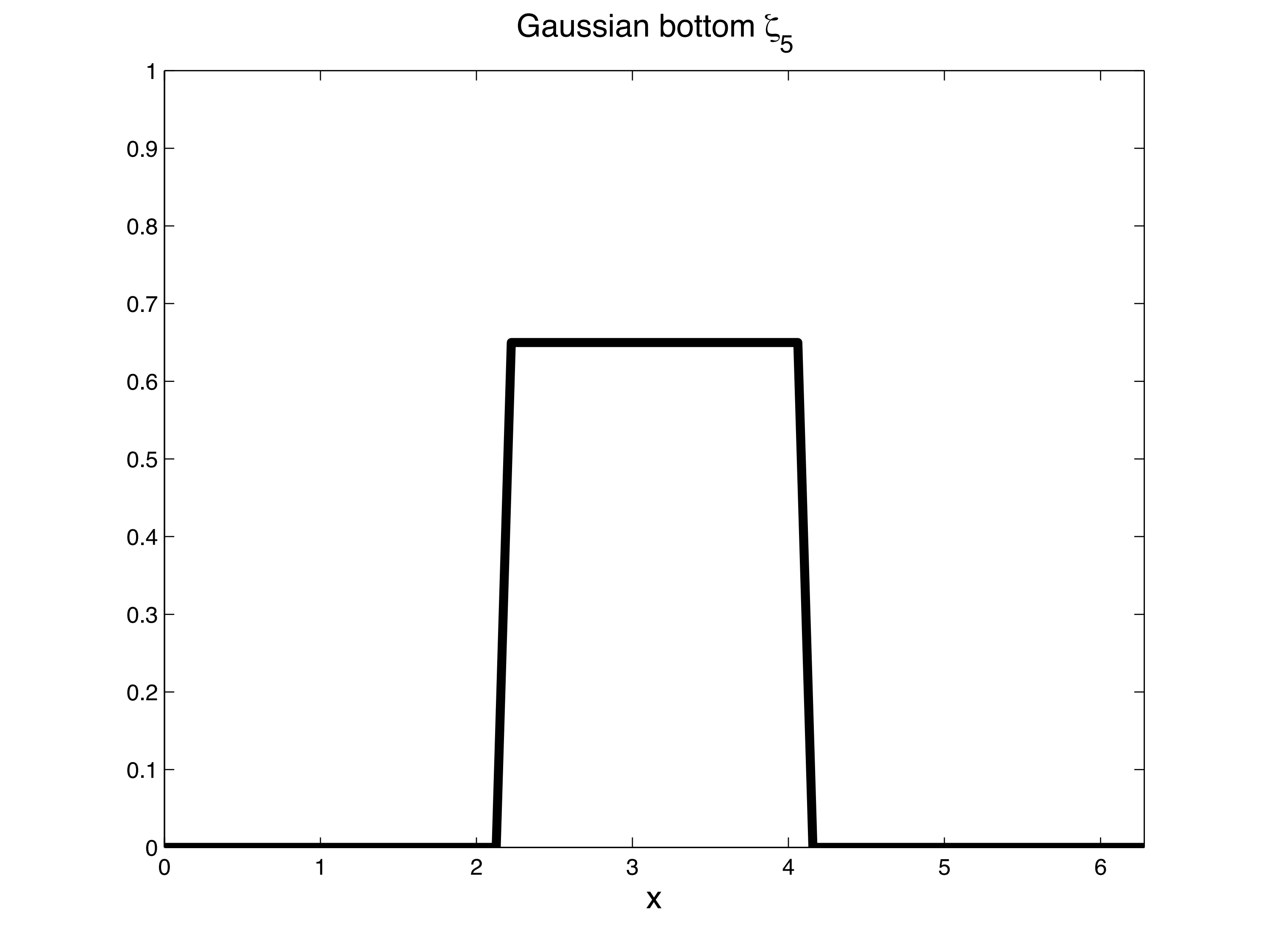

k
λ
0
200
400
600
800
1000
c)



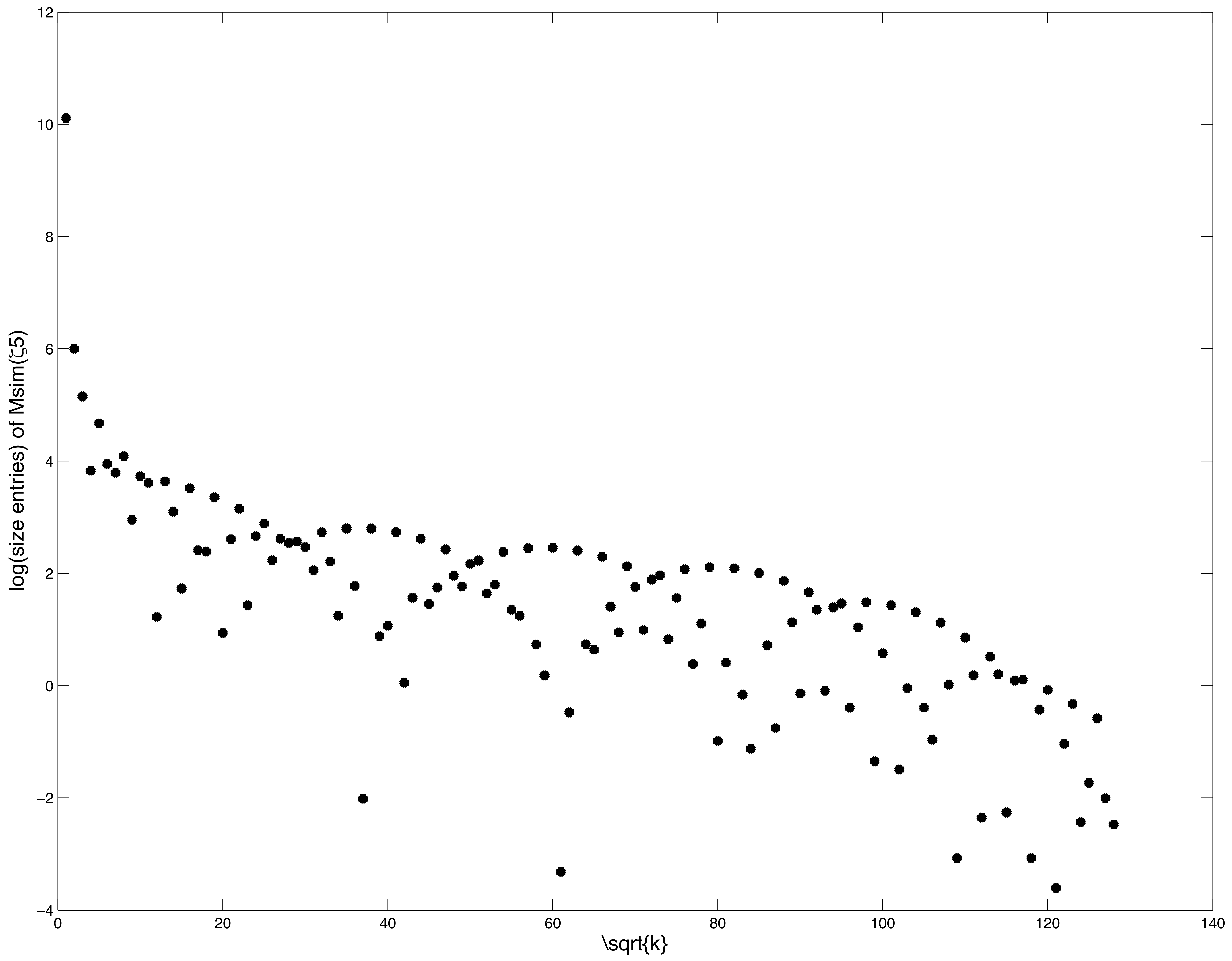
κMax
2^9

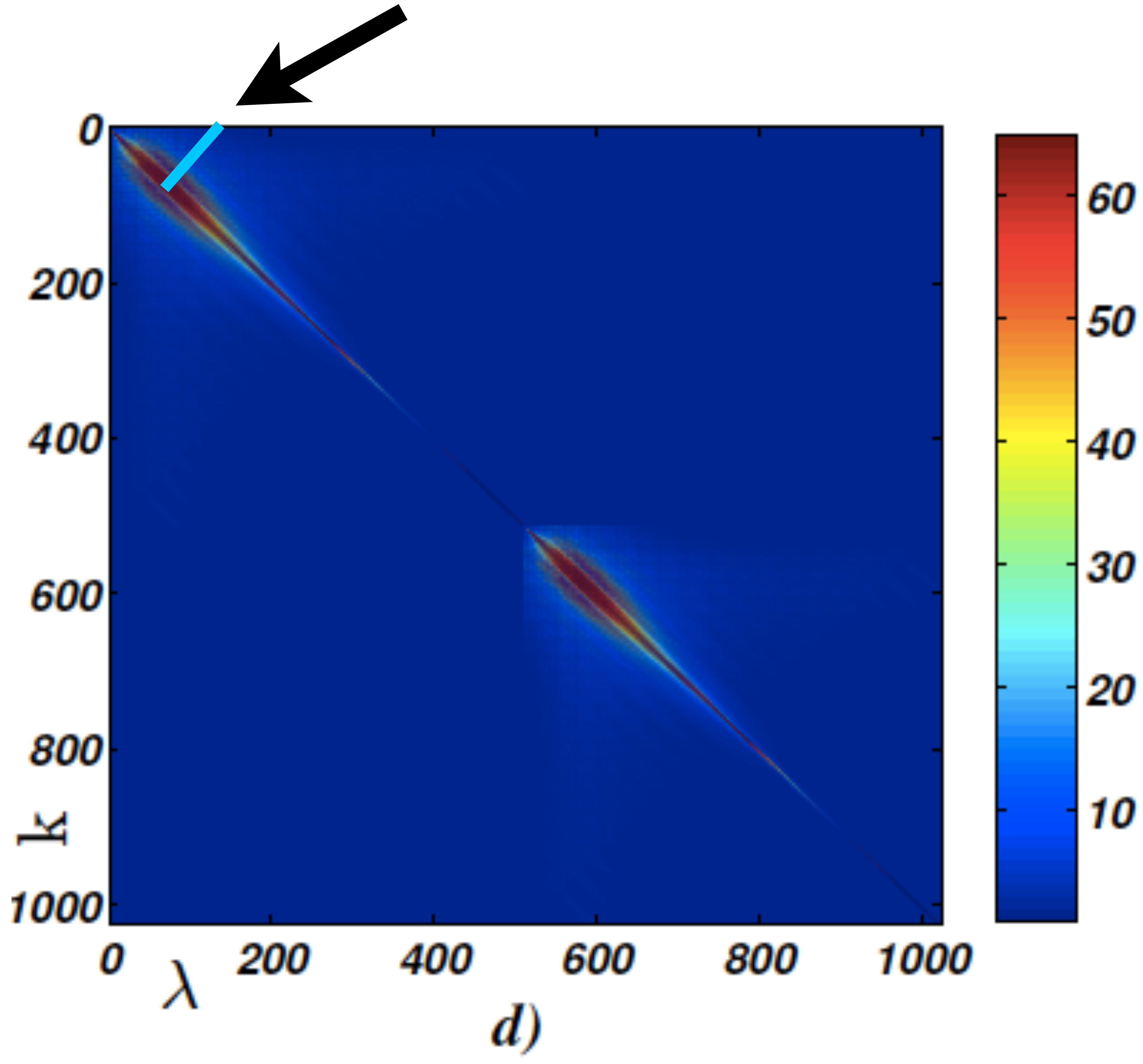
ε
.01

Bottom profile



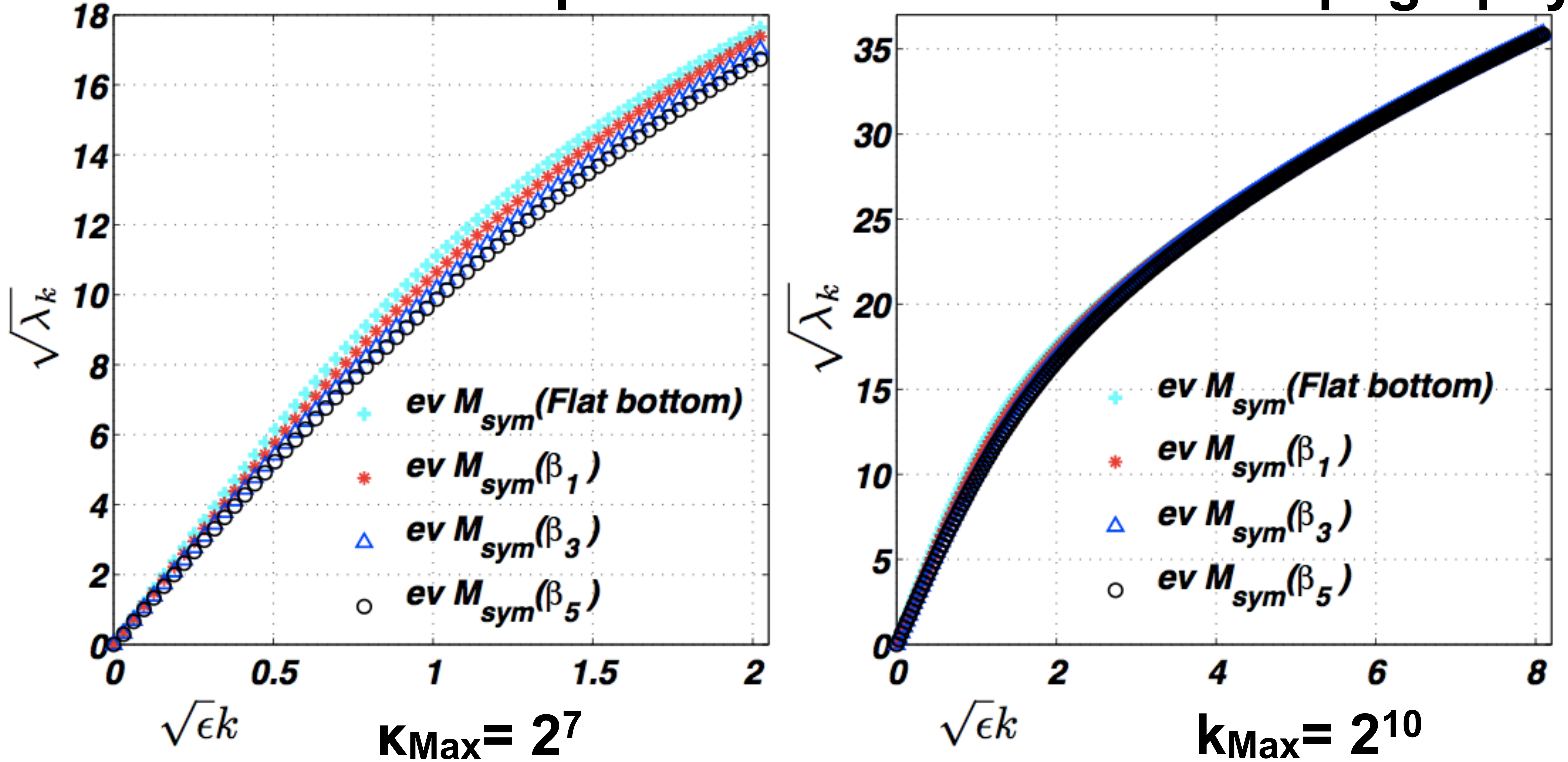
We see an enlargement around the diagonal that finishes in an skinny line, first the bright region "opens" and then "closes" around the diagonal for larger k.


log(size entries) of Msim(ζ5)
\sqrt{k}


k
λ
d)

Wave number $\kappa \longrightarrow \lambda_\kappa \longrightarrow \sqrt{\lambda_\kappa}$

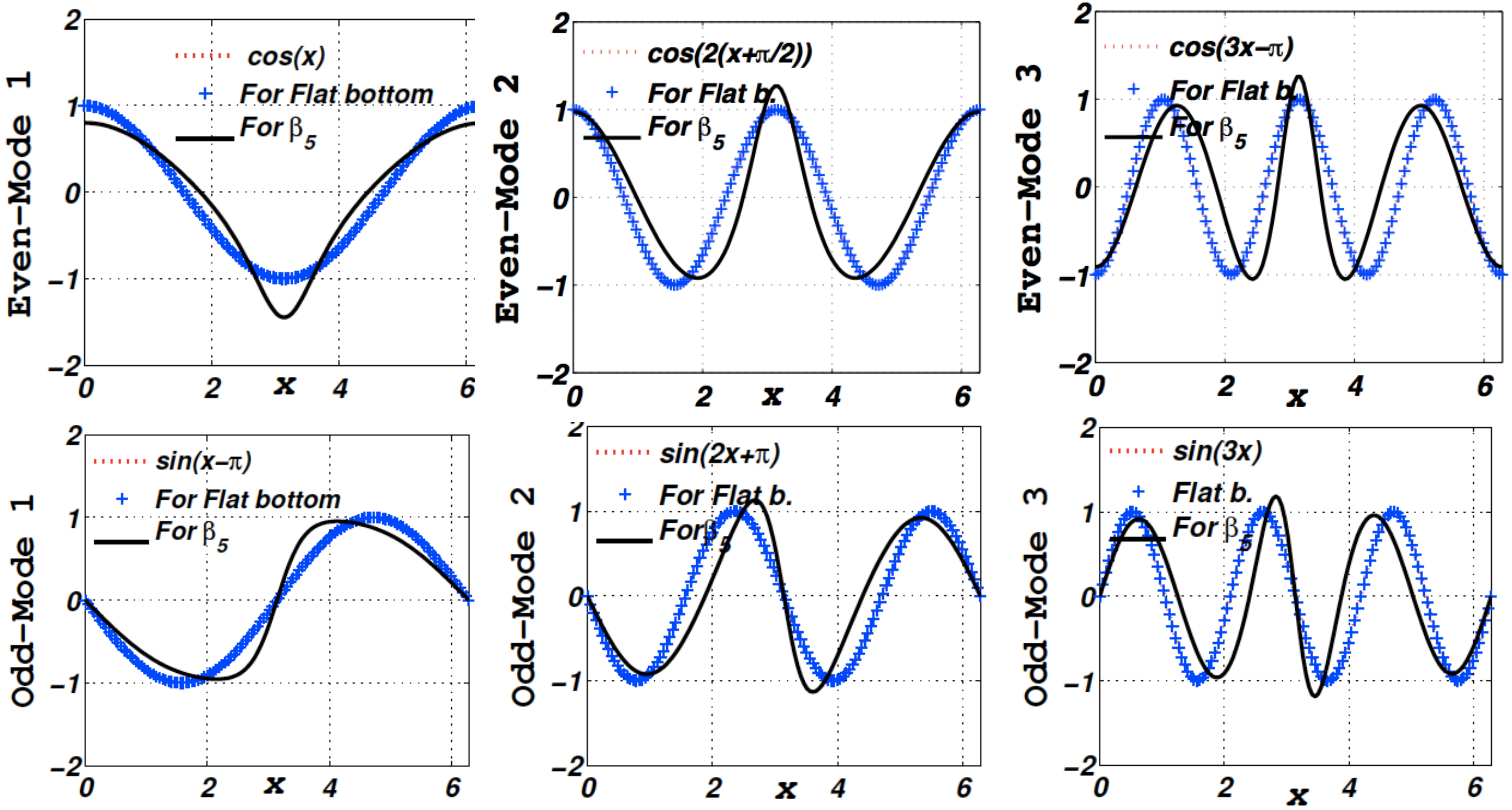
Generalized Dispersive Curve : Gaussian Topography



$\kappa_{Max} = 2^7$ $\quad$ $k_{Max} = 2^{10}$

The eigenfrequencies decrease with bathymetry.
There is evidence of monotonicity in the Curve with the depth. When you increase the amplitude on the depth, all the eigenvalues goes down.
And in the limit as κ you go to infinity we have that we approach the constant depth dispersion relation

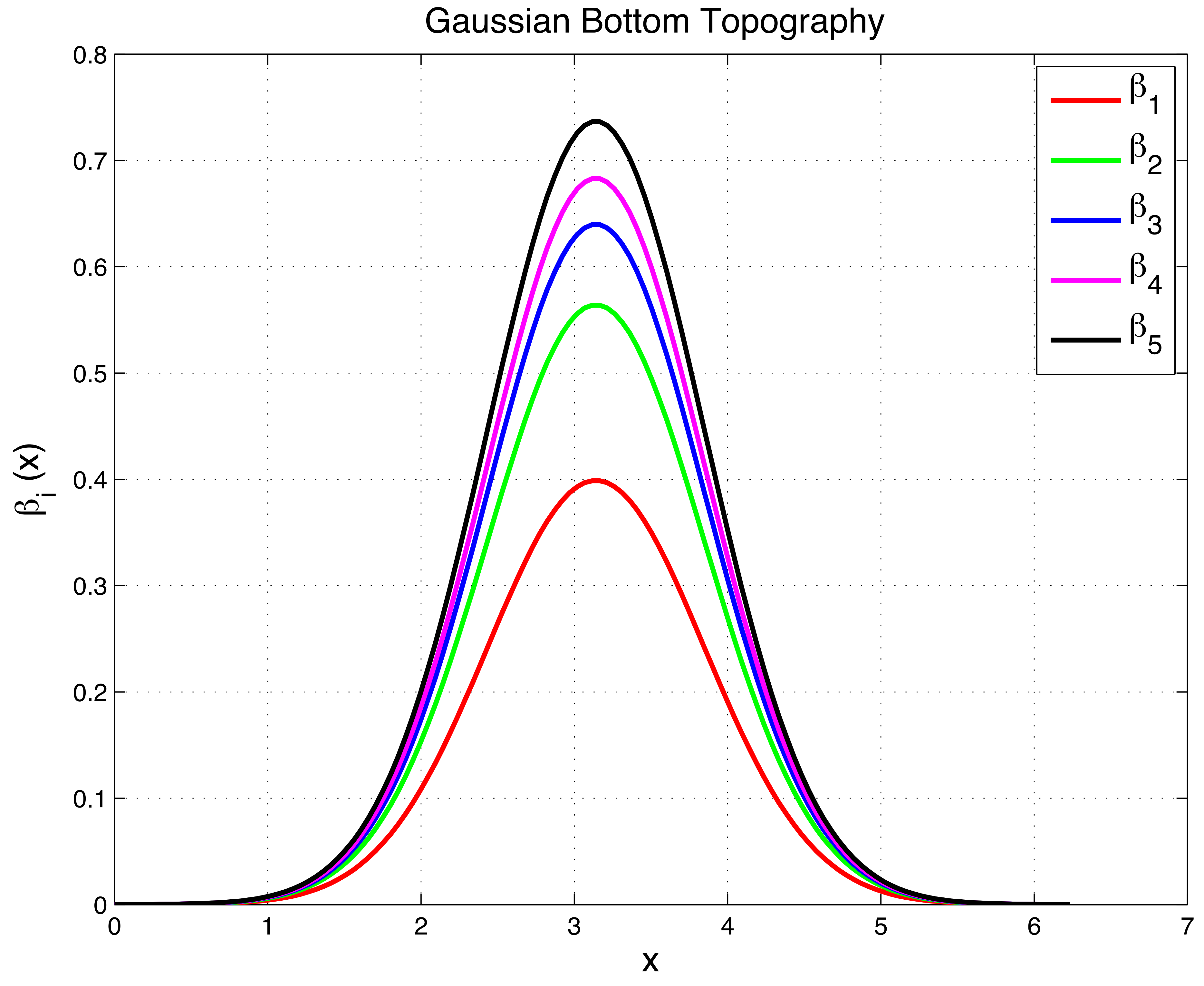
Steepening I.

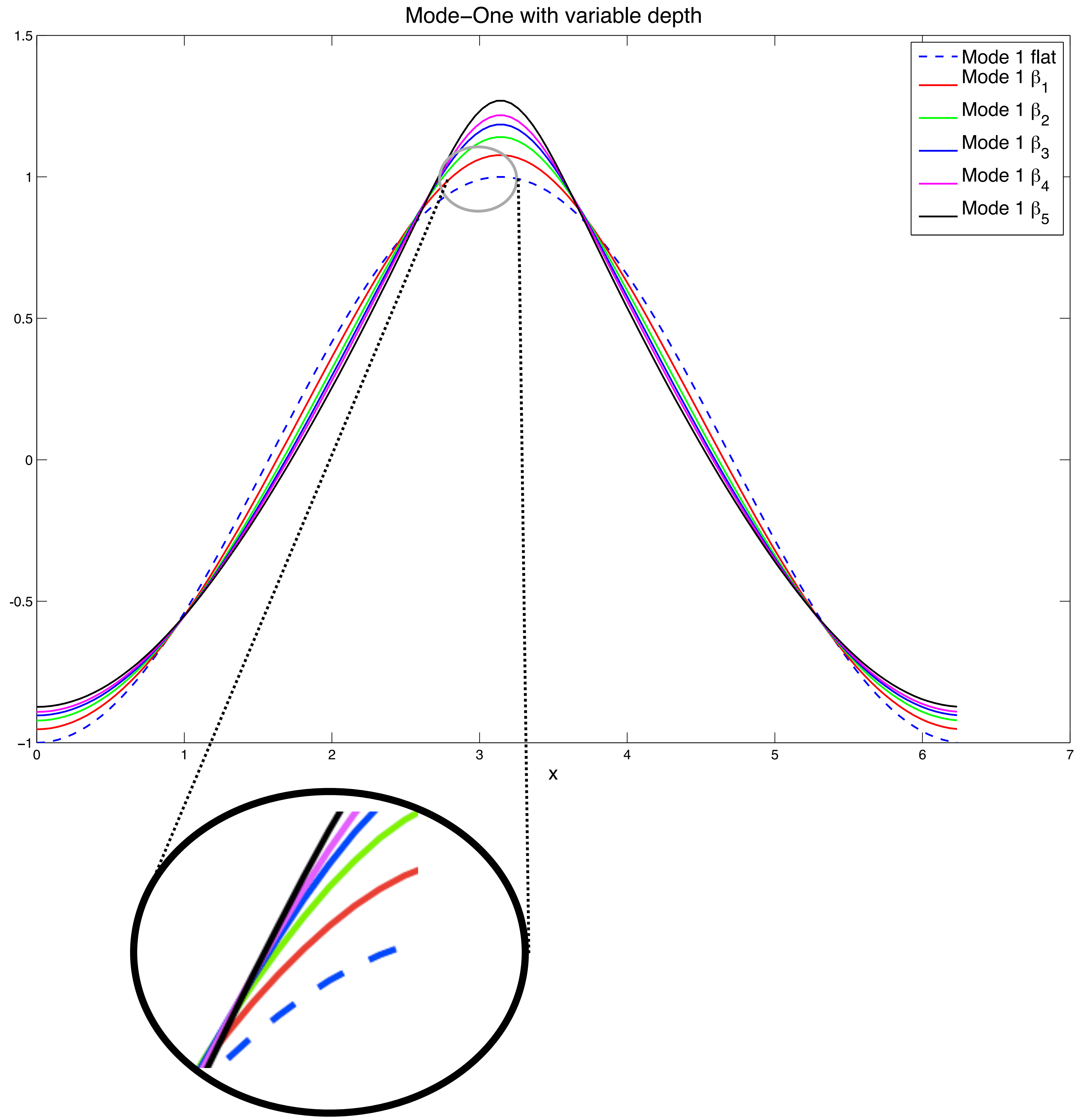


The spectral analysis of matrices pointing to show evidences that the depth variation produce significant effects on the eigenmodes:

Steepening II.

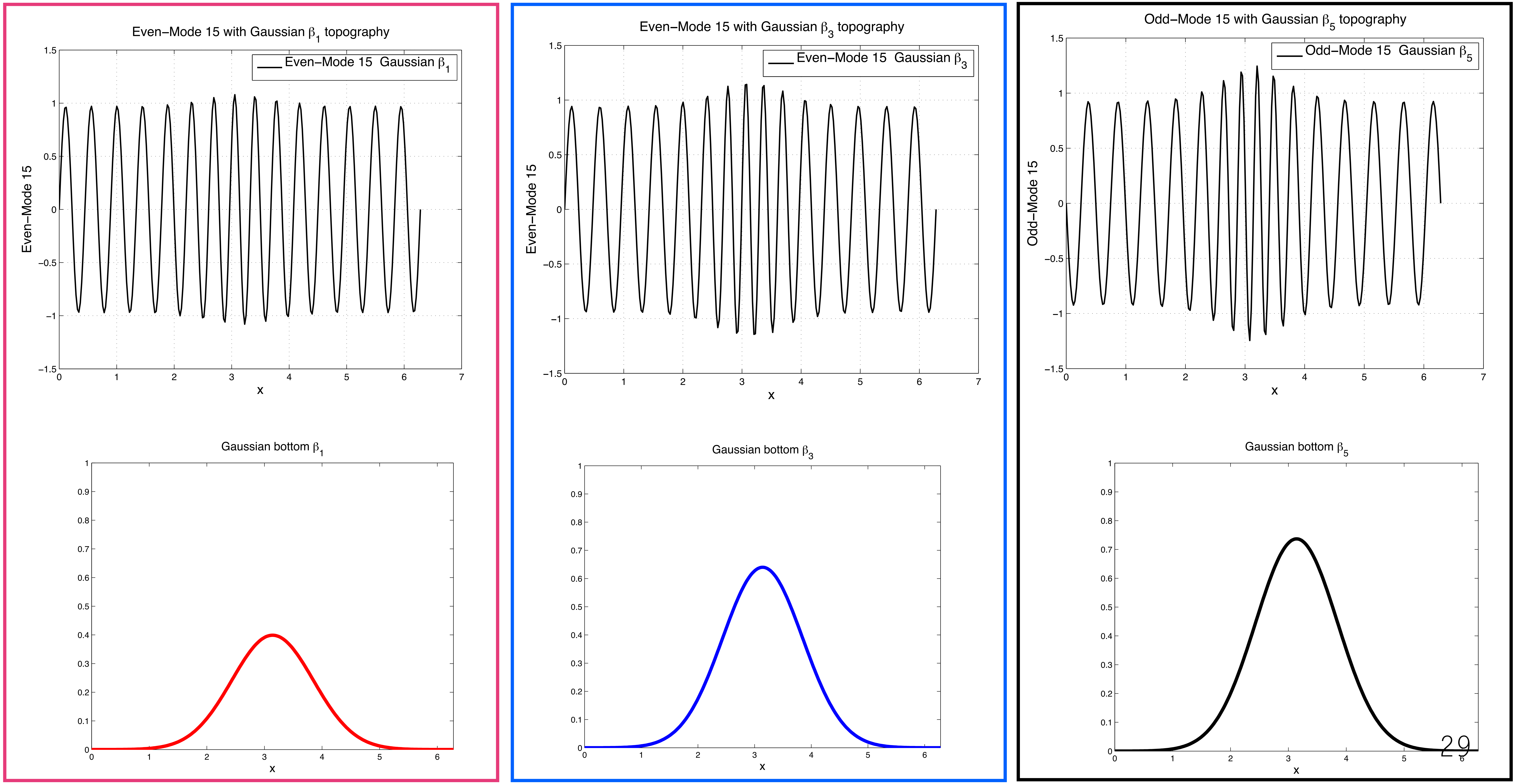
Topography leads to steeper profiles!



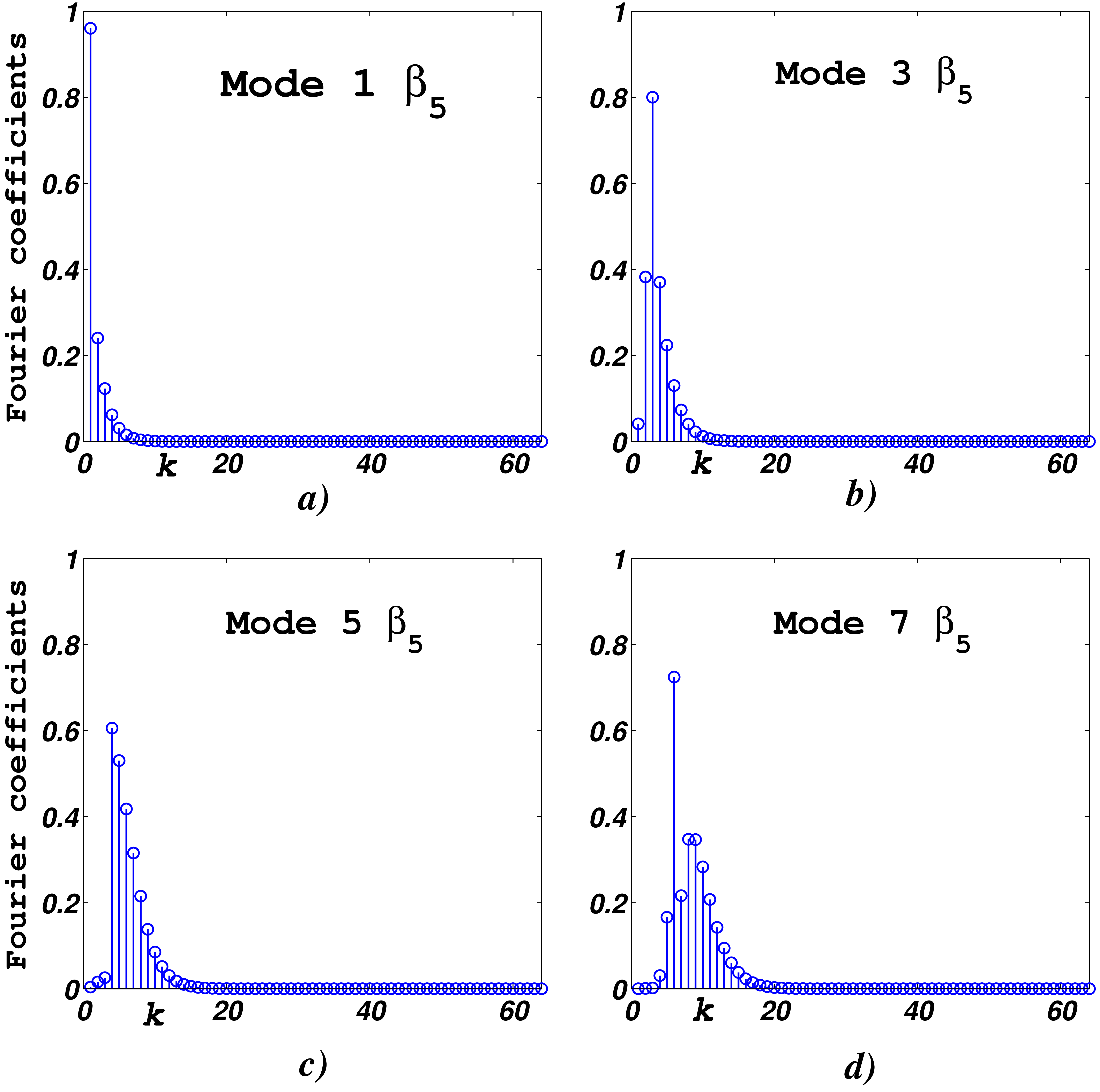


Modulation

Mode 15 $k_{Max} = 2^7$



Band width


Fourier coefficients
Mode 1 β5
Mode 3 β5
Mode 5 β5
Mode 7 β5
k
0
20
40
60
0
0.2
0.4
0.6
0.8
1
a)
b)
c)
d)

¿Do there exist a 3-wave resonance?

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3}$$

$$k_1 + k_2 = k_3 + \Delta_\kappa$$

Using the Bathymetry can we have solutions of this kind for not small wave number values?

Numerical integration of the evolution of some initial wave-profiles over different topographies.

$$H_2 = \frac{1}{2}\int_{\mathbb{R}} [\xi(Sym(\frac{D}{\sqrt{\epsilon}}\tanh(\sqrt{\epsilon}h(x)D)) + \epsilon D\eta D)\xi + \eta^2]dx,$$

$$\begin{cases} \partial_t \hat{\eta}_k = \frac{\partial H}{\partial \hat{\xi}_k^*} \\ \partial_t \hat{\xi}_k = -\frac{\partial H}{\partial \hat{\eta}_k^*} \end{cases}, \quad k \in J_M, \quad \text{with} \quad J_M = [1, ..., M] \, .$$

- $a_k = Re(\eta_k)$
- $\beta_k = Im(\eta_k)$
- $\gamma_k = Re(\hat{\xi}_k)$
- $\delta_k = Im(\xi_k)$

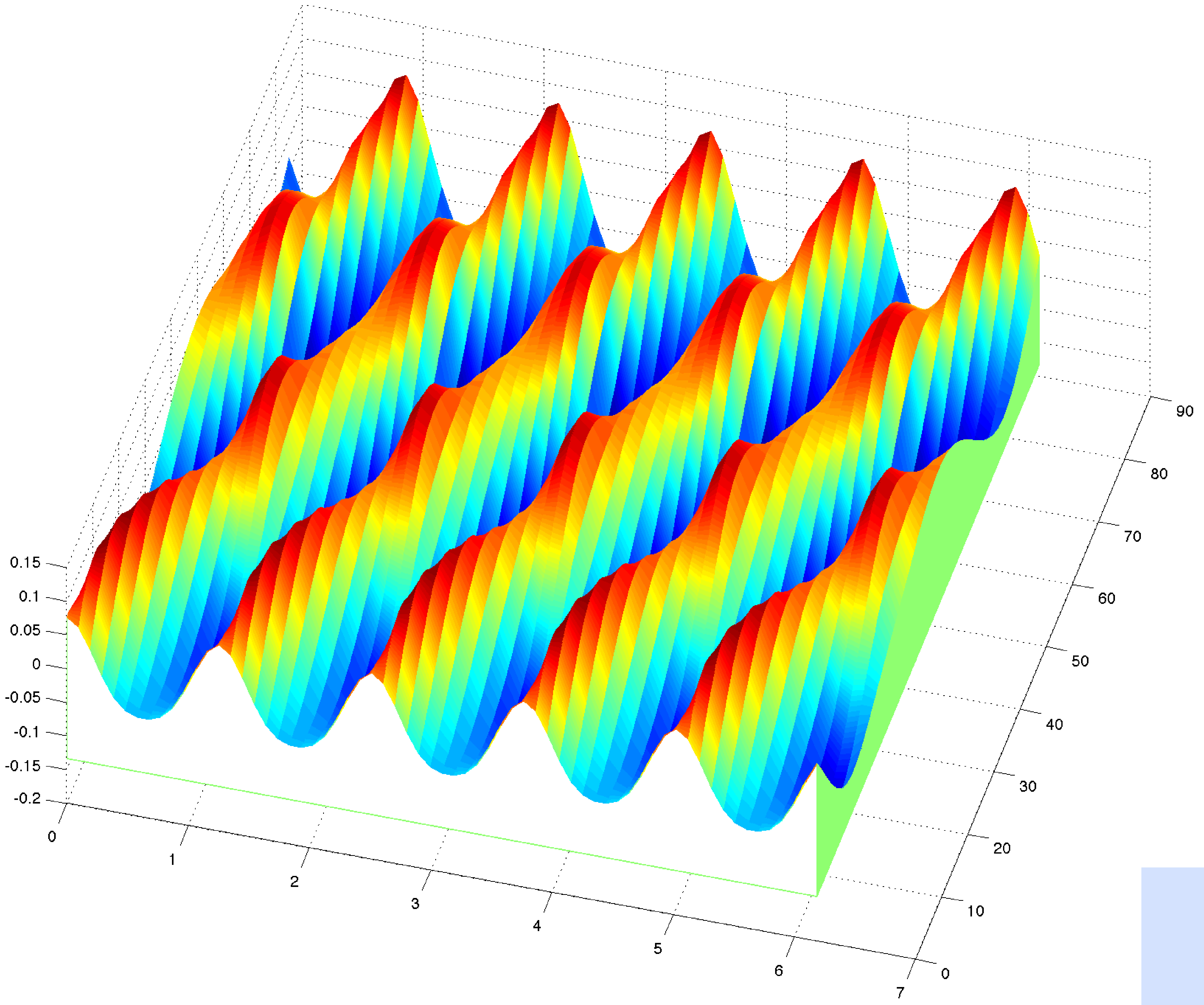
$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} g_1(\alpha,\beta,\gamma,\delta) \\ g_2(\alpha,\beta,\gamma,\delta) \\ g_3(\alpha,\beta,\gamma,\delta) \\ g_4(\alpha,\beta,\gamma,\delta) \end{pmatrix}$$

FULLY SPECTRAL EQUATIONS.

FOURTH-FIFTH ORDER ADAMS-BASHFOR/MOULTON

Second-order approximation of Stokes wavetrain

$k_{Max} = 2^5$ $dt = 0.001$ $\varepsilon = 0.01$

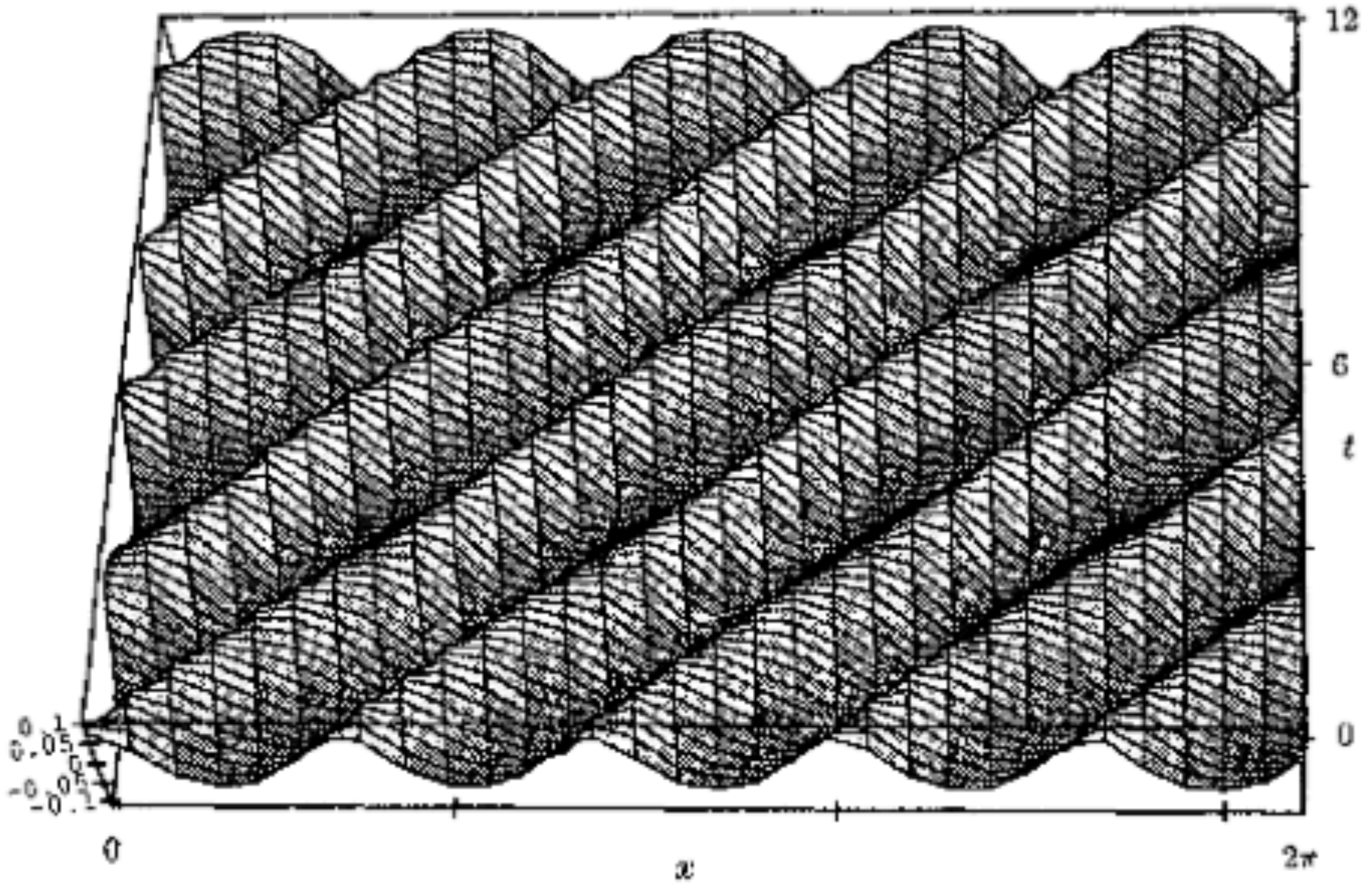


$$\eta_0(x) = a\cos(\lambda x) + \mu_2 a^2 \cos(2\lambda x),$$
$$\xi_0(x) = \nu_1 a \cosh(\lambda(\eta_0 + h))\sin(\lambda x) + \nu_2 a^2 \cosh(2\lambda(\eta_0 + h))\sin(2\lambda x).$$

$$\mu_2 = \frac{1}{2}\lambda\coth(h\lambda)\left(1 + \frac{3}{2\sinh(\lambda h)}\right),$$

$$\nu_1 = \frac{\omega}{\lambda\sinh(h\lambda)}, \qquad \nu_2 = \frac{3}{8}\frac{3\omega}{\sinh^4(h\lambda)}.$$

$a = 0.065,\ \lambda = 5.$



Craig and Sulem, **Numerical Simulation of gravity waves.**,1992,

Flat bottom

Our numerical simulations using the Whitham–Boussinesq model suggest that variable depth has significant effects on the dynamics of surface waves.

Second-order approximation of Stokes wavetrain

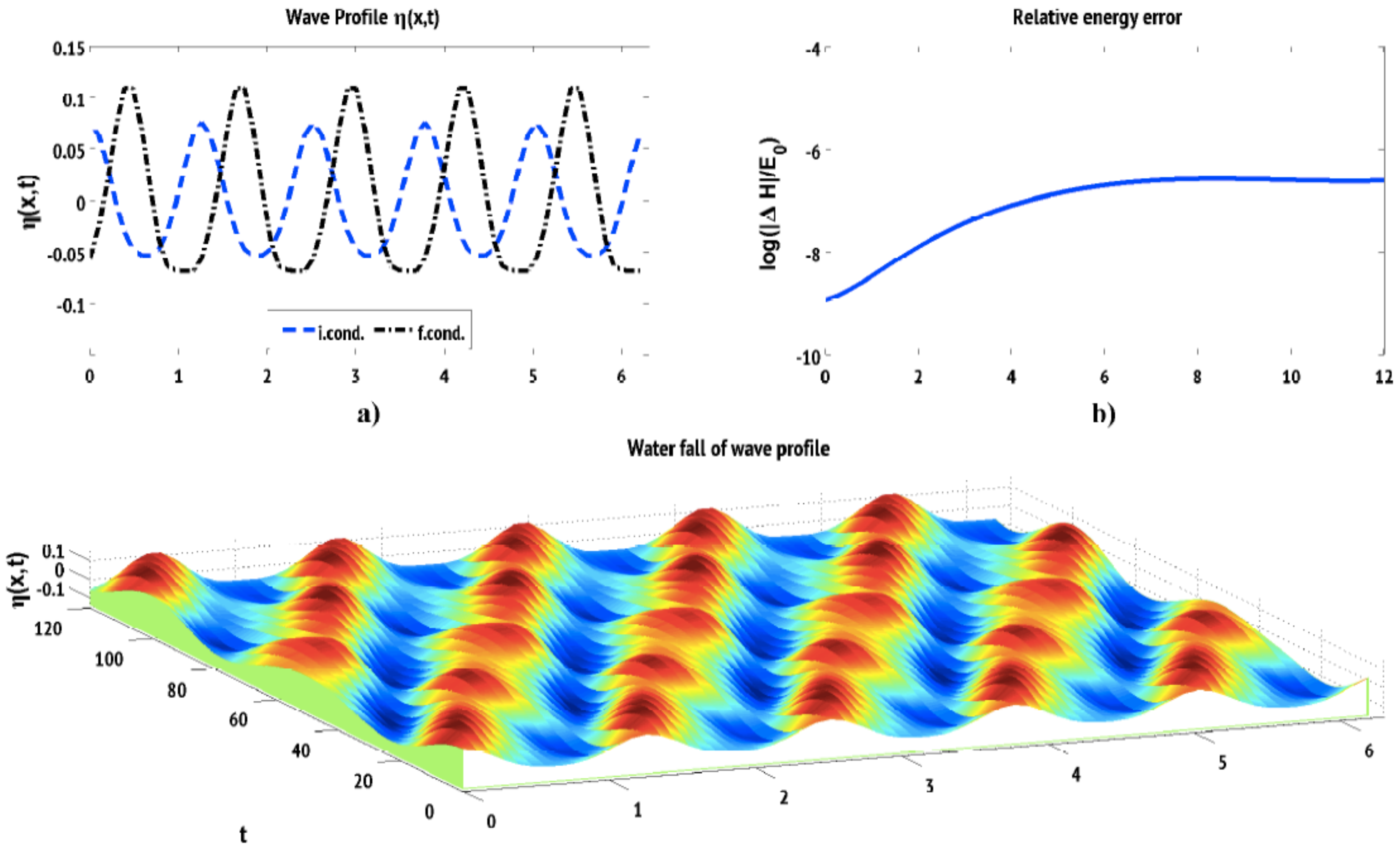
$k_{Max} = 2^5$ $\varepsilon = 0.01$ $dt = 0.001$ $t_{final} = 120$

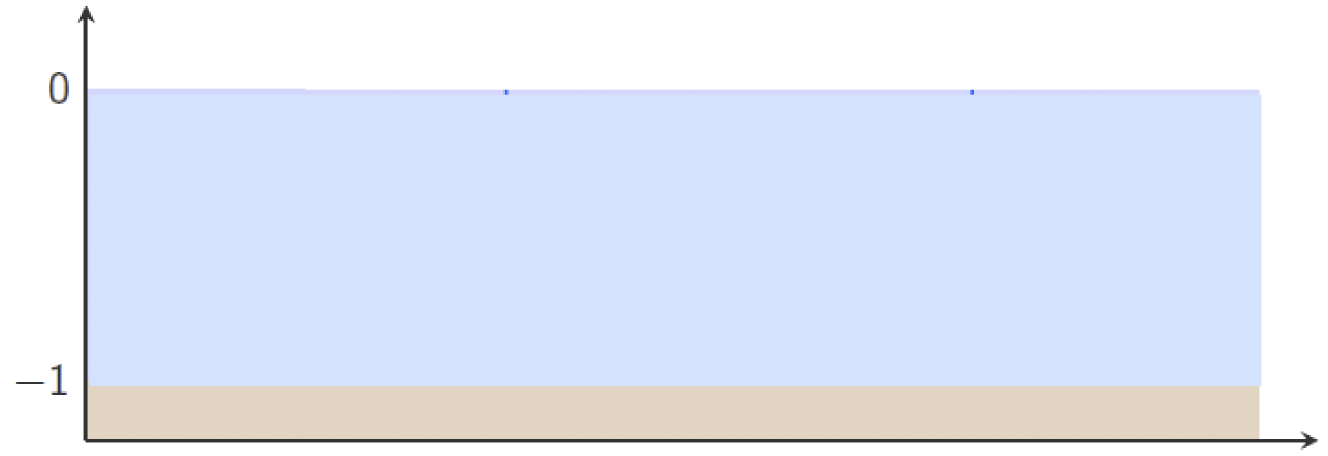
$a = 0.065$

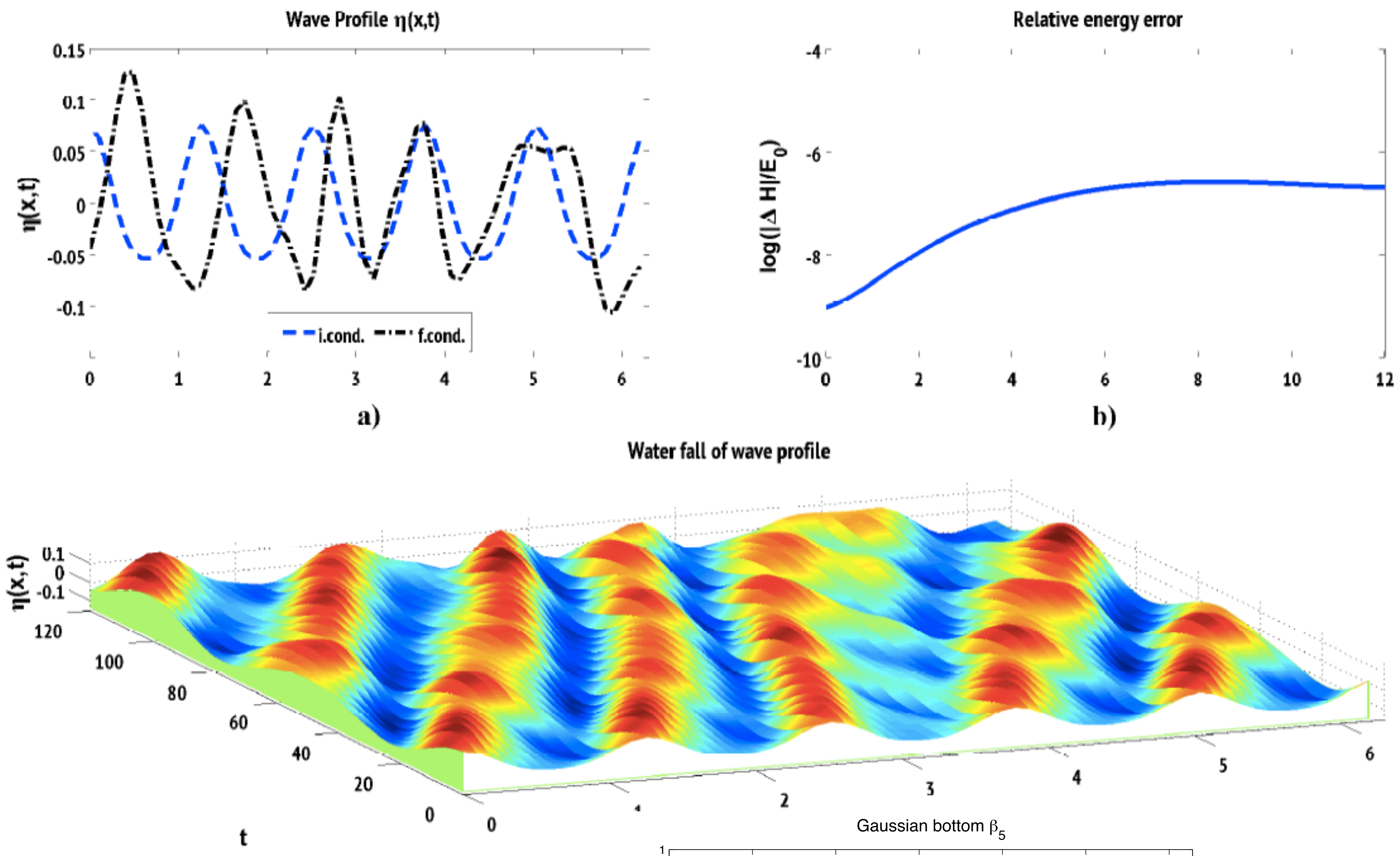
$\lambda = 5$

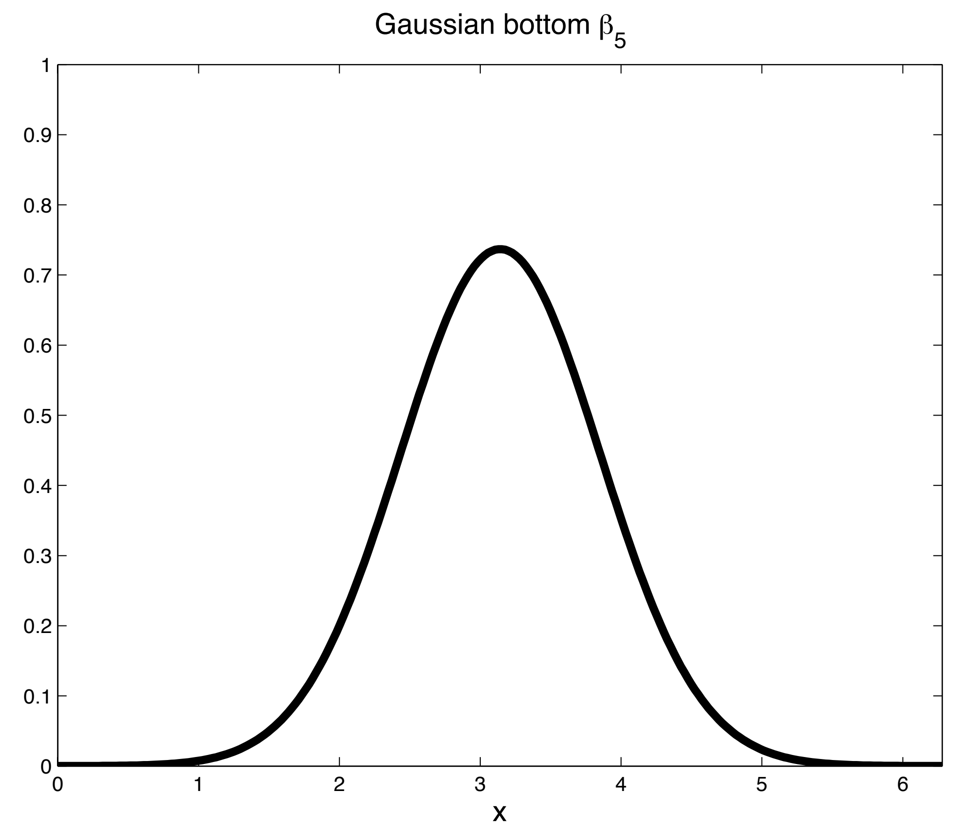
$$\eta_0(x) = a\cos(\lambda x) + \mu_2 a^2 \cos(2\lambda x),$$
$$\xi_0(x) = \nu_1 a \cosh(\lambda(\eta_0 + h))\sin(\lambda x) + \nu_2 a^2 \cosh(2\lambda(\eta_0 + h))\sin(2\lambda x$$

$$\mu_2 = \frac{1}{2}\lambda\coth(h\lambda)\left(1 + \frac{3}{2\sinh(\lambda h)}\right),$$









Modulated wave packet

$$\eta_0(x) = 0,01e^{-\frac{4}{3}(x-\pi)^2}\cos(10x),$$
$$\xi_0(x) = 0,\ \forall x \in [0, 2\pi].$$

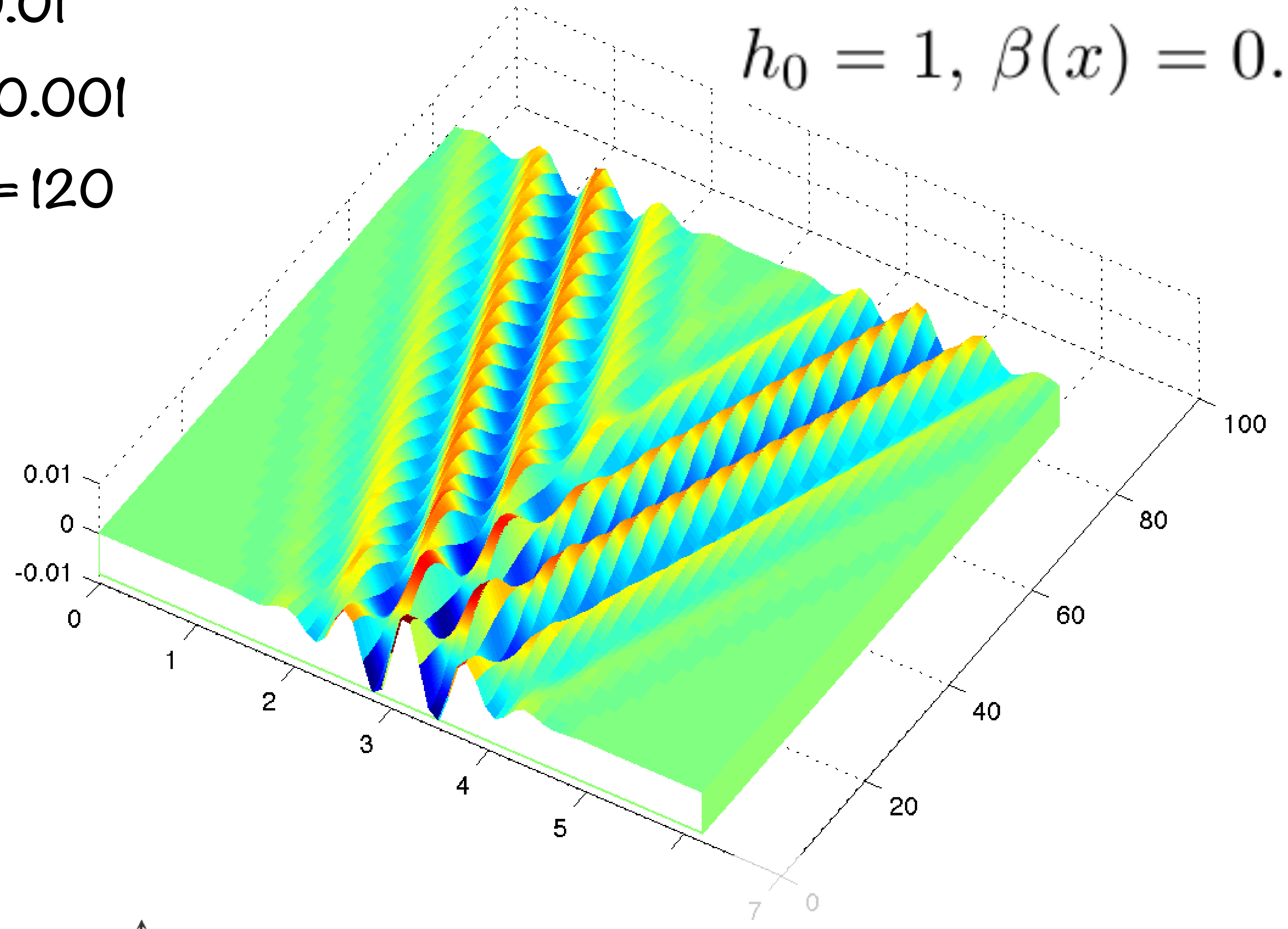
$k_{Max} = 2^5$

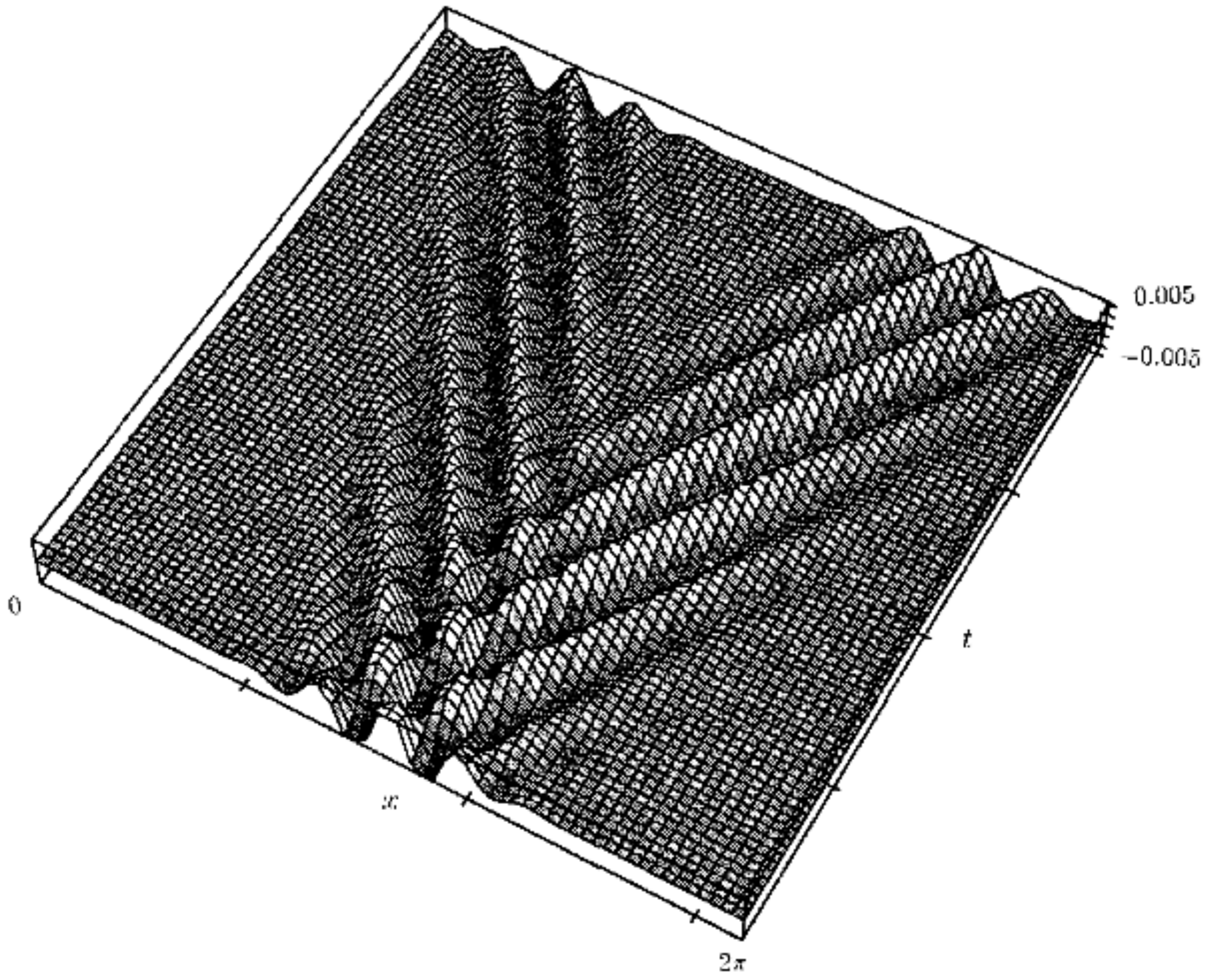
$\varepsilon = 0.01$

$dt = 0.001$

$t_{final} = 120$

$$h_0 = 1,\ \beta(x) = 0.$$





Craig and Sulem, **Numerical Simulation of gravity waves.**, 1992,

Modulated wave packet

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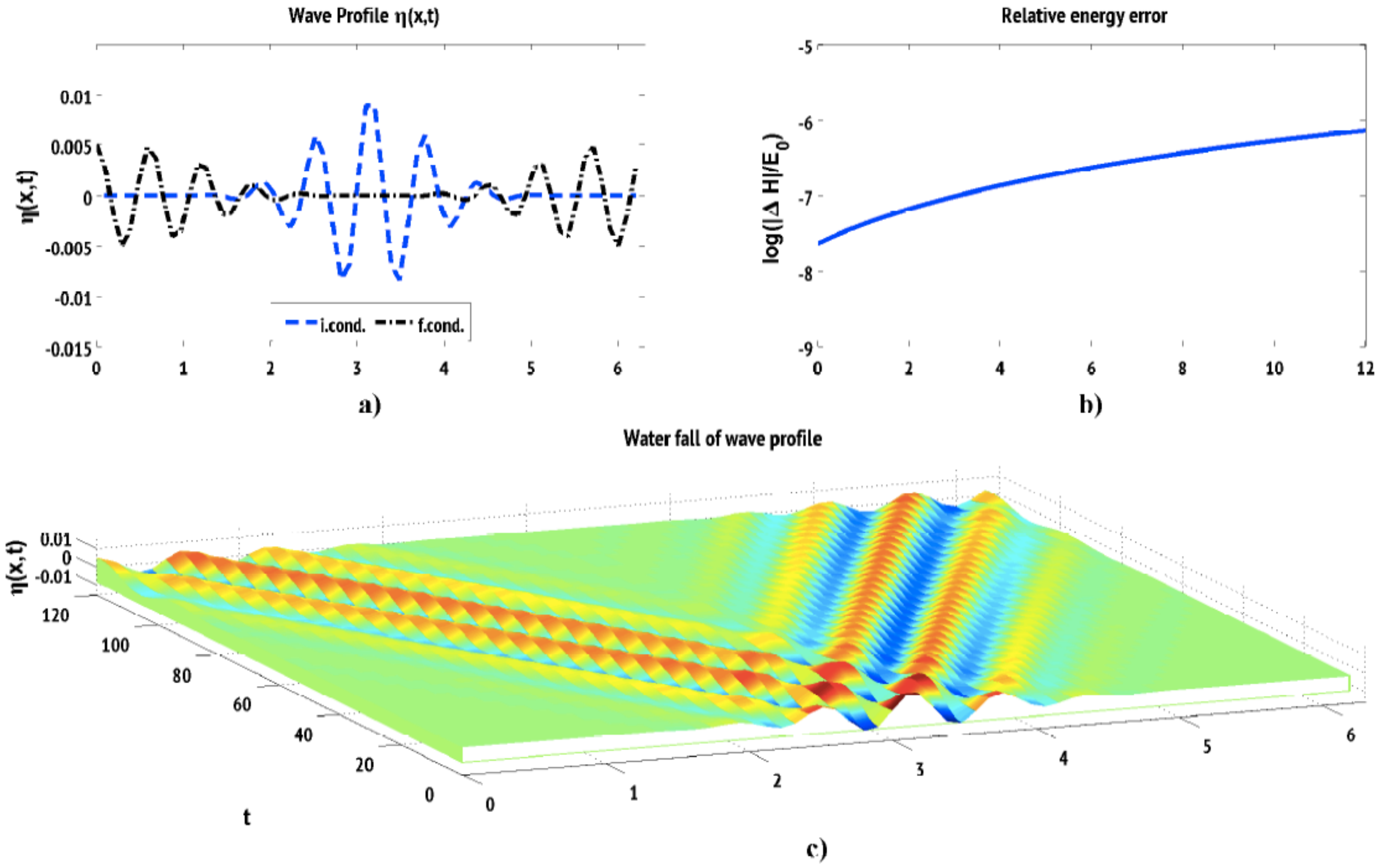
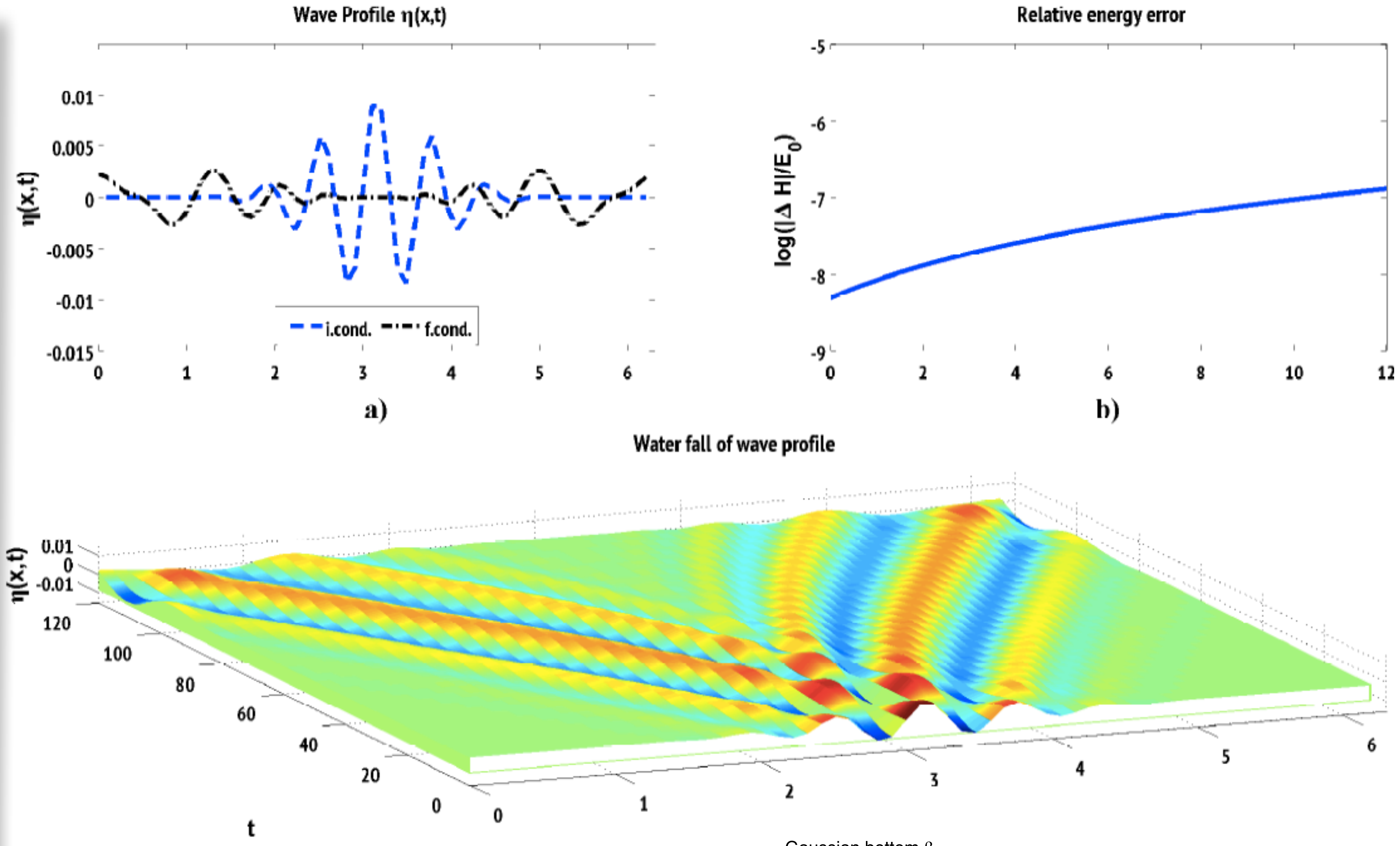
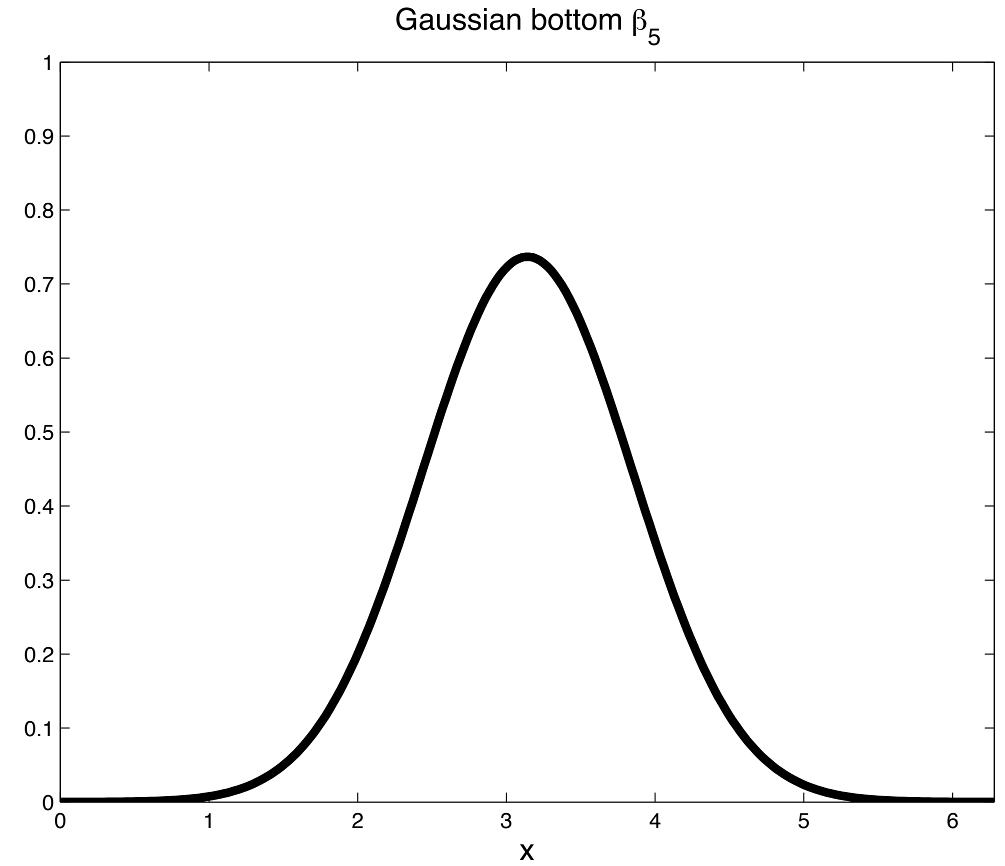
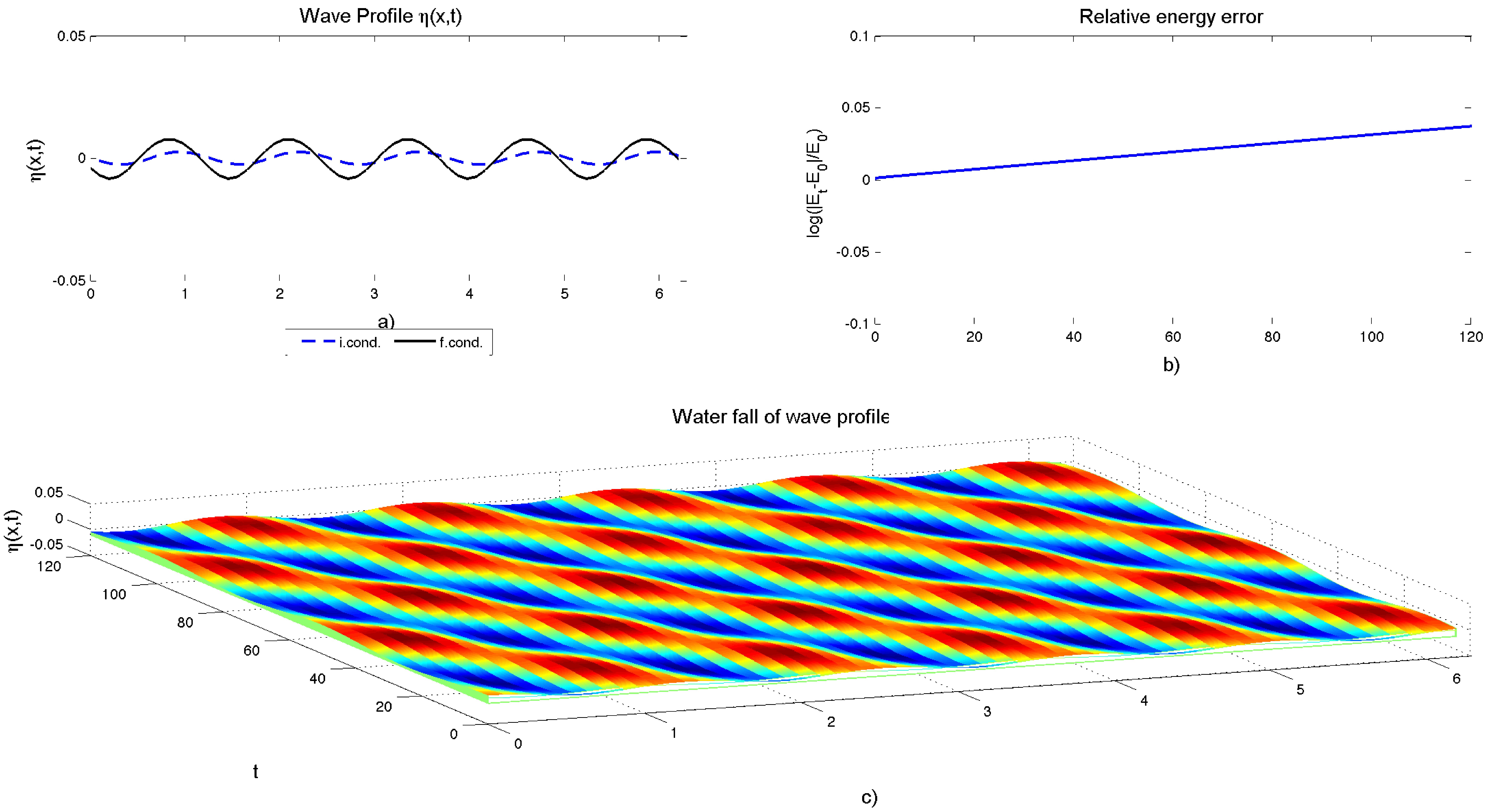
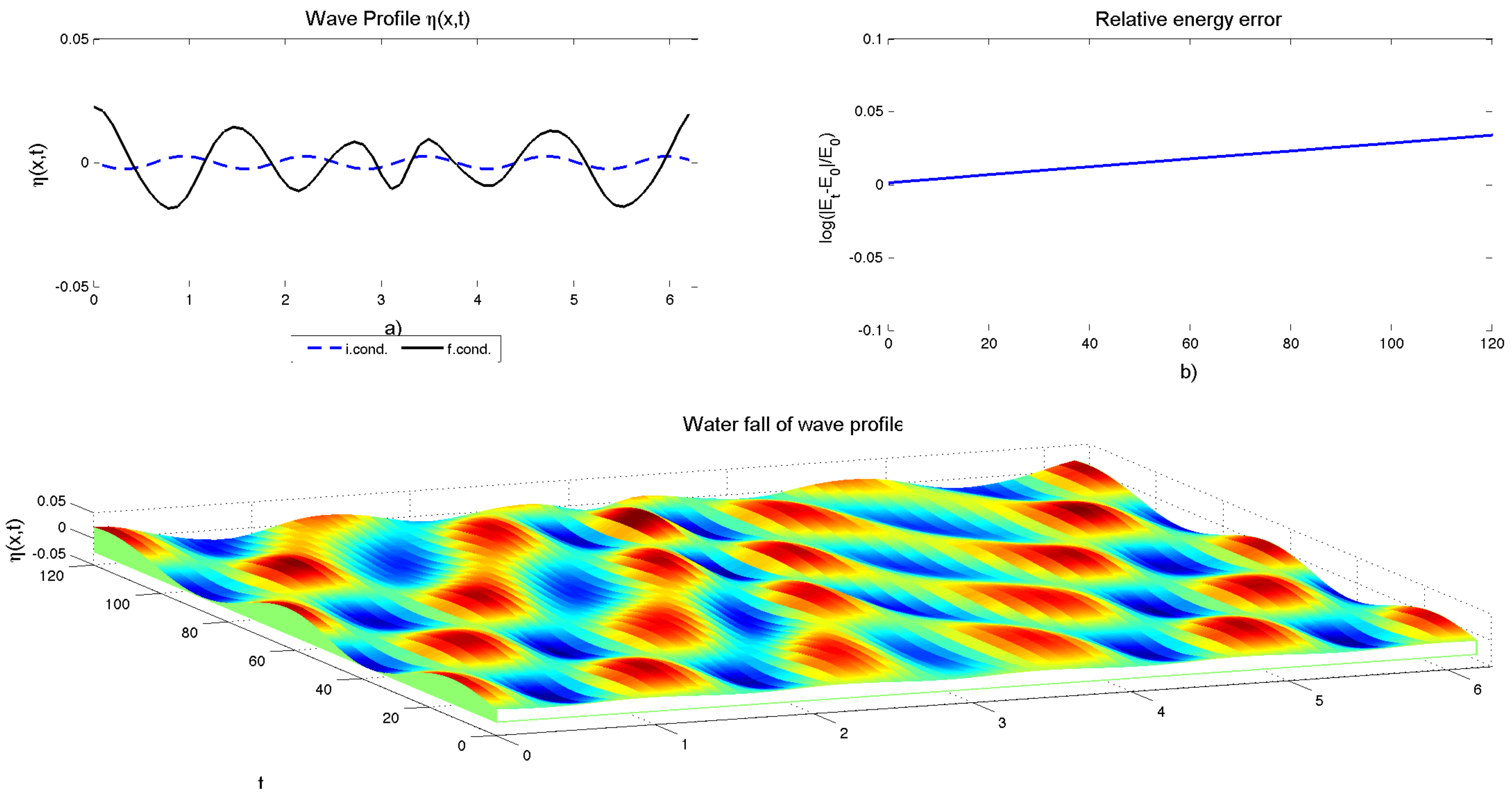


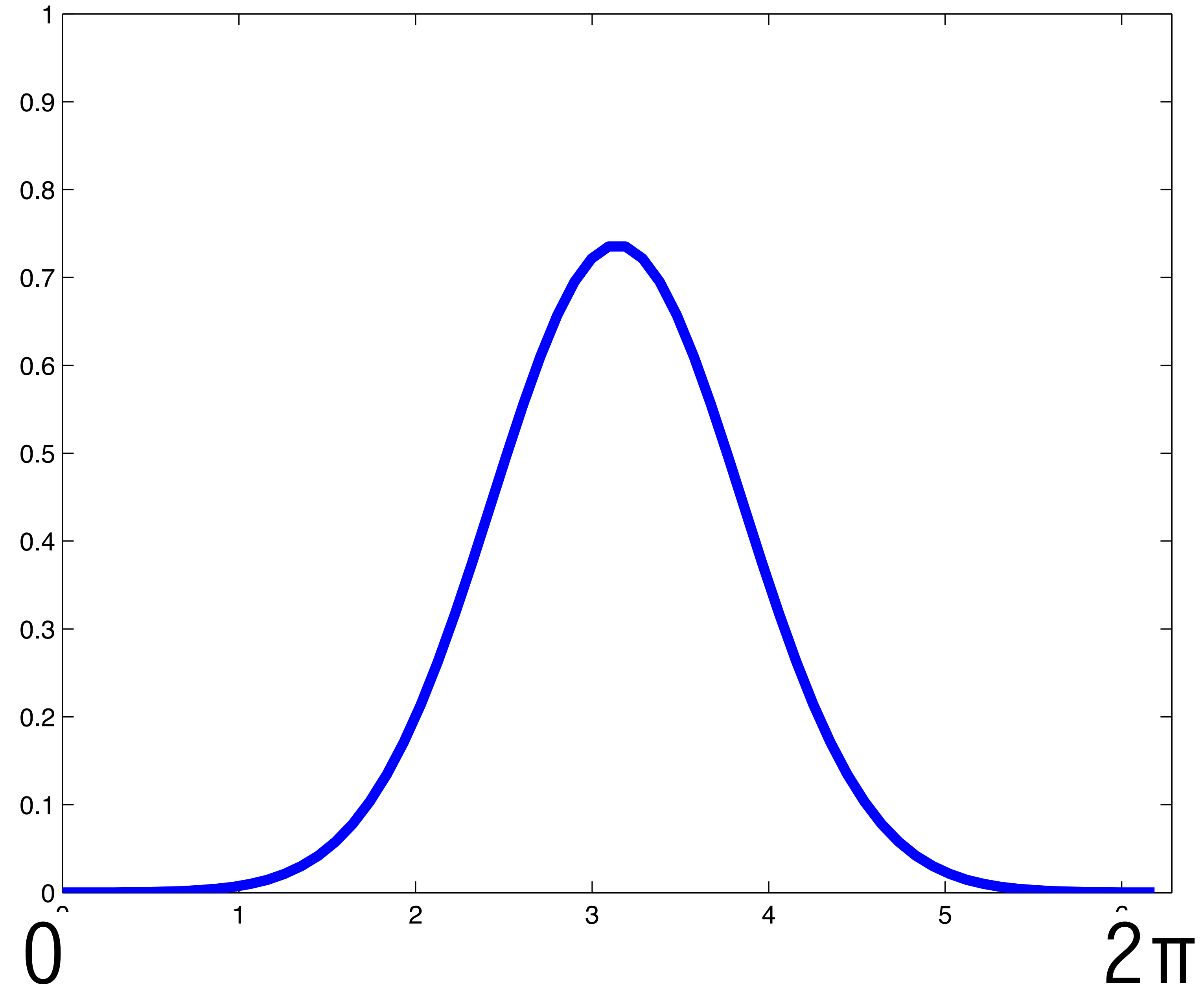


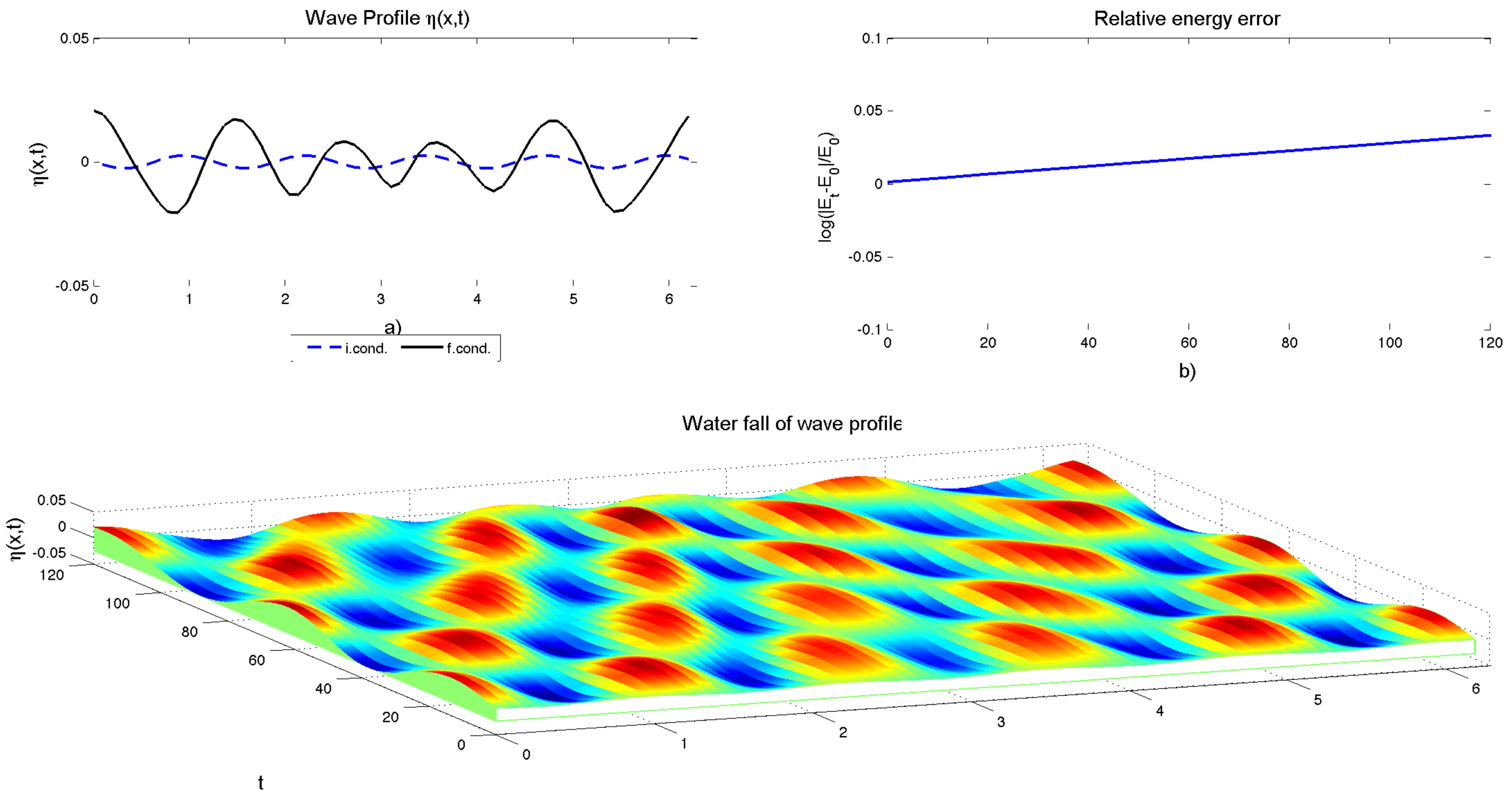


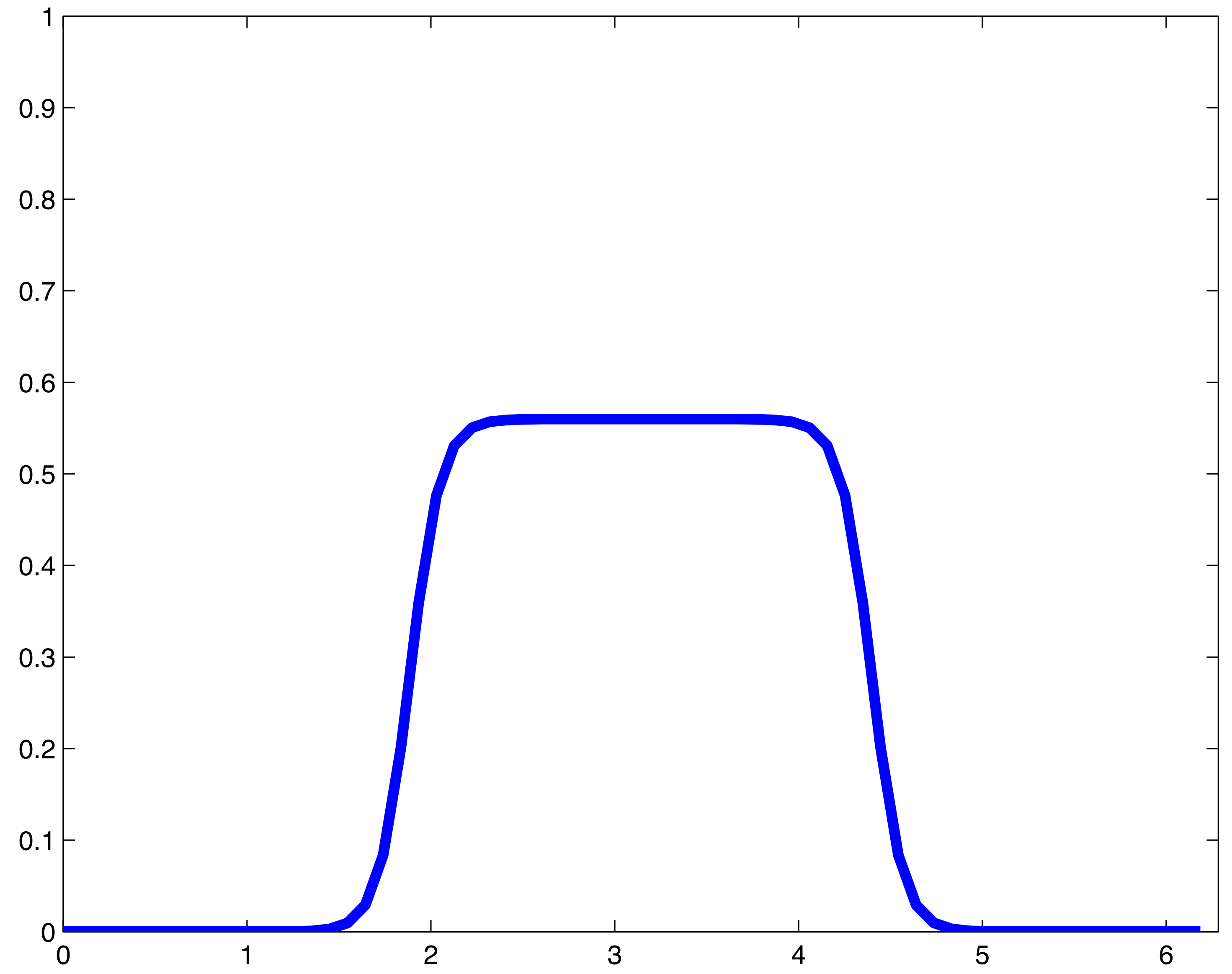
$$\eta = \frac{\varepsilon\nu_0}{k}\cos(k\tau) + \frac{\nu_0\varepsilon^2 B}{2k}\cos(2k\tau) - \frac{\varepsilon^2}{2k^2}\cos^2(k\tau) \qquad \xi(\tau) = \varepsilon\frac{-1}{k^2}\sin(k\tau) + \varepsilon^2\frac{-B}{4k^2}\sin(2k\tau)$$

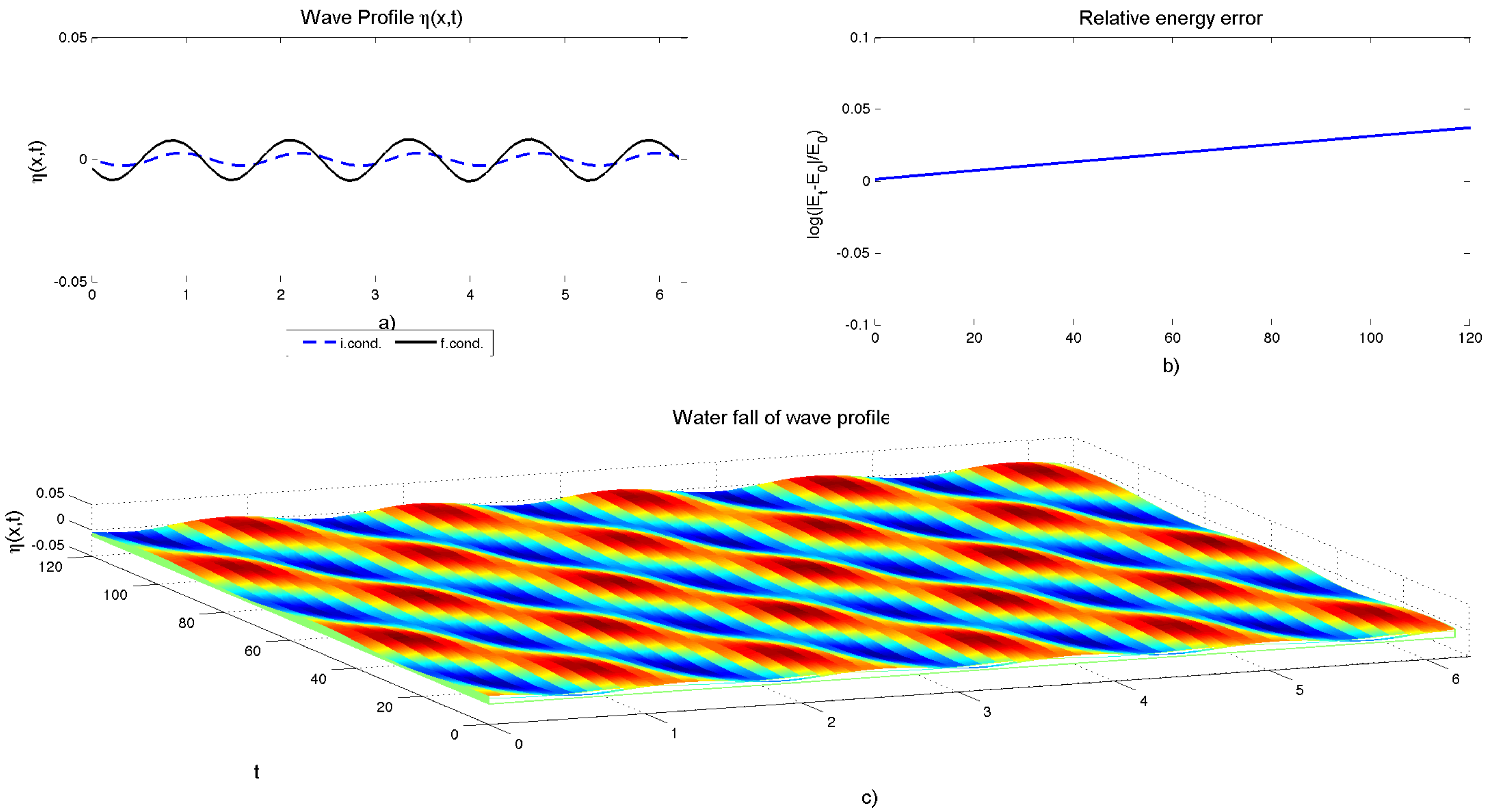



Wave Profile η(x,t)
η(x,t)
a)
i.cond.
f.cond.
Relative energy error
log(|E_t-E_0|/E_0)
b)
Water fall of wave profile
η(x,t)
t

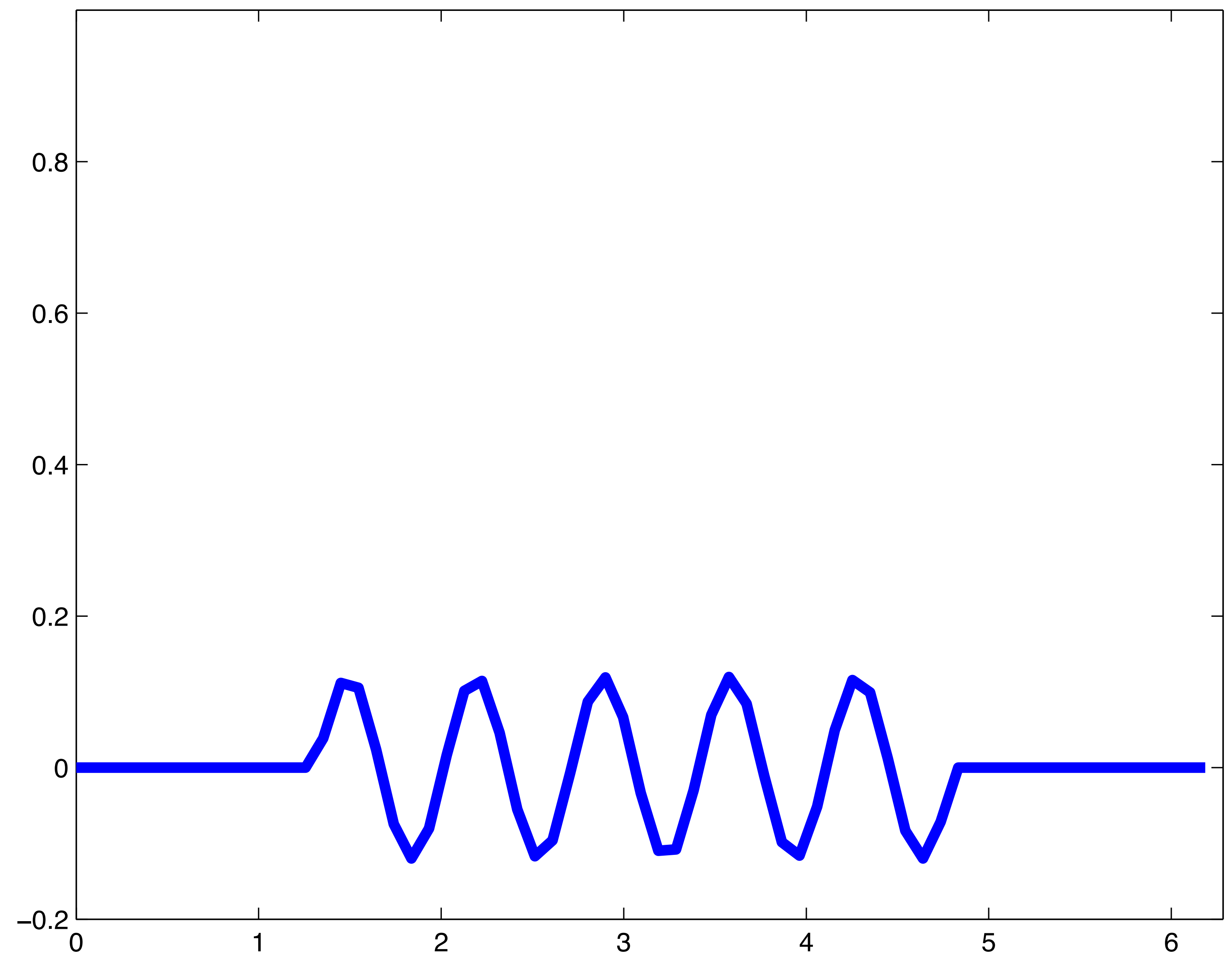

0
2π


Wave Profile η(x,t)
η(x,t)
a)
i.cond.
f.cond.
Relative energy error
log(|E_t-E_0|/E_0)
b)
Water fall of wave profile
η(x,t)
t



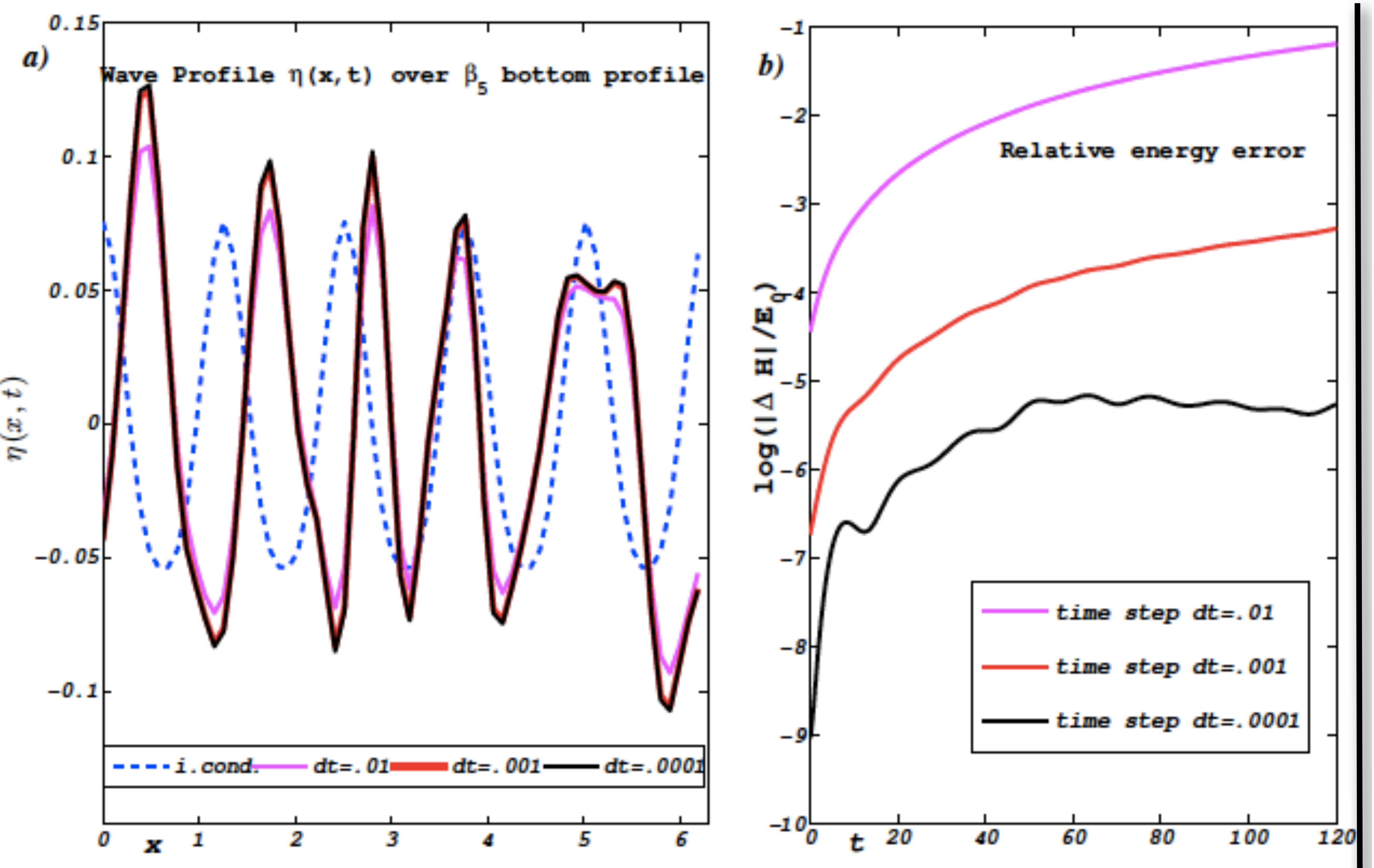


Wave Profile η(x,t)
η(x,t)
0.05
0
-0.05
0 1 2 3 4 5 6
a)
i.cond.
f.cond.
Relative energy error
log(|E_t-E_0|/E_0)
0.1
0.05
0
-0.05
-0.1
0 20 40 60 80 100 120
b)
Water fall of wave profile
η(x,t)
0.05
0
-0.05
120 100 80 60 40 20 0
t
0 1 2 3 4 5 6
c)

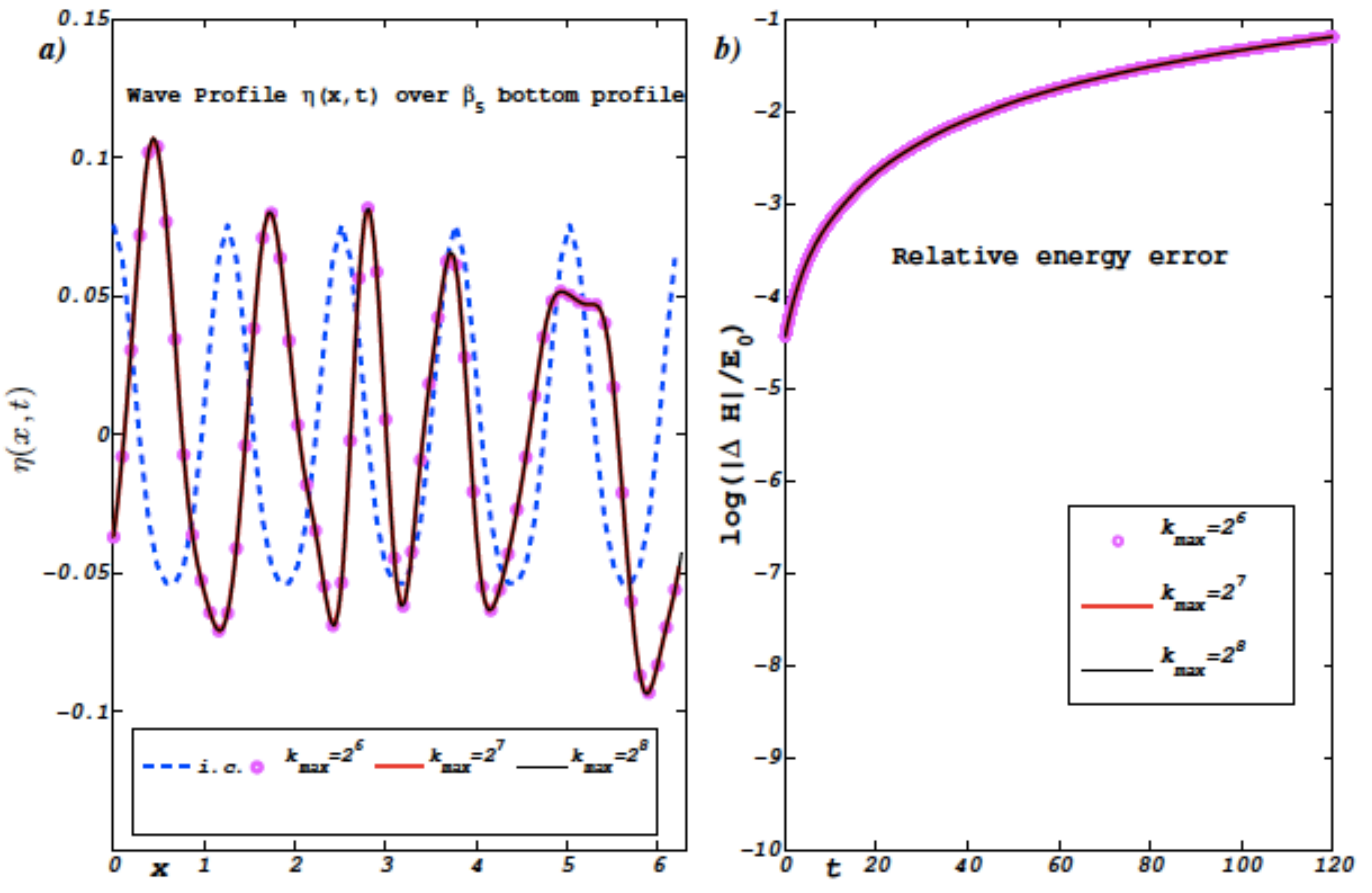


0.8
0.6
0.4
0.2
0
−0.2
0 1 2 3 4 5 6

Remarks on the accuracy of the numerical integration



Dependence of our results on the numerical time step Δ_t.



Dependence of our results on k_{Max}=M

5 Work in progress

I. Existence of Trapped Modes. Normal modes in a channel of arbitrary cross section. With Panayotaros and Minzoni

II. Comparison of two approaches to the DN operator: PDO approach and CGNS without computing the L(β) operator.

III. The question of whether the particular A_{G0} or other approximations of the Dirichlet–Neumann operator that avoid expansions in the depth variation can be evaluated with an efficiency that is comparable to that of pseudospectral methods.

IV. Looking for triad resonance considering bathymetry.

V Global bifurcation Theorem of the Stokes waves. With Garcia-Azpeitia and Panayotaros.

Work in progress I. Trapped Ursell Modes with Panayotaros and Minzoni

We are looking for solutions that correspond to an harmonic wave propagating in the x direction without any attenuation or distorsion. The second condition expresses the fact that the transverse energy of the wave is finite and in fact confined, as we shall see later,in some neighbourhood of the coast.

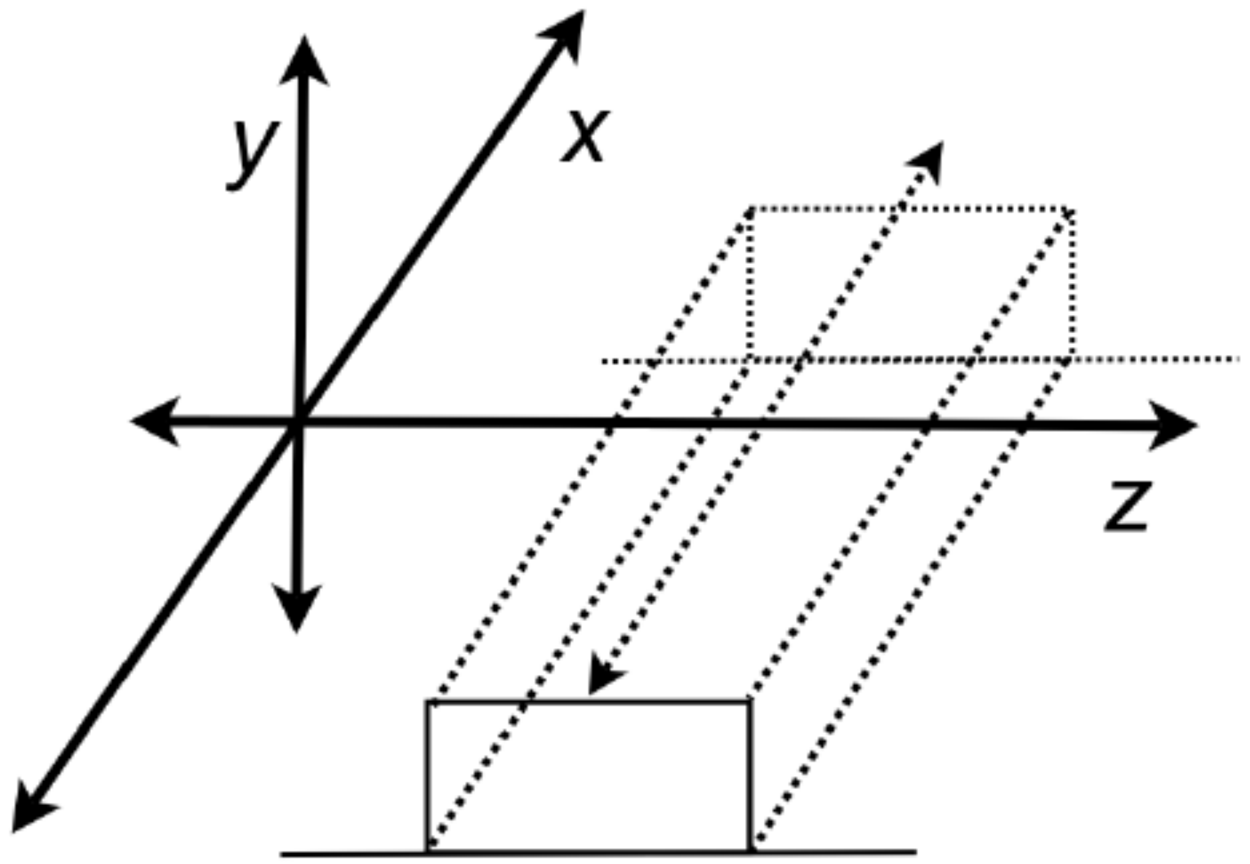


Figura 1: Stepped Ridge

$$\phi(x,z,y,t) = \mathbf{Re}(\psi(z,y)e^{i(\omega t-\beta z)}) = \psi(z,y)\cos(\omega t - \beta z),$$

$$\int_\Omega (|\ \psi(z,y)\ |^2 + |\ \nabla\psi(z,y)\ |^2)dxdy < +\infty$$

$$\begin{cases} -\Delta\psi = -\beta^2\psi \text{ en } \Omega, \\ \frac{\partial\psi}{\partial y} = \overline{f} \text{ en } \Omega\cap\Gamma_F, \\ \frac{\partial\psi}{\partial n} = 0 \text{ en } \Omega\cap\Gamma_B \end{cases}$$

$$\omega^2\psi(z,0) = \overline{f}$$

Constant Depth

$$G_{\mathcal{A}_0}(\beta)f = \left[\sqrt{\beta_0 + \mathbf{D}^2}\tanh(h_0\sqrt{\beta_0 + \mathbf{D}^2})\right] f$$

Variable Depth

$$[\sqrt{\beta_0^2 + \mathbf{D}^2}\tanh(h(X)\sqrt{\beta_0^2 + \mathbf{D}^2})]$$

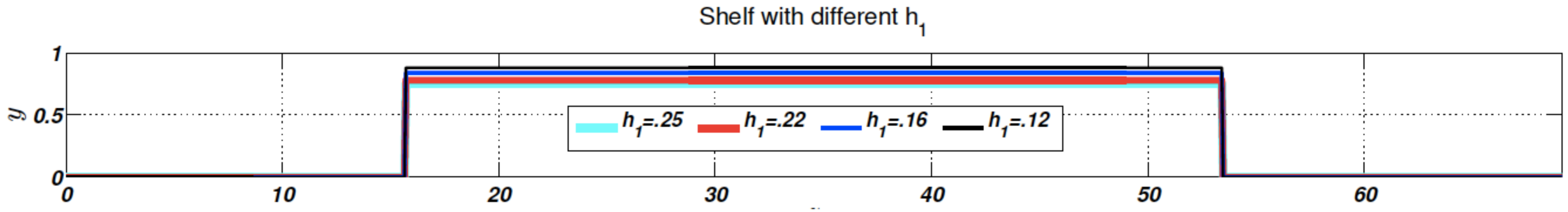


Figura 4: Flatted Shelf with five different highs

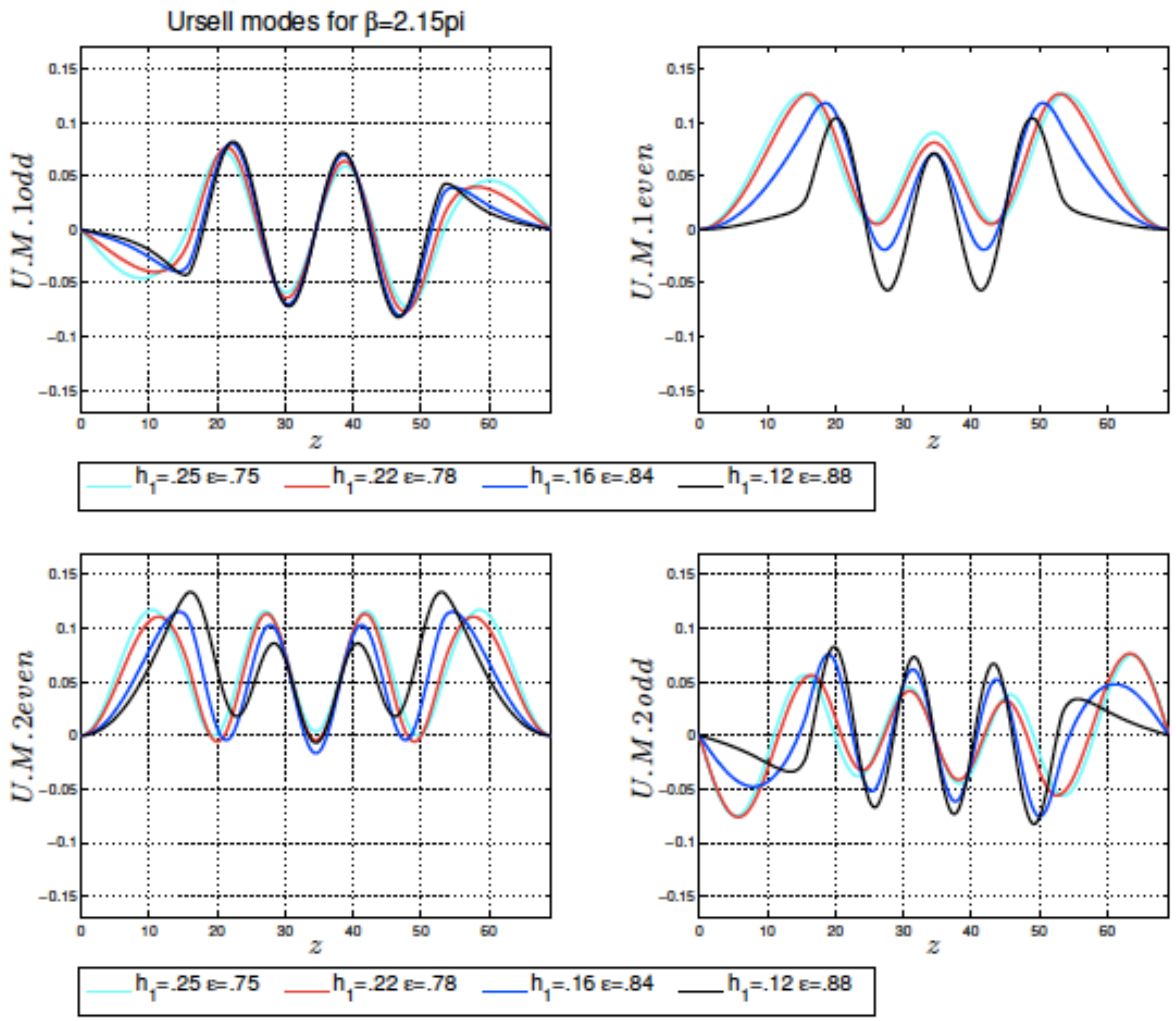


Figura 10: Even and Odd Trapped Modes for n=2 and n=3 with $\beta_0 = 2{,}15\pi$

Normal modes in a channel of arbitrary cross section.

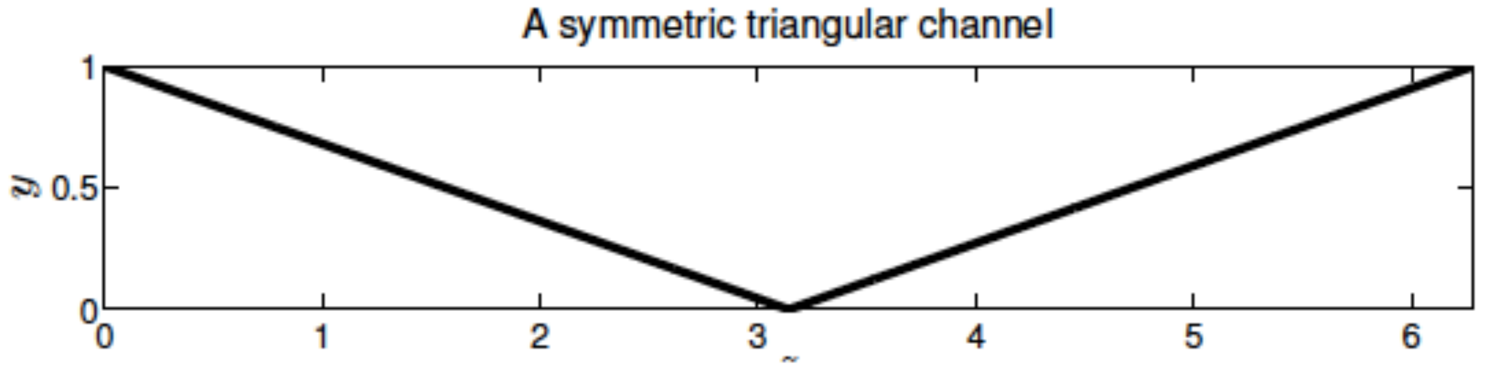
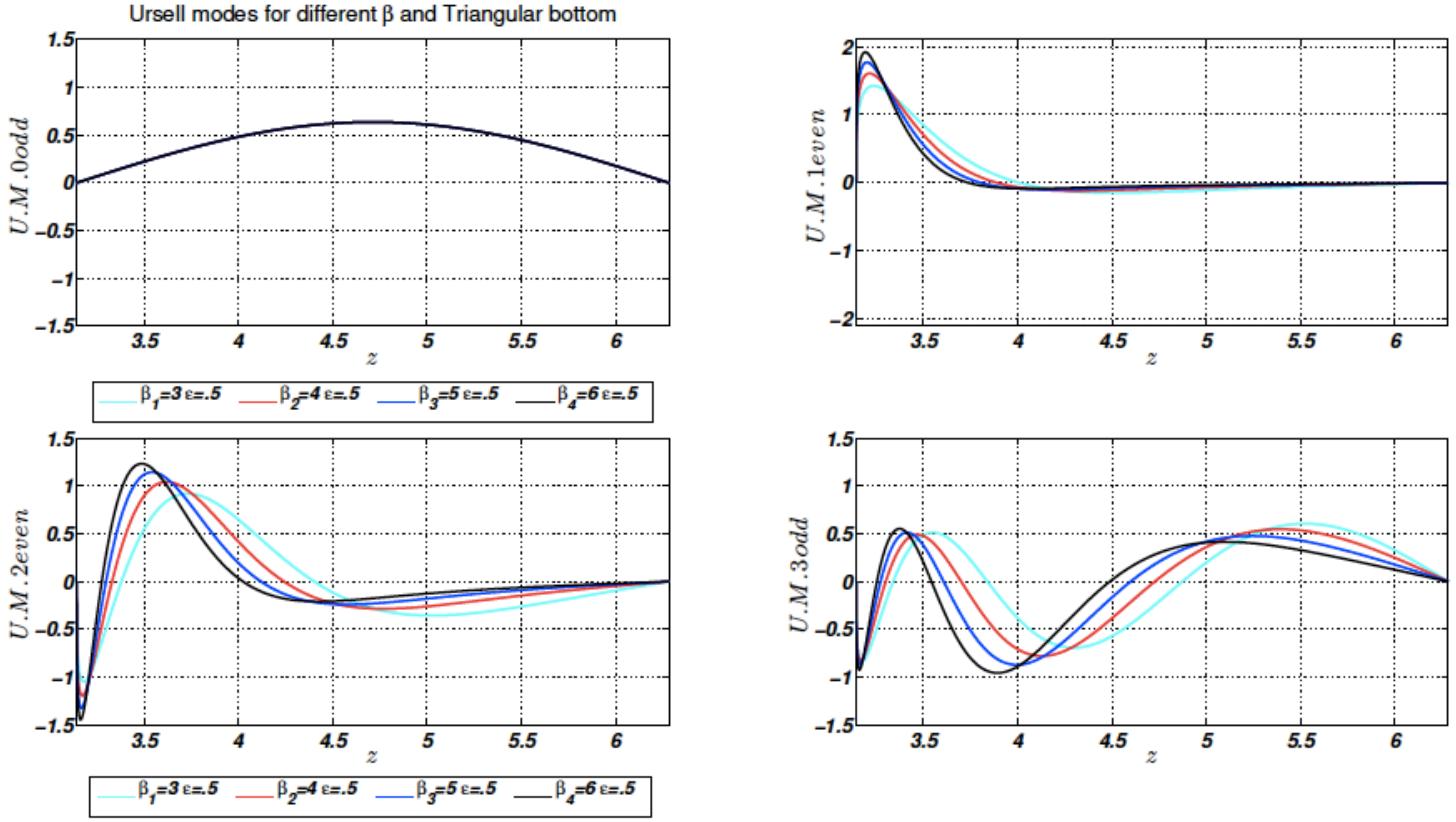


Figura 11: Symmetric triangular channel



Mark D. Groves HAMILTONIAN LONG-WAVE THEORY FOR WATER WAVES IN A CHANNEL 1994

II. Spectral comparison of the *two operators approaches* to the DN operator 1. By Craig,Guyenne Nicholls Sulem and 2.The PDO approach.

We would like to compare the spectrum of:

1. The operator derived from the expansion of the DN operator by Craig, Guyenne, Nicholls and Sulem:

$$\mathcal{O}p_{CS} = \frac{1}{\sqrt{\delta}}(D\tanh(\sqrt{\delta}D)\xi - \frac{1}{\sqrt{\delta}}D(B_\delta[\gamma\beta]A_\delta[\gamma\beta])\xi$$

$$A(\beta)\xi = \int_{\mathbb{R}} e^{ikx}\sinh(\beta(x)k)\operatorname{sech}(hk)\hat{\xi}(k)dk,$$

$$B(\beta) = C(\beta)^{-1}.$$

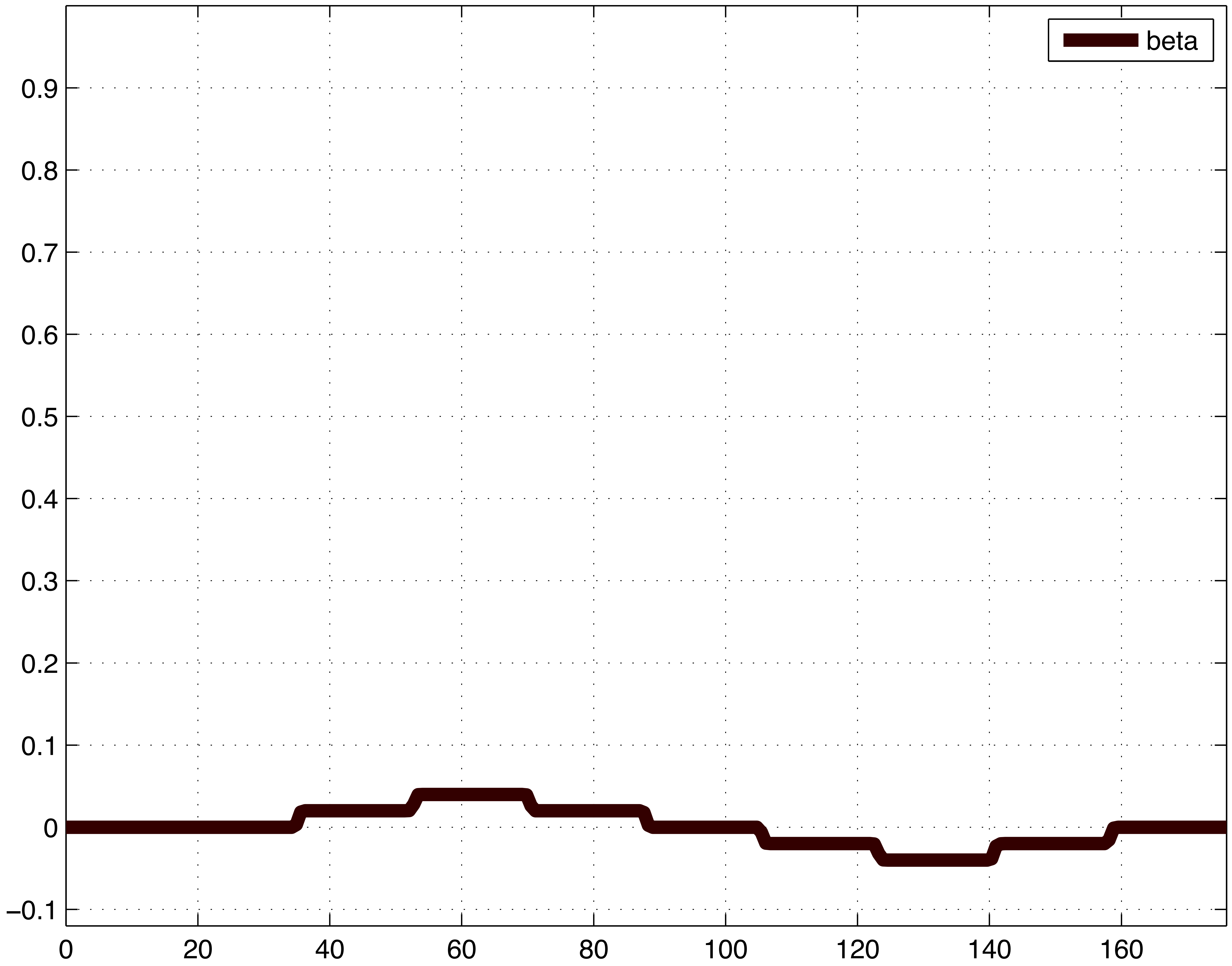
$$C(\beta)\xi = \int_{\mathbb{R}} e^{ikx}\cosh((-h_0+\beta(x))k)\hat{\xi}(k)dk,$$

2. The operator involving the PDO approach.

$$\mathcal{A}_{G_0} = \frac{1}{\sqrt{\delta}}D\tanh(\sqrt{\delta}h(x)D) = \frac{1}{\sqrt{\delta}}D\tanh(\sqrt{\delta}(1-\beta(x))D)$$

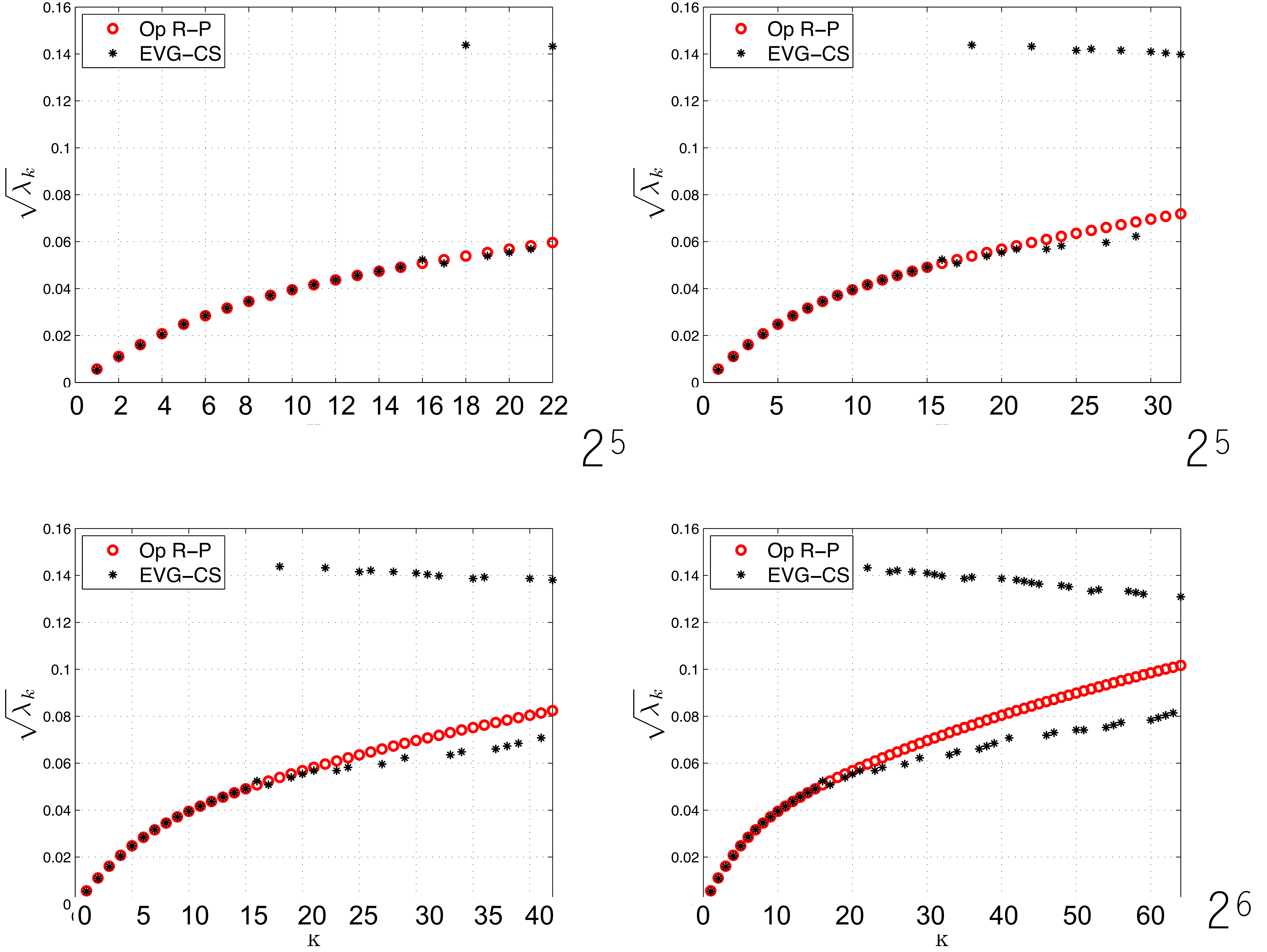
By solving the generalized eigenvalue problem presented below

$$\begin{aligned}
\frac{1}{\sqrt{\delta}}((D\tanh(\sqrt{\delta}D) - D(B_\delta[\gamma\beta]A_\delta[\gamma\beta]))\xi &= \lambda\xi \\
((D\tanh(\sqrt{\delta}D) - D(B_\delta[\gamma\beta]A_\delta[\gamma\beta]))\xi &= \sqrt{\delta}\lambda\xi \\
((\tanh(\sqrt{\delta}D) - (B_\delta[\gamma\beta]A_\delta[\gamma\beta]))\xi &= \sqrt{\delta}\lambda D^{-1}\xi \\
((C\tanh(\sqrt{\delta}D) - \beta]A_\delta[\gamma\beta]))\xi &= \sqrt{\delta}\lambda C D^{-1}\xi
\end{aligned}$$

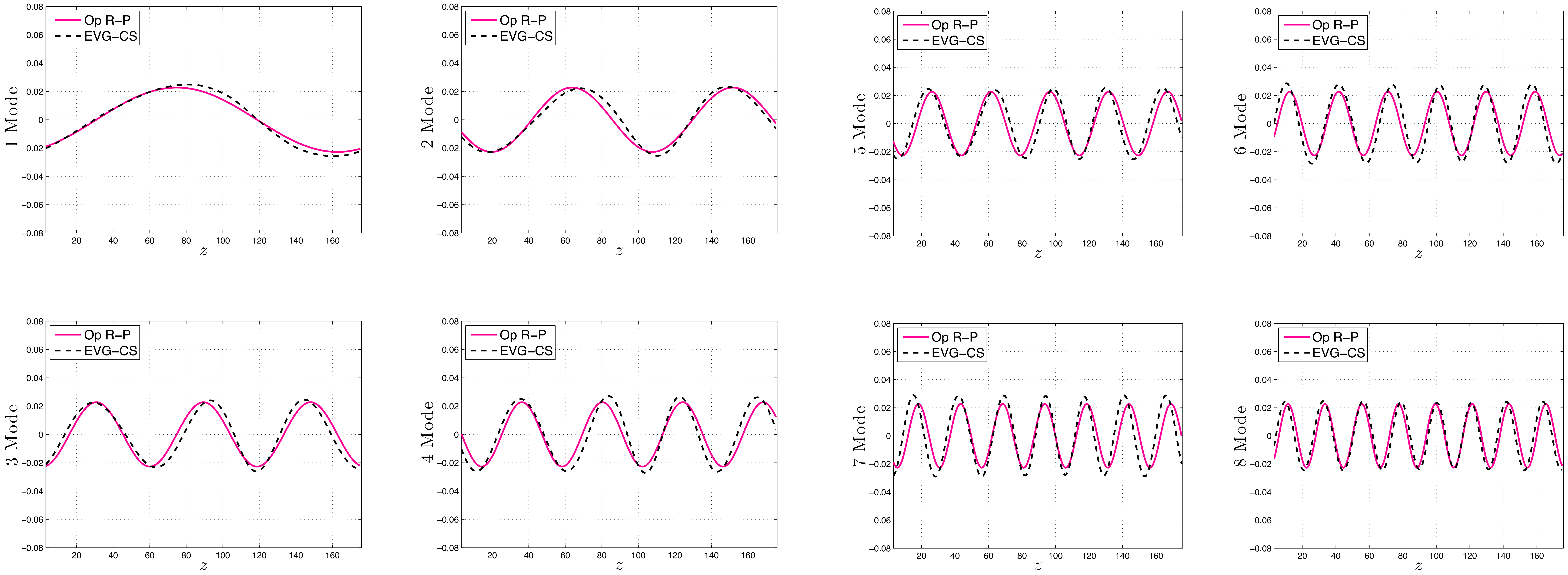


beta
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
0
20
40
60
80
100
120
140
160

Generalized Dispersive Curve. PDO operator vs CGNS approach.



Eigenmodes PDO operator vs CGNS approach.



III. Global Bifurcation in Stokes waves for the Whitham-Boussinesq equations with Garcia- Aspeitia , Panayotaros

The existence of travelling waves can be setting in a problem where the global Rabinowitz alternative can be applied. Garcia-Azpeitia using an appropiate operator for the Whitham-Boussinesq equations, proves that this operator has a global bifurcation in Stokes waves in an appropiate space, he use the fact that this operator is O(2) equivariant.

Thank you!