Relationships between pressure, bathymetry, and wave-height

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- The AMS-Simons Research Travel Grants
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Warning System

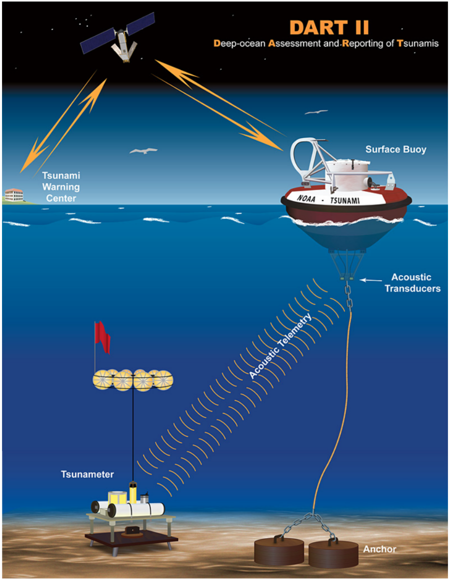
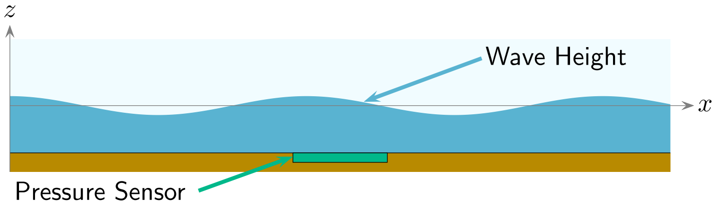


Figure: Source: `http://www.srh.noaa.gov/jetstream/tsunami/dart.htm`

Overview of the Problem



Given the pressure at the bottom of a fluid, can we reconstruct the surface elevation?

Objective - Measure the Wave

The critical first step is measuring the wave.

Given p (pressure at the bottom), find the height of the water.

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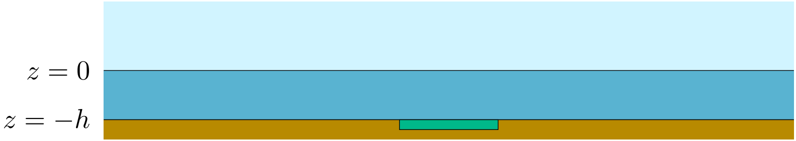

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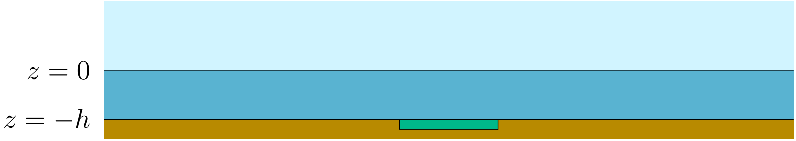

In physics, we learned

$$p = \rho g h$$

where ρ is density and g is acceleration due to gravity.

This is called the hydrostatic pressure.

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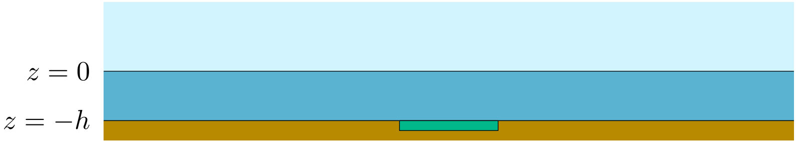

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This is called the hydrostatic pressure. Given p, we could solve for h to find

$$h = \frac{p}{\rho g}.$$

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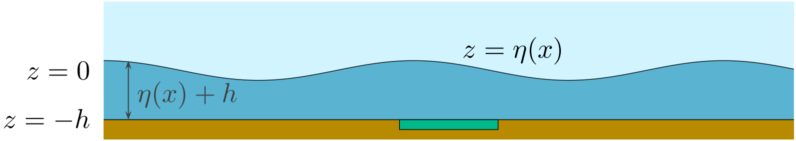
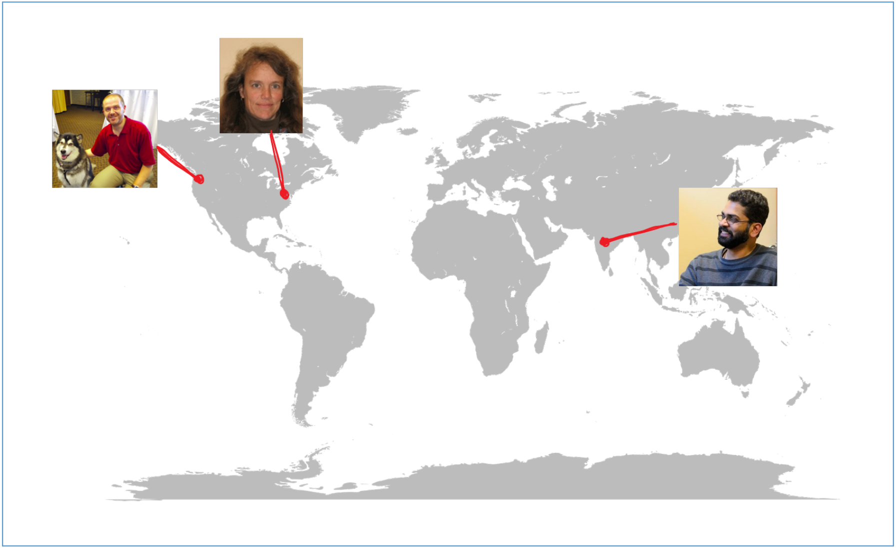


Is the hydrostatic approximation $p = \rho g(\eta + h)$ still a valid model?

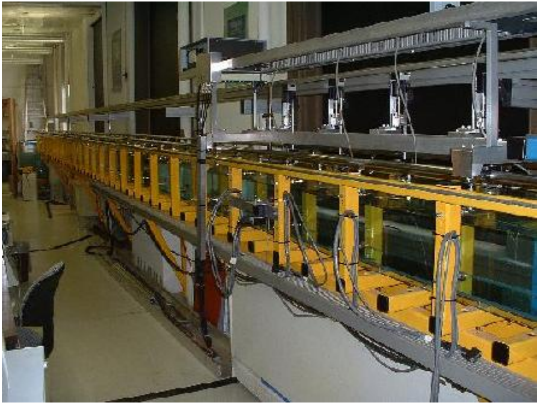
Now, assume we know h as well. The only unknown is η. If this is a good model, then

$$\eta = \frac{p}{\rho g} - h$$

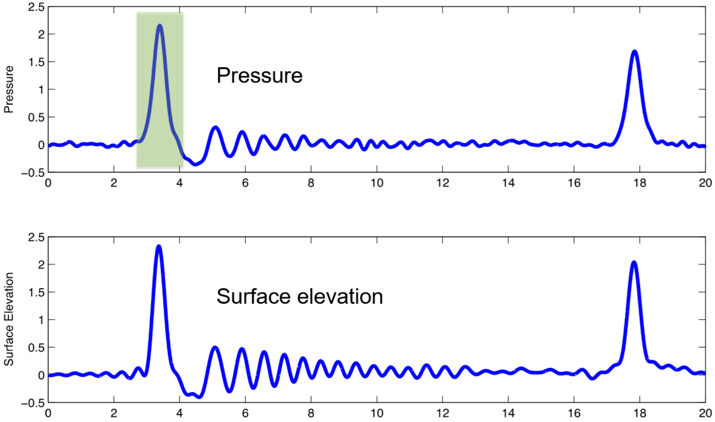
Collaborators



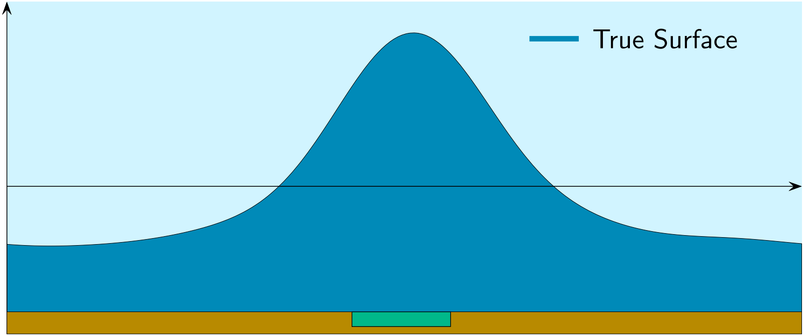
Experimental Set-Up



Experimental Data



Testing the Hydrostatic Approximation



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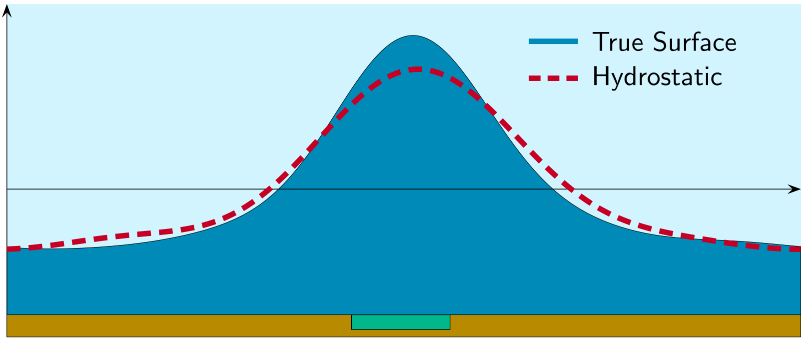

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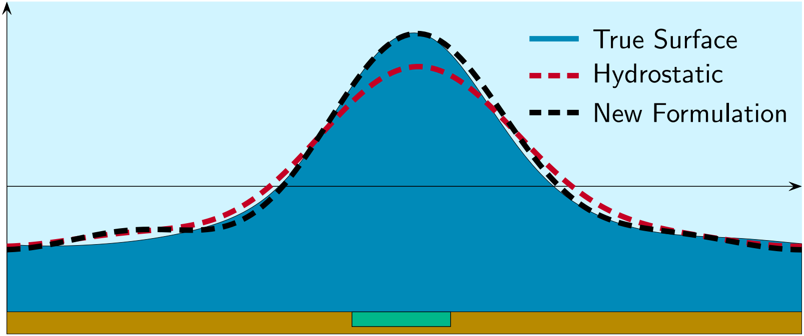
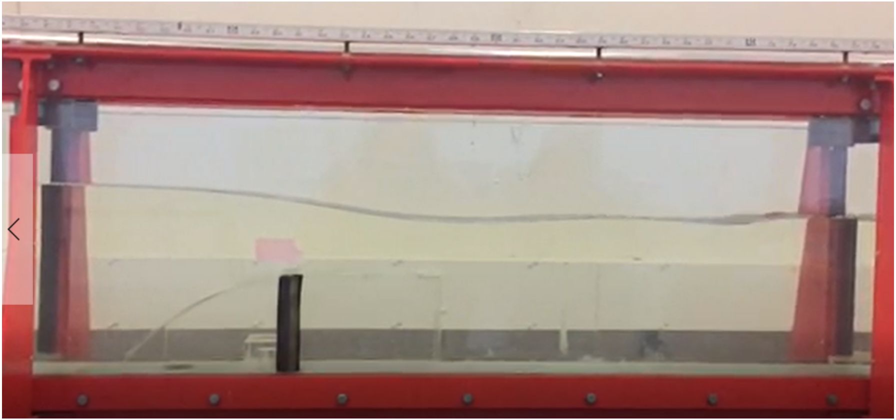
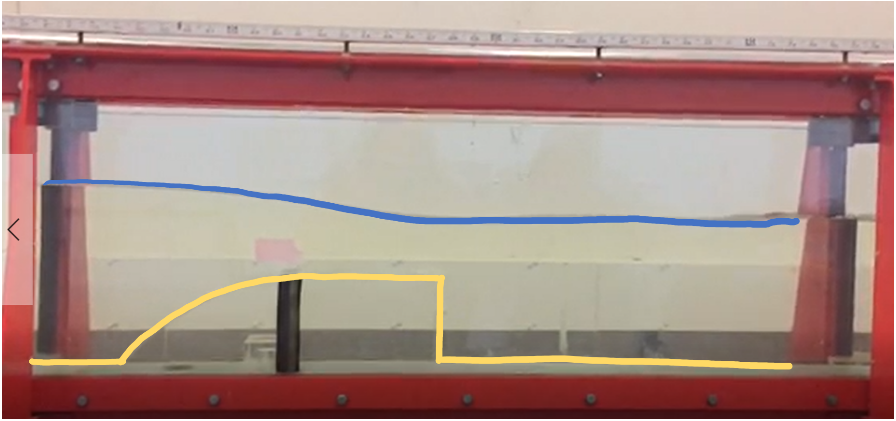


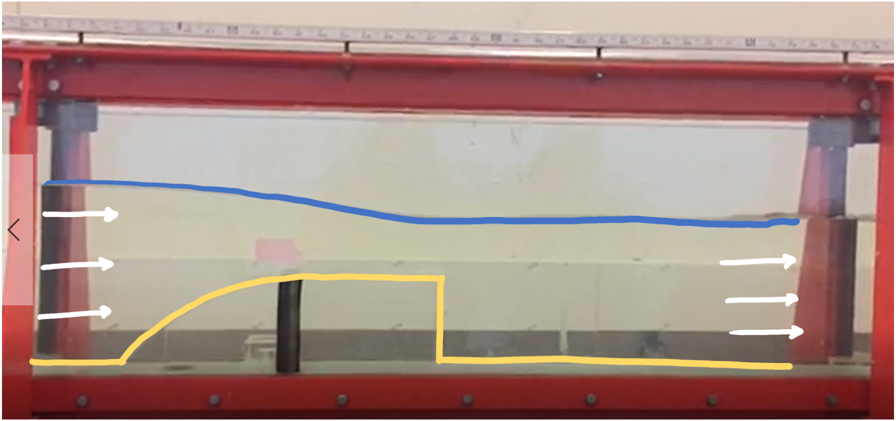
$$\eta(x) = \frac{\mathcal{F}^{-1}\left\{\hat{p}(k)\mathsf{cosh}(\mu k)\right\}}{1 - \epsilon\mu\mathcal{F}^{-1}\left\{\hat{p}(k)\mathsf{sinh}(\mu k)\right\}}$$

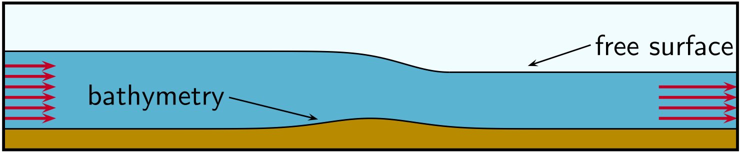
We “achieved” our goal.

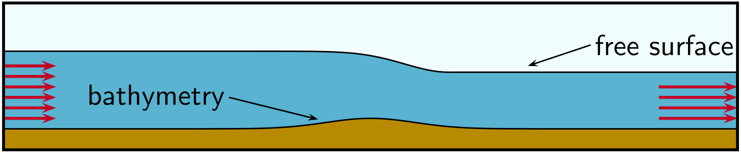
Movie Time!








free surface
bathymetry



Can we say something *new* relating the surface to the bathymetry?

Overview of the talk

Derivation of the Nonlinear Formulae

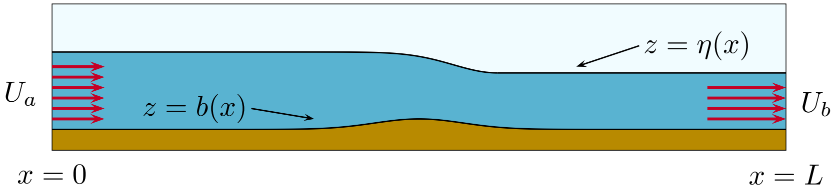
Asymptotic Expansions & Results

Summary & Future Work

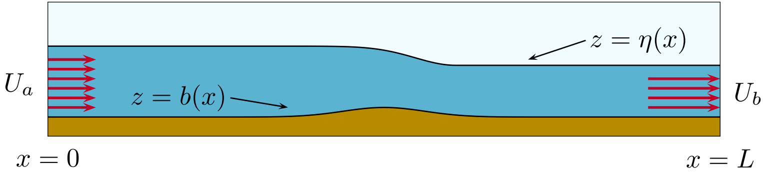
Model Assumptions

We begin with Euler's equations for an invicid, irrotational fluid.

- One-dimensional
- Irrotational & Invicid
- Constant Density
- Stationary Flow
- No friction/boundary layer effects
- Zero Atmospheric Pressure



Equations of Motion



We begin with Euler's equations for irrotational free-surface flow given by

$$\begin{aligned}
&\phi_{xx} + \phi_{zz} = 0, && (x, z) \in D,\\
&\phi_z - b_x \phi_x = 0, && z = b(x),\\
&\phi_z - \eta_x \phi_x = 0, && z = \eta(x),\\
&\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + g\eta = \frac{B}{2}, && z = \eta(x),
\end{aligned}$$

with the horizontal boundary conditions

$$\begin{aligned}
&\phi_z = 0, && x = 0, \text{ and } x = L,\\
&\phi_x = U_a && x = 0,\\
&\phi_x = U_b && x = L.
\end{aligned}$$

Equations of motion - at the free surface

At the free surface, we have

$$\phi_z\Big|_{z=\eta} = \eta_x \phi_x\Big|_{z=\eta}$$

and

$$\frac{1}{2}\phi_x^2\Big|_{z=\eta} + \frac{1}{2}\phi_z^2\Big|_{z=\eta} + g\eta = \frac{1}{2}B,$$

where B is the Bernoulli Constant.

Combining the two relationships, we find

$$\boxed{\phi_x = \pm\frac{\sqrt{B-2g\eta}}{\sqrt{1+\eta_x^2}}.}$$

Thus, we can express the velocity potential at the surface in terms of the Bernoulli constant, and the surface elevation.

Equations of motion - at the bottom

At the bottom, we have

$$\phi_z\Big|_{z=b} = b_x \phi_x\Big|_{z=b}$$

and

$$\frac{1}{2}\phi_x^2\Big|_{z=b} + \frac{1}{2}\phi_z^2\Big|_{z=b} + g \cdot b(x) + p_b(x) = \frac{1}{2}B$$

where B is the Bernoulli Constant and $p_b(x) = p(x, b(x))$.

Combining the two relationships, we find

$$\boxed{\phi_x = \pm \frac{\sqrt{B - 2\,b(x) - 2\,p_b(x)}}{\sqrt{1 + b_x^2}}}$$

Thus, we can express the velocity potential at the bottom in terms of the Bernoulli constant, the pressure along the bottom, and the surface elevation.

Equations of Motion: Boundary Conditions

At the Surface	At the Bottom
$\phi_x = \pm \frac{\sqrt{B - 2g\eta}}{\sqrt{1 + \eta_x^2}}$	$\phi_x = \pm \frac{\sqrt{B - 2\,b(x) - 2\,p_b(x)}}{\sqrt{1 + b_x^2}}$
$\phi_z = \pm \eta_x \frac{\sqrt{B - 2g\eta}}{\sqrt{1 + \eta_x^2}}$	$\phi_z = \pm b_x \frac{\sqrt{B - 2\,b(x) - 2\,p_b(x)}}{\sqrt{1 + b_x^2}}$

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A note about the $\pm$ signs:

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$\phi_x = \pm \dfrac{\sqrt{B - 2g\eta}}{\sqrt{1+\eta_x^2}}$	$\phi_x = \pm \dfrac{\sqrt{B - 2\,b(x) - 2\,p_b(x)}}{\sqrt{1+b_x^2}}$
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A note about the $\pm$ signs:

Since we are assuming that the fluid is irrotational, then ϕ_x must be sign-definite throughout the domain. For simplicity, we choose the $+$ sign.

Equations of Motion: Boundary Conditions

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To summarize, we have expressed the gradient of the velocity

- at the surface in terms of the surface elevation, and
- at the bottom in terms of the pressure and bathymetry.

Connecting the dots

Recall that the connecting glue is

$$\phi_{xx} + \phi_{zz} = 0, \quad \text{for } b(x) < z < \eta(x).$$

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$$\psi_z(\phi_{xx} + \phi_{zz}) - \phi_z(\psi_{xx} + \psi_{zz}) = 0$$

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This can be arranged to

$$(\psi_z\phi_x + \psi_z\phi_x)_x - (\phi_x\psi_x - \phi_z\psi_z)_z = 0$$

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Integrating over the entire domain yields:

$$\iint_D (\psi_z \phi_x + \psi_z \phi_x)_x - (\phi_x \psi_x - \phi_z \psi_z)_z dA = 0$$

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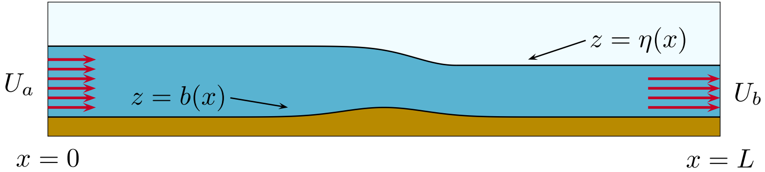
Evaluating the integral



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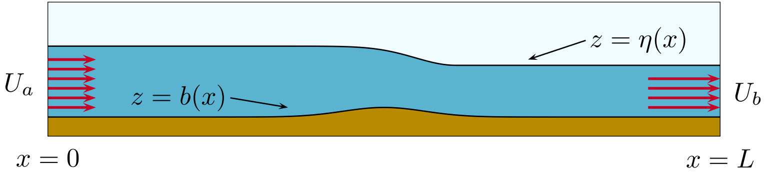

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$\phi_z = \eta_x \dfrac{\sqrt{B - 2g\eta}}{\sqrt{1 + \eta_x^2}}$	$\phi_z = \dfrac{b_x \sqrt{B - 2gb(x) - 2\, p_b(x)}}{\sqrt{1 + b_x^2}}$

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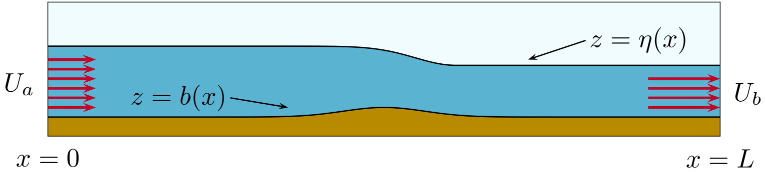

$$\oint_{\partial D} \left[(\phi_x \psi_z + \phi_z \psi_x) dz + (\phi_x \psi_x - \phi_z \psi_z) dx \right] = 0$$

Using the boundary conditions, the above integral becomes

$$\int_0^L \left[\sqrt{(1 + b_x^2)(B - 2gb(x) - p_b(x))}\, \psi_x \Big|_{z=\eta} \right] dx + U_a\, \psi \Big|_{(0,b(0))}^{(0,\eta(0))}$$

$$- \int_0^L \left[\sqrt{(1 + \eta_x^2)(B - 2g\eta(x))}\, \psi_x \Big|_{z=\eta} \right] dx - U_b\, \psi \Big|_{(L,b(L))}^{(L,\eta(L))} = 0,$$

To summarize...

We can related the quantities of interest via the equation

$$\int_0^L \left[\sqrt{(1+b_x^2)(B-2gb(x)-p_b(x))}\, \psi_x \Big|_{z=\eta} \right] dx + U_a\, \psi \Big|_{(0,b(0))}^{(0,\eta(0))}$$

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At this point, assume that we know $b(x)$, so that $p(x)$, $\eta(x)$, and B are all unknown.

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If we choose $\psi = z$,

$$U_a \cdot (\eta(0) - b(0)) - U_b \cdot (\eta(L) - b(L)) = 0.$$

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Conservation of Mass

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If we choose $\psi = x$,

$$\int_0^L \left[\sqrt{(1+b_x^2)(B-2gb(x)-2p_b(x))} - \sqrt{(1+\eta_x^2)(B-2g\eta)} \right] dx = 0$$

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Average value of tangential velocity along the surface and bathymetry.

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How many relationships are enough to directly relate $b(x)$ and $\eta(x)$?

Relating the quantities of interest

Following the work of [Ablowitz, *et. al*], [O., Vasan, Deconinck& Henderson], let

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Following the work of [Ablowitz, *et. al*], [O., Vasan, Deconinck& Henderson], let

$$\psi_1 = e^{-ikx}\sinh(kz), \quad \text{and} \quad \psi_2 = e^{-ikx}\cosh(kz).$$

Then for ψ_1, we find,

$$\mathcal{S}(b(x),k)\left\{\sqrt{(1+b_x^2)(B-2gb(x)-2p_b(x))}\right\} \; + \; U_a\left(\sinh\left(k\eta(0)\right)-\sinh\left(kb(0)\right)\right)$$
$$-\; \mathcal{S}(\eta(x),k)\{\sqrt{(1+\eta_x^2)(B-2g\eta(x))}\} - \; U_b\left(\sinh\left(k\eta(L)\right)-\sinh\left(kb(L)\right)\right) = 0$$

where

$$\mathcal{S}(f(x),k)\{g(x)\} = -ik\int_0^L \left[e^{-ikx}\sinh(kf(x))g(x)\right]\, dx.$$

We can find a similar expression for ψ_2 where we introduce the operator $\mathcal{C}(f(x),k)\{g(x)\}$.

Relating the quantities of interest

$$\mathcal{S}(b(x),k)\left\{\sqrt{(1+b_x^2)(B-2gb(x)-2p_b(x))}\right\} \;+\; U_a\left(\sinh\left(k\eta(0)\right)-\sinh\left(kb(0)\right)\right)$$
$$-\;\mathcal{S}(\eta(x),k)\{\sqrt{(1+\eta_x^2)(B-2g\eta(x))}\} -\; U_b\left(\sinh\left(k\eta(L)\right)-\sinh\left(kb(L)\right)\right)=0$$

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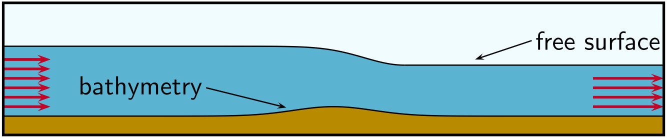
These equations form a “system” of 2 equations for 3 unknowns ($p_b(x)$, $\eta(x)$, and B. While B can be considered an unknown, we can relate B to the values of $\eta(x), b(x)$ and $p_b(x)$ at $x=0$ and $x=L$ via the Bernoulli Equation.

The question you may be asking...



Given $b(x)$, is this system of equations actually solvable for the other two parameters? Specifically, given $b(x)$, will you find the *correct* $p_b(x)$ and $\eta(x)$?

Outline of the Proof (for traveling waves/flat bottom)

The principle idea behind the proof is to use the implicit function theorem.

- We define the appropriate Banach spaces.
- Use the implicit function theorem to determine the existence of a map.
- Establish that if the pressure corresponds to a true solution of the water-wave problem, then the maps gives the true surface elevation as a function of the pressure.

In other words, given small amplitude true pressure data, we can determine the true surface elevation for a fixed wave speed.

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We are currently working to extend these results to this problem.

Overview of the talk

Derivation of the Nonlinear Formulae

Asymptotic Expansions & Results

Summary & Future Work

Asymptotic Formulae

We can non-dimensionalize the equations of motion by introducing the dimensionless parameters:

$$\epsilon = \frac{a}{h}, \qquad \mu = \frac{h}{L}$$

where $h = \dfrac{1}{L}\displaystyle\int_0^L \eta(x) - b(x)\, dx$. This yields

$$\begin{aligned}
&\mathcal{S}(\epsilon\tilde{b}(x) - 1)\left\{\sqrt{(1+\epsilon^2\mu^2 b_x^2)(B - 2\epsilon(\tilde{b}(x) - \tilde{p}_b(x))}\right\} \\
&\quad -\mathcal{S}(\epsilon\eta(x))\{\sqrt{(1+\epsilon^2\mu^2\eta_x^2)(B-2\epsilon\eta(x))}\} \\
&\quad + U_a\left(\text{sinh}\,(\mu k\tilde{\eta}(0)) - \text{sinh}\left(\mu k(\epsilon\tilde{b}(0) - 1)\right)\right) \\
&\quad - U_b\,(\text{sinh}\,(\epsilon\mu k\eta(2\pi)) - \text{sinh}\,(\mu k(\epsilon b(2\pi) - 1))) = 0
\end{aligned}$$

where we have defined $\tilde{p}_b(x) = gh - agp_b(\tilde{x})$, $b(x) = -h + ab(\tilde{x})$ and $\eta(x) = a\eta(\tilde{x})$.

We find a simlar equation for the cosh equation.

Asymptotics (KdV Limit)

If we assume shallow-water conditions where $\mu^2 = \epsilon$ and expand in powers of ϵ we find the leading order behavior

$$\tilde{p}_b(x) = \tilde{\eta}(x) - \tilde{b}(x) + \mathcal{O}(\epsilon),$$

along with

$$ik \int_0^{2\pi} e^{-ikx} \tilde{\eta}(x)\, dx = \frac{ik}{B-1} \int_0^{2\pi} e^{-ikx} \tilde{b}(x)\, dx + \frac{6\sqrt{B}\,(U_b - U_a)}{B-1} + \mathcal{O}(\epsilon)$$

If we assume $U_a = U_b = U$, then we find

$$\tilde{\eta}(x) = \frac{\tilde{b}(x)}{B-1} + \mathcal{O}(\epsilon)$$

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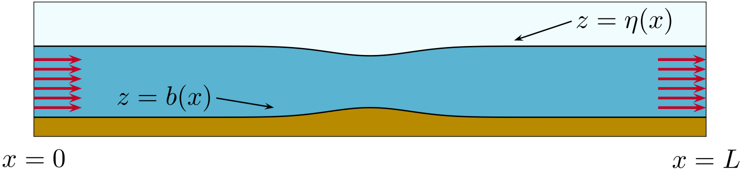

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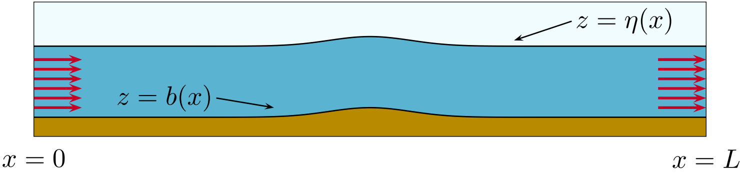

Small Amplitude

$$\sum_{k=-\infty}^{\infty} \hat{\eta}_{1_n} e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{b}_{1_n} e^{ikx} \frac{kU_a^2}{kU_a^2 \mathsf{cosh}(hk) - g \mathsf{sinh}(hk)}.$$

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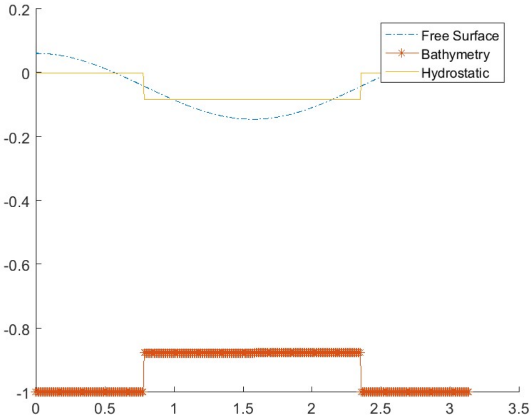

Figure: Square wave with subcritical flow

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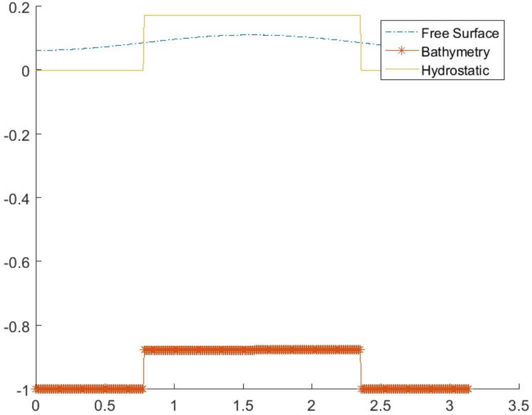

Figure: Square wave with supercritical flow

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Summary & Future Work

Summary:

We have developed a relationship between η, $b(x)$, and $p_b(x)$ based on the fully-nonlinear model.

This formulation provides an easy mechanism for asymptotic reductions.

Unfortunately, you need to know too much information.

Future Work:

Investigate various asymptotic models and compare with experimental results.

Aim to eliminate various quantities from the formulation.

Investigate higher dimensional versions of this formulation.

Thank you for your attention!