Nonuniqueness of weak solutions to the SQG equation

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The SQG equation

- Inviscid Surface Quasi-Geostrophic (SQG) equation introduced in Constantin-Majda-Tabak (1994)

$$\partial_t \theta + u \cdot \nabla \theta = 0 \,,$$
$$u = \mathcal{R}^\perp \theta := \nabla^\perp \Lambda^{-1} \theta \,,$$

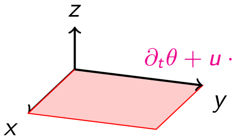
- $\theta = \theta(x,t)$, where $(x,t) \in \mathbb{T}^2 \times \mathbb{R} = [-\pi,\pi]^2 \times \mathbb{R}$.
- $\Lambda = (-\Delta)^{1/2}$, $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ is the vector of Riesz-transforms
- $\nabla^\perp = (-\partial_2, \partial_1)$, and $x^\perp = (-x_2, x_1)$ for any vector $x = (x_1, x_2)$.
- example of "active scalar equation"

Geophysics

- Derivation in Held, Pierrehumbert, Garner, Swanson (1995)
- θ is temperature (or surface buoyancy)
- Model for rapidly rotating, stratified fluids
- Uniform potential vorticity
- Applications in meteorology and oceans

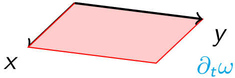
SQG and 2-D Euler

- Stream function ψ
- Velocity $u = \nabla^{\perp}\psi$
  - SQG: $-\Delta\psi = 0$ in $\{z > 0\}$ and $\theta = \frac{\partial\psi}{\partial z} = \Lambda\psi$, $\theta = \Lambda^{-1}\psi$



$$\partial_t\theta + u\cdot\nabla\theta = 0 \text{ in } \{z = 0\}, \quad u = \nabla^{\perp}\Lambda^{-1}\theta$$

  - 2-D Euler: $-\Delta\psi = \omega$ in $\mathbb{R}^2$, so $\omega = (-\Delta)^{-1}\psi$



$$\partial_t\omega + u\cdot\nabla\omega = 0 \text{ in } \{z = 0\}, \quad u = \nabla^{\perp}(-\Delta)^{-1}\omega$$

SQG and 3-D Euler

- 3-D Euler: Let $\omega = \operatorname{curl} \boldsymbol{u}$
  - $\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = \nabla \boldsymbol{u} \cdot \omega$
- 2-D SQG: Let $W = \nabla^{\perp} \theta$
  - $\partial_t W + \boldsymbol{u} \cdot \nabla W = \nabla \boldsymbol{u} \cdot W$
- SQG has a very similar behavior as 3-D Euler
- Global existence of smooth solutions?
- Finite-time blow-up?

A few prior results

- Local existence of smooth solutions, θ_0 in H^s, $s > 2$, or $C^{1,\alpha}$, $\alpha > 0$, Constantin-Majda-Tabak (1994)
- Numerical simulations indicated a collapsing hyperbolic saddle blow-up scenario for θ-contours
- Cordoba (1998) and Cordoba-Fefferman (2002) ruled this out
- Constantin-Lai-Sharma-Tseng-Wu (2012) resolved numerical simulation well past predicted blow-up time

SQG Hamiltonian and Conservation Laws

- $\dot{H}^{-1/2}$ Hamiltonian:
  - Compute L^2 inner product of SQG equation with $\Lambda^{-1}\theta$
  - integrate by parts in nonlinear term and use that $u \cdot \nabla\Lambda^{-1}\theta = \nabla^{\perp}\Lambda^{-1}\theta \cdot \nabla\Lambda^{-1}\theta = 0$
  - Then,

$$\mathcal{H}(t) := \|\theta(\cdot, t)\|^2_{\dot{H}^{-1/2}(\mathbb{T}^2)} = \|\theta_0\|^2_{\dot{H}^{-1/2}(\mathbb{T}^2)}$$

  - Isett-Vicol (2015) showed $\theta \in L^3_{t,x}(\mathbb{T}^2 \times \mathbb{R})$ implies $\mathcal{H}(t)$ conserved
- Casamir functions:
  - Since θ is transported by the incompressible vector field u, then

$$\|\theta(\cdot, t)\|_{L^p(\mathbb{T}^2)} = \|\theta_0\|_{L^p(\mathbb{T}^2)}\,, \quad 1 \leq p \leq \infty$$

Weak solutions to inviscid SQG

- L^2 weak solution
  - **Definition 1.** $\theta \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{T}^2))$ is weak solution if

$$\int\int_{\mathbb{R}\times\mathbb{T}^2} (\theta\partial_t\phi + \theta u\cdot\nabla\phi)\, dx\, dt = 0 \quad \forall\phi \in C^\infty(\mathbb{T}^2\times\mathbb{R})$$

- Resnick (1995) proved existence of global L^2 weak solutions
- $\dot{H}^{-1/2}$ distributional solution
  - **Definition 2.** $\theta \in L^2_{\mathrm{loc}}(\mathbb{R}; \dot{H}^{-1/2}(\mathbb{T}^2))$ is weak solution if

$$\int_{\mathbb{R}} \langle \mathcal{R}_i^\perp\theta, \partial_t\Lambda^{-1}\phi^i\rangle + \langle \mathcal{R}_j^\perp\theta, \mathcal{R}_i^\perp\Lambda^{-1}\theta\partial_j\phi^i\rangle - \frac{1}{2}\langle \mathcal{R}_i\mathcal{R}_j^\perp\theta, [\Lambda, \phi^i]\mathcal{R}_j^\perp\Lambda^{-1}\theta\rangle\, dt = 0$$

  for any $\phi \in C_0^\infty(\mathbb{T}^2\times\mathbb{R})$ such that $\operatorname{div}\phi = 0$, where $\langle\cdot,\cdot\rangle$ denotes the $\dot{H}^{-1/2}$-$\dot{H}^{1/2}$ duality pairing.
- Marchand (2008) proved existence of global L^p weak solutions, $p > 4/3$
- $\theta \in \dot{H}^{-1/2}$ remains open

Nonuniqueness of weak solutions to inviscid SQG

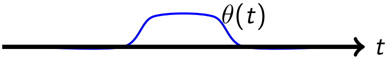
- Uniqueness remained open and was Challenge Problem 11 in De Lellis-Székelyhidi (2002), Bulletin AMS

Theorem 1

Suppose $\mathfrak{h} : \mathbb{R} \to \mathbb{R}^+$ *is a smooth function with compact support. Then for every* $1/2 < \beta < 4/5$ *and* $\sigma < \beta/(2-\beta)$*, there exist weak solution* θ*, with* $\Lambda^{-1}\theta \in C_t^\sigma C_x^\beta$*, satisfying*

$$\mathcal{H}(t) = \int_{\mathbb{T}^2} \left| \Lambda^{-1/2}\theta(x,t) \right|^2 \, dx = \mathfrak{h}(t) \quad \forall t \in \mathbb{R}.$$

- $\theta \equiv 0$ not the only weak solution



Dissipative SQG equation

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda^\gamma = 0\,, \qquad \gamma > 0$$
$$u = \mathcal{R}^\perp\theta := \nabla^\perp\Lambda^{-1}\theta\,,$$

- The distribution $\theta \in L^2_{\rm loc}(\mathbb{R}; \dot H^{-1/2}(\mathbb{T}^2))$ is a weak solution of the dissipative SQG equation if

$$\int_{\mathbb{R}} \langle \mathcal{R}_i^\perp\theta, \partial_t\Lambda^{-1}\phi^i\rangle + \langle \mathcal{R}_j^\perp\theta, \mathcal{R}_i^\perp\Lambda^{-1}\theta\partial_j\phi^i\rangle$$
$$- \frac{1}{2}\langle \mathcal{R}_i\mathcal{R}_j^\perp\theta, [\Lambda, \phi^i]\mathcal{R}_j^\perp\Lambda^{-1}\theta\rangle - \langle \mathcal{R}_i^\perp\theta, \Lambda^{\gamma-1}\phi^i\rangle\, dt = 0$$

for any $\phi \in C_0^\infty(\mathbb{T}^2\times\mathbb{R})$ such that $\operatorname{div}\phi = 0$, where $\langle\cdot,\cdot\rangle$ denotes the $\dot H^{-1/2}$-$\dot H^{1/2}$ duality pairing.

Natural scaling symmetry

- $\theta_\lambda(x,t) = \lambda^{\gamma-1}\theta(\lambda x, \lambda^\gamma t)$ is a $[-\frac{\pi}{\lambda}, \frac{\pi}{\lambda}]^2$-periodic solution with initial datum $\theta_{0,\lambda}(x) = \lambda^{\gamma-1}\theta_0(\lambda x)$
- L^∞_x-norm scale invariant for $\gamma = 1$
- $\gamma > 1$ subcritical and semilinear – global regularity Constantin-Wu (1999)
- $\gamma = 1$ critical and quasilinear – global regularity Kiselev-Nazarov-Volberg (2007), Cafarelli-Vasseur (2010), Constantin-Vicol (2012)
- $\gamma < 1$ supercritical, open
- $\gamma > 0$, weak solutions for $\theta_0 \in \dot{H}^{-1/2}$, Marchand (2008)

Nonuniqueness of weak solutions to dissipative SQG

Theorem 2

Suppose $\mathfrak{h} : \mathbb{R} \to \mathbb{R}^+$ *is a smooth function with compact support. Then for every* $1/2 < \beta < 4/5$, $0 < \gamma < 2 - \beta$ *and* $\sigma < \beta/(2-\beta)$*, there exists a weak solution* θ*, with* $\Lambda^{-1}\theta \in C_t^\sigma C_x^\beta$*, satisfying*

$$\int_{\mathbb{T}^2} \left|\Lambda^{-1/2}\theta(x,t)\right|^2 dx = \mathfrak{h}(t) \quad \forall t \in \mathbb{R}\,.$$

- The restriction $\gamma + \beta < 2$ is sharp, in the sense that the $C_t^0 C_x^\beta$ norm for $\Lambda^{-1}\theta$ is scale invariant when $\gamma + \beta = 2$.
- For supercritical scaling, parabolic smoothing does not hold.
- First instance of convex integration for subcritical semilinear equation $1 < \gamma \le 6/5$

Inviscid Hydrodynamical Systems via Arnold (1966)

- Inviscid hydrodynamical systems are geodesics with respect to right invariant metrics on

$$\mathcal{D}_\mu = \text{ group of volume-preserving diffeomorphisms}$$

with

$$\text{metric } = \int_{\mathbb{T}^2} Au \cdot v dx\,, \quad A > 0 \text{ linear, self-adjoint}$$

- Apply Euler-Poincaré variational principle to find the system of PDE:

$$\begin{aligned} \partial_t v^i + \partial_j v^i u^j + \partial_i u^j v^j &= -\nabla \widetilde{p} \\ \operatorname{div} u &= 0 \\ v &= Au \end{aligned}$$

Reformulation of SQG

- SQG: $A = \Lambda^{-1}$
- metric $= \int_{\mathbb{T}^2} \Lambda^{-1} u \cdot v dx$
- $v = \Lambda^{-1} u =$ potential velocity, $\quad u =$ transport velocity

$$\begin{aligned} \partial_t v^i + \partial_j v^i u^j - \partial_i v^j u^j &= -\partial_i p \\ \operatorname{div} u &= 0 \\ u &= \Lambda v \end{aligned}$$

- $\theta = -\nabla^\perp v \Longrightarrow \partial_t \theta + u \cdot \nabla \theta = 0$
- Very nice form for convex integration – no odd (in frequency) multiplier

Onsager conjecture for SQG

Conjecture 1

(a) *If $v \in C(\mathbb{R}; C^\alpha(\mathbb{T}^2))$ is a weak solution of the SQG equation with $\alpha > 1$, then the Hamilitonian is conserved. (Isett-Vicol (2015) using θ formulation)*

(b) *For any $1/2 < \alpha < 1$, there exist infinitely many weak solutions of the SQG equation, with $v \in C(\mathbb{R}; C^\alpha(\mathbb{T}^2))$, such that the Hamiltonian is not conserved.*

- Euler Onsager with Hölder exponent $1/3$
- For Euler, part a) was proven by Constantin-E-Titi (1994) (cf. Eyink 1994, Cheskidov-Constantin-Friedlander- Shvydkoy 2008)
- Part b) was recently resolved: $C^0_{x,t}$ De Lellis-Székelyhidi Jr. (2012); $C^{1/10-}_{x,t}$ DeSz (2012); $C^{1/5-}_{x,t}$ Isett (2013); $C^{1/5-}_{x,t}$ BuDeSz (2013); $C^{1/3-}_{x}$ a.e. in time; Bu (2015); $L^1_t C^{1/3-}_x$ BuDeSz (2016); 3-D $C^{1/3-}_{x,t}$ Isett (2016)
- SQG regularity gap $\alpha \in [4/5, 1)$ is similar to 2-D Euler gap $\alpha \in [1/5, 1/3)$

Regularity exponents

- Consider a scale of Banach spaces $B^\alpha = C_t C_x^\alpha$

$$\alpha_{WP} = \text{ well-posedness} \qquad \alpha_* = \text{ scale invariant},$$
$$\alpha_U = \text{ uniqueness} \qquad \alpha_O = \text{ Onsager scale},$$
$$\alpha_N = \text{ Nash scale} \; .$$

- Expectation of ordering: $\alpha_* \leq \alpha_O \leq \alpha_N \leq \alpha_U \leq \alpha_{WP}$ (Klainerman (2016))
- Euler: $\alpha_{WP} = 1$, $\alpha_O = 1/3$, $\alpha_* = 0$, conjectured that $\alpha_U = 1$
- Inviscid SQG (for v): $\alpha_O = 1$, $\alpha_{WP} = 2$
- Critical SQG ($\gamma = 1$): Conjecture that $1 = \alpha_* = \alpha_O = \alpha_N = \alpha_U = \alpha_{WP}$
  - First fluids equation where α are the same!

SQG Momentum Equation

- Stong form: $\partial_t v^i + \partial_j v^i \Lambda v^j - \partial_i v^j \Lambda v^j + \Lambda^\gamma v^i = -\partial_i p$ with div $v = 0$
- Weak form: $v \in L^2_{t,loc} \dot{H}^{1/2}$ is a weak solution of SQG if

$$\int_{\mathbb{R}} \left\{ \langle v^i, \partial_t \phi^i \rangle + \langle \Lambda v^j, v^i \partial_j \phi^i \rangle + \frac{1}{2} \langle \partial_j v^i, [\Lambda, \phi^i] v^j \rangle + \int_{\mathbb{T}^2} v^i \Lambda^\gamma \phi^i dx \right\} dt = 0$$

  for all $\phi \in C_0^\infty(\mathbb{T}^2 \times \mathbb{R})$ such that div $\phi = 0$
- $L^\infty_t \dot{H}^{1/2}_x$ global weak solutions for v, Marchand (2008)

Convex integration scheme

- We construct a sequence of solutions $(v_q, p_q, \mathring{R}_q)$ to the *relaxed SQG momentum equation*

$$
\begin{aligned}
\partial_t v_q + u_q \cdot \nabla v_q - (\nabla v_q)^T \cdot u_q + \nabla p_q + \Lambda^\gamma v_q &= \operatorname{div} \mathring{R}_q \\
\operatorname{div} v_q &= 0 \\
u_q &= \Lambda v_q
\end{aligned}
$$

- $\mathring{R}_q$ is a symmetric trace-free 2×2 matrix (Reynolds stress)
- The goal is to obtain $\mathring{R}_q \to 0$ as $q \to \infty$ (in a suitable topology), and show that a limiting function $v_q \to v$ exists, and solves SQG.

Iteration Scheme

- Iteration: potential velocity: $v_{q+1} = v_q + w_{q+1}$
- Each v_q is localized at frequency λ_q
- Given $\lambda_0 \gg 1$, $\lambda_q = \lambda_0^q$
- Each v_q has amplitude $\lambda_0^{1-q\beta}$
- We fix $\beta = \frac{4}{5} - \epsilon$ to be the Hölder exponent that we expect for our weak solution
- Perturbation w_{q+1} lives at frequency λ_0^{q+1} and must be chosen to cancel low frequency Reynolds stress R_q error, and $|w_{q+1}| \sim \lambda_0^{1-\beta(q+1)}$

Decomposition of Reynolds Stress error

Setting $w_{q+1} := v_{q+1} - v_q$ we have

$$
\begin{aligned}
\operatorname{div} \mathring{R}_{q+1} = & \left(\partial_t w_{q+1} + u_q \cdot \nabla w_{q+1}\right) \\
& + \left(\Lambda w_{q+1} \cdot \nabla v_q - (\nabla v_q)^T \cdot \Lambda w_{q+1} - (\nabla u_q)^T \cdot w_{q+1}\right) \\
& + \Lambda^\gamma w_{q+1} \\
& + \left(\operatorname{div} \mathring{R}_q + \Lambda w_{q+1} \cdot \nabla w_{q+1} - (\nabla w_{q+1})^T \cdot \Lambda w_{q+1}\right) \\
& + \nabla \widetilde{p}_{q+1} \\
=: & \operatorname{div} R_T + \operatorname{div} R_N + \operatorname{div} R_D + \operatorname{div} R_O + \nabla \widetilde{p}_{q+1}
\end{aligned}
$$

Heuristic estimates for one term of Nash error

- Assume w_{q+1} is of frequency $\sim \lambda_0^{q+1}$ and $|w_{q+1}| \sim \lambda_0^{-\beta(q+1)}$ for $\frac{1}{2} < \beta < 1$, so that $v_q \to v \in C^\beta$.
- Then

$$\begin{aligned}\left\| \mathrm{div}^{-1} \left((\nabla u_q)^T \cdot w_{q+1} \right) \right\|_{C^0} &\lesssim \lambda_0^{-(q+1)} \|u_q\|_{C^1} \|w_{q+1}\|_{C^0} \\ &\lesssim \lambda_0^{-(q+1)} \|v_q\|_{C^2} \|w_{q+1}\|_{C^0} \\ &\lesssim \lambda_0^{-(q+1)} \lambda_0^{q(2-\beta)} \lambda_0^{-(q+1)\beta} = \lambda_0^{(q+2)(1-2\beta)} \lambda_0^{3(\beta-1)}\end{aligned}$$

- The allowable error must be proportional to $|\Lambda w_{q+1} \otimes w_{q+1}| \sim \lambda_0^{(q+2)(1-2\beta)}$. Thus we obtain the restriction $\beta < 1$.

The construction of w_{q+1}

- The perturbation will be a sum of approximate Beltrami plane waves (similar to the convex integration scheme for 2D Euler: Choffrut-De Lellis-Székelyhidi (2012), Choffrut (2012)):

$$w_{q+1} \approx \sum_k a_k(x,t) b_k(\lambda^{q+1} x)$$

- finite number of $k = (k_1, k_1)$ in $\mathbb{S}^1$
  - $k := (1,0)\,, (3/5, 4/5)\,, (3/5, -4/5)$, etc.
- Beltrami plane waves $b_k(\xi) := ik^\perp e^{ik\cdot\xi}$ are eigenfunctions of Λ

The oscillation error

Recall

$$\operatorname{div} R_O = \operatorname{div} \mathring{R}_q + \Lambda w_{q+1} \cdot \nabla w_{q+1} - (\nabla w_{q+1})^T \cdot \Lambda w_{q+1}$$

Setting $w_k := a_k(x,t) b_k(\lambda^{q+1} x)$ so that $w_{q+1} \approx \sum_k w_k$.

$$\operatorname{div} R_O = \underbrace{\operatorname{div} \mathring{R}_q + \sum_k \left(\Lambda w_k \cdot \nabla w_{-k} - (\nabla w_k)^T \cdot \Lambda w_{-k} \right)}_{\operatorname{div} R_{O,\text{low}}}$$

$$+ \underbrace{\sum_{k+k' \neq 0} \left(\Lambda w_k \cdot \nabla w_{k'} - (\nabla w_k)^T \cdot \Lambda w_{k'} \right)}_{\operatorname{div} R_{O,\text{high}}}$$

It is not difficult to show that $\operatorname{div} R_{O,\text{high}}$ is a gradient of pressure plus a small error.

The low frequency oscillation error

We wish to write

$$\operatorname{div} \mathring{R}_q + \sum_k \left(\Lambda w_k \cdot \nabla w_{-k} - (\nabla w_k)^T \cdot \Lambda w_{-k}\right) = \operatorname{div} \left(\mathring{R}_q + \sum_k \mathcal{Q}_k\right) + \nabla \mathcal{P}_k$$

(Easy for the Euler equations, but complicated for SQG.) Setting $\vartheta_{j,k} = \nabla^\perp \cdot w_k$. With quite a bit of work, one can show

$$Q_k^{m\ell} = \mathcal{S}^m(\Lambda^{-1}\vartheta_k, \mathcal{R}^l \vartheta_{-k})$$

for some bilinear integral operator $\mathcal{S}$. Using the structure of w_k, expanding in frequency

$$Q_k^{m\ell} = \frac{\lambda^{q+1}}{2} |a_k|^2 k^\perp \otimes k^\perp + \text{ error}\,.$$

(If a_k was a constant then there would be no error.)

Transport Error

Recall

$$\operatorname{div} R_T = \partial_t w_{q+1} + u_q \cdot \nabla w_{q+1}$$

To ensure the transport error is small, we replace our previous ansatz

$$w_{q+1} \approx \sum_k a_k(x,t) b_k(\lambda^{q+1} x)$$

with

$$w_{q+1} \approx \sum_{j,k} a_{j,k}(x,t) b_k(\lambda^{q+1} \Phi_j(x,t))$$

where $a_{j,k}(x,t)$ is zero outside the range $(j-1)\tau_{q+1} < t < (j+2)\tau_{q+1}$ and

$$\begin{cases} \partial_t \Phi_j + u \cdot \nabla \Phi_j = 0 \\ \Phi_j(x, j\tau_{q+1}) = x \,. \end{cases}$$