Circular instability of a standing surface wave: numerical simulation and wave tank experiment.

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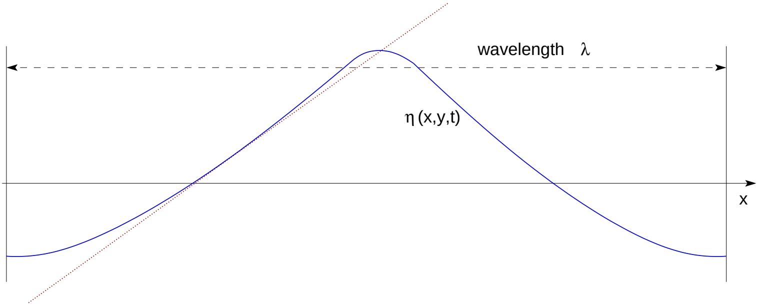
BIRS, Canada

Water waves. Problem formulation.

Let us consider a potential flow of an ideal incompressible $((\nabla \vec{v}) = \Delta\phi = 0)$ fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t)$, $\vec{r} = (x, y)$; $\vec{v} = \nabla\phi$ and surface elevation $\eta(\vec{r}, t)$.



Steepness of the surface $\mu = \sqrt{\langle |\nabla \eta(\vec{r}, t)|^2 \rangle} \simeq 0.1$ — average slope of the surface.

Hamiltonian expansion.

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$

where $\psi = \phi(\vec{r}, \eta(\vec{r}, t), t)$ is a velocity potential on the surface of the fluid. In order to calculate the value of ψ we have to solve the Laplace equation in the domain with varying surface η. One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here $\Delta = \nabla^2$ and $\hat{k} = \sqrt{-\Delta}$)

$$H = \frac{1}{2}\int \left(g\eta^2 + \psi \hat{k} \psi \right) \mathrm{d}^2 r + \\ + \frac{1}{2}\int \eta \left[|\nabla \psi|^2 - (\hat{k}\psi)^2 \right] \mathrm{d}^2 r + \\ + \frac{1}{2}\int \eta(\hat{k}\psi) \left[\hat{k}(\eta(\hat{k}\psi)) + \eta \Delta \psi \right] \mathrm{d}^2 r.$$

Canonical variables.

$\psi(\vec{r},t)$ and $\eta(\vec{r},t)$ are real valued functions, $\Rightarrow \psi_{\vec{k}} = \psi^*_{-\vec{k}}, \eta_{\vec{k}} = \eta^*_{-\vec{k}}$ — Hermitian symmetry.

It is convenient to introduce the canonical (normal) variables $a_{\vec{k}}$ as shown below

$$a_{\vec{k}} = \sqrt{\frac{\omega_k}{2k}}\eta_{\vec{k}} + \mathrm{i}\sqrt{\frac{k}{2\omega_k}}\psi_{\vec{k}}, \text{ where } \omega_k = \sqrt{gk}.$$

$$\dot{a}_{\vec{k}} = -\mathrm{i}\frac{\delta H}{\delta a^*_{\vec{k}}} \text{ — Hamiltonian equations,}$$

$a_{\vec{k}}$ — is an elementary excitation (plane wave).

Resonant conditions

Let us get rid of the linear rotation of phase (fastest motion):

$$(a_{\vec{k}_1} a_{\vec{k}_2} a^*_{\vec{k}_0} + a^*_{\vec{k}_1} a^*_{\vec{k}_2} a_{\vec{k}_0})\delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_0)$$

$$a_{\vec{k}}(t) = A_{\vec{k}}(t)\mathrm{e}^{\mathrm{i}\omega_k t} \Rightarrow a^*_{\vec{k}_0} a_{\vec{k}_1} a_{\vec{k}_2} = A^*_{\vec{k}_0} A_{\vec{k}_1} A_{\vec{k}_2} \mathrm{e}^{\mathrm{i}(\omega_{k_0} - \omega_{k_1} - \omega_{k_2})t}$$

Resonant conditions for 3-waves interaction (decaying and merging):

$$\omega_{k_0} = \omega_{k_1} + \omega_{k_2}, \quad \vec{k}_0 = \vec{k}_1 + \vec{k}_2.$$

Resonant conditions for 4-waves interaction (two into two scattering):

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4. \tag{1}$$

Standing wave instability, general case.

Universal and very interesting is the case of interaction of two waves $a_{\vec{k}_0}$ and $a_{-\vec{k}_0}$ in the presence of a 4-waves interaction. In this case resulting waves $\vec{k}_3$ and $\vec{k}_4$ has to obey the following relation

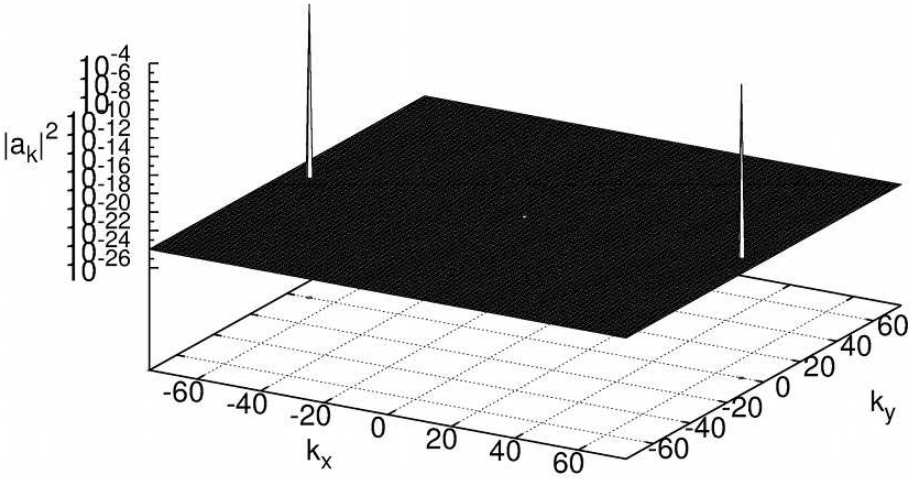
$$\vec{k}_0 + (-\vec{k}_0) = \vec{0} = \vec{k}_3 + \vec{k}_4, \Rightarrow \vec{k}_3 = -\vec{k}_4.$$

If we have a dispersion relation depending only on the magnitude of the wavevector, the condition on the resonance of the frequencies gives us

$$2\omega_{k_0} = 2\omega_{k_3},$$

which in case of capillary and gravity waves results in $|\vec{k}_3| = |\vec{k}_0|$, with arbitrary direction.
In other words resonant curve is a circle with the center at zero wave number vector and of radius $|\vec{k}_0|$. It is clear that such a process is general for any isotropic dispersion.

$k_0 = 68.\ T = 0.$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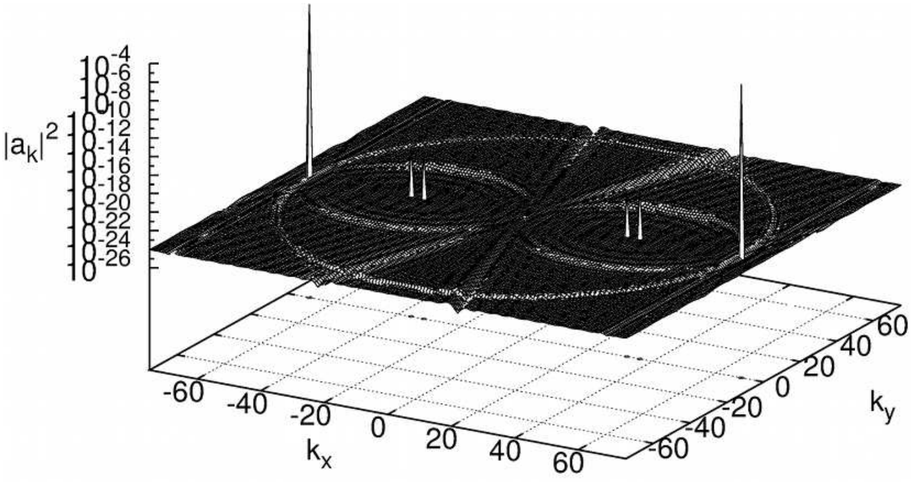
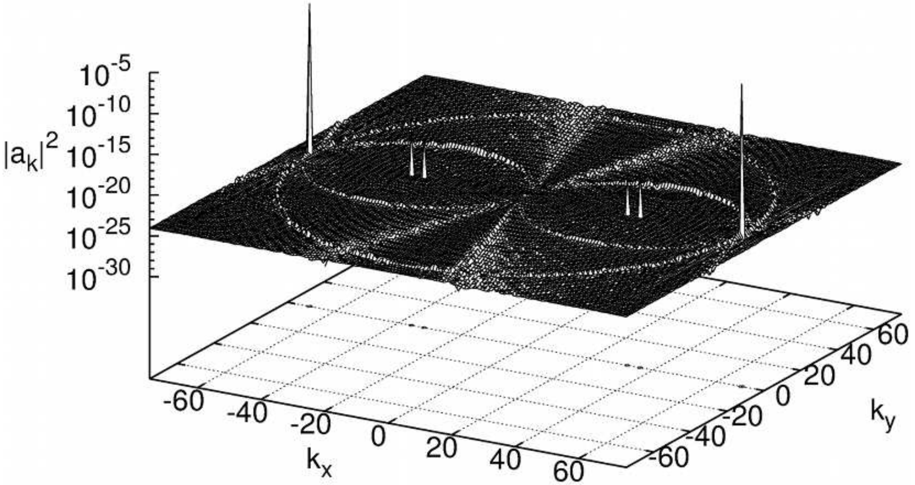
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-18
10^-20
10^-22
10^-24
10^-26
10^0
k_x
-60 -40 -20 0 20 40 60
k_y
-60 -40 -20 0 20 40 60

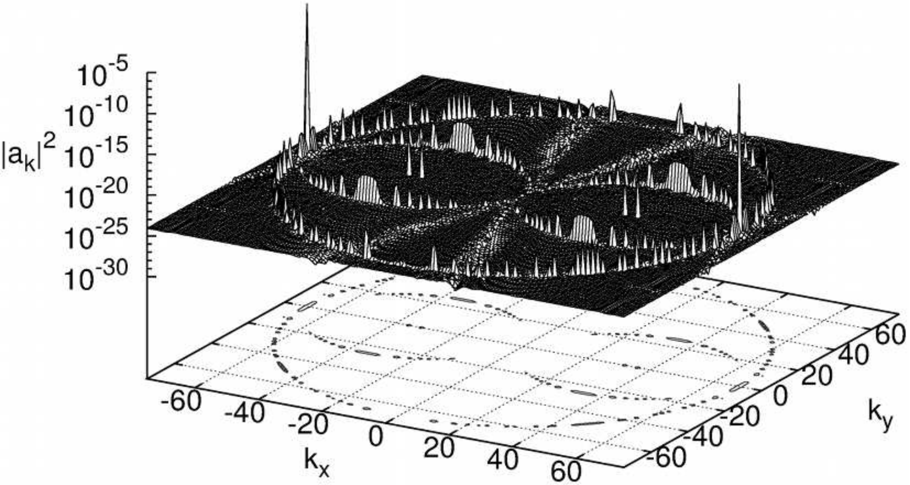
$k_0 = 68.\ \ T = 14T_0.$


$|a_k|^2$
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-18
10^-20
10^-22
10^-24
10^-26
-60 -40 -20 0 20 40 60
k_x
-60 -40 -20 0 20 40 60
k_y

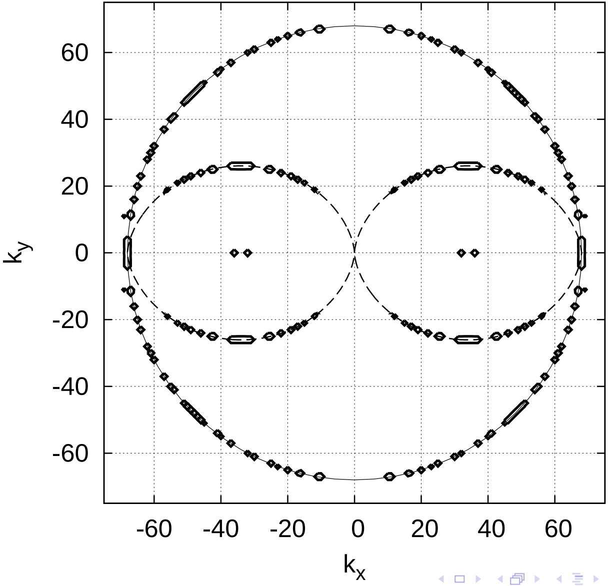
$k_0 = 68.\ \ T = 57\,T_0.$



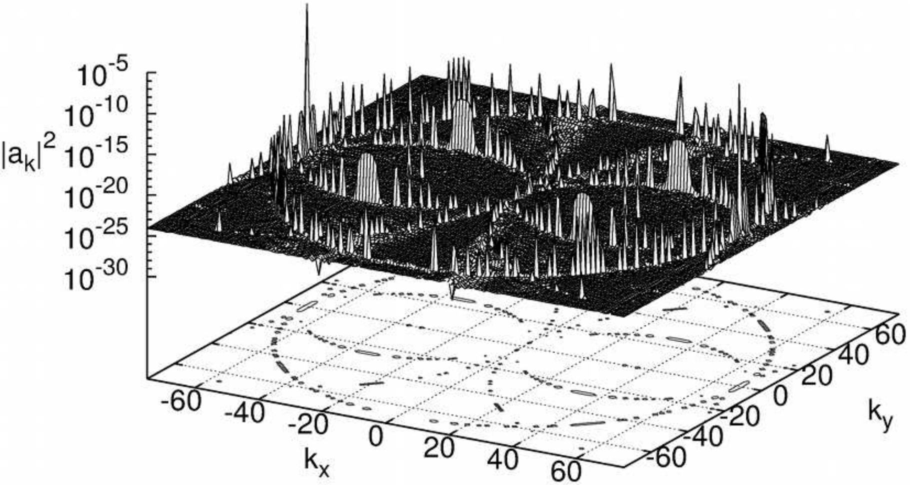
$k_0 = 68.\ \ T = 283\,T_0.$


$|a_k|^2$
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
-60
-40
-20
0
20
40
60
k_x
60
40
20
0
-20
-40
-60
k_y

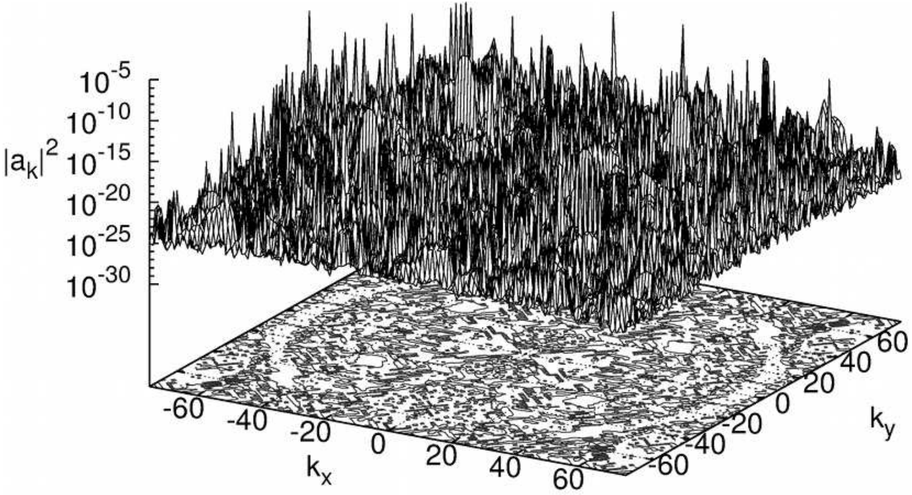
$k_0 = 68.\ \ T = 283\, T_0.$



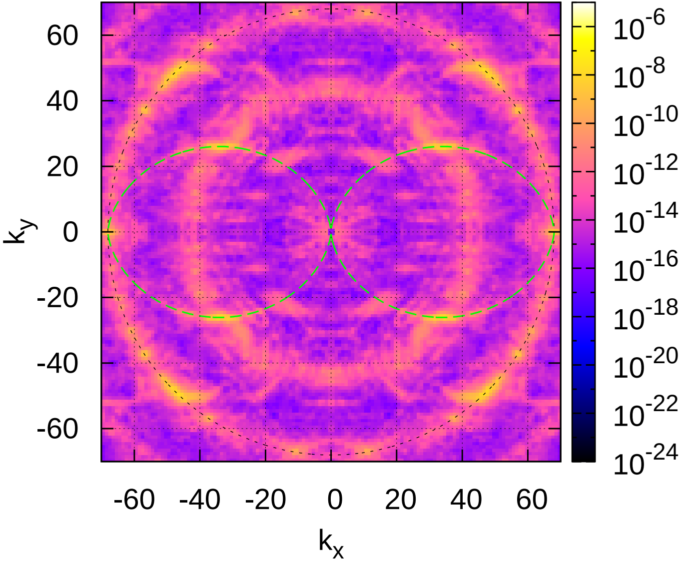
$k_0 = 68.\ T = 509\,T_0.$


$|a_k|^2$
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
k_x
-60
-40
-20
0
20
40
60
k_y
-60
-40
-20
0
20
40
60

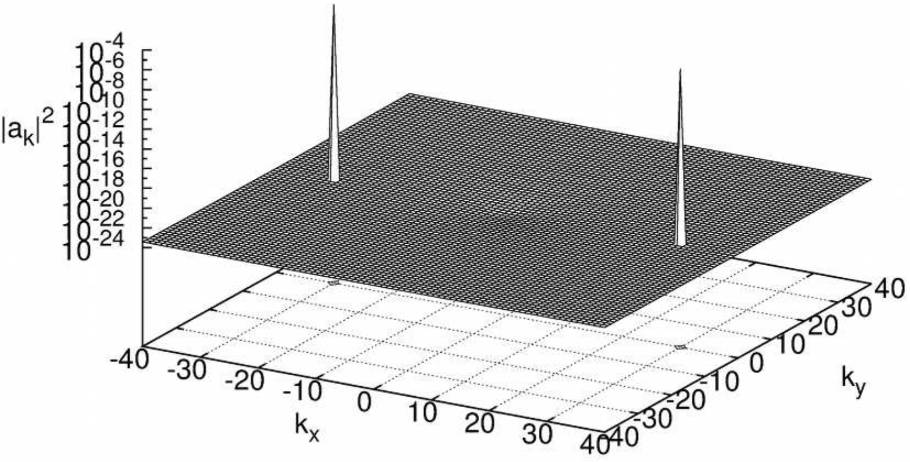
$k_0 = 68. \ T = 1018\,T_0.$



$k_0 = 68. \ T = 2587\,T_0.$

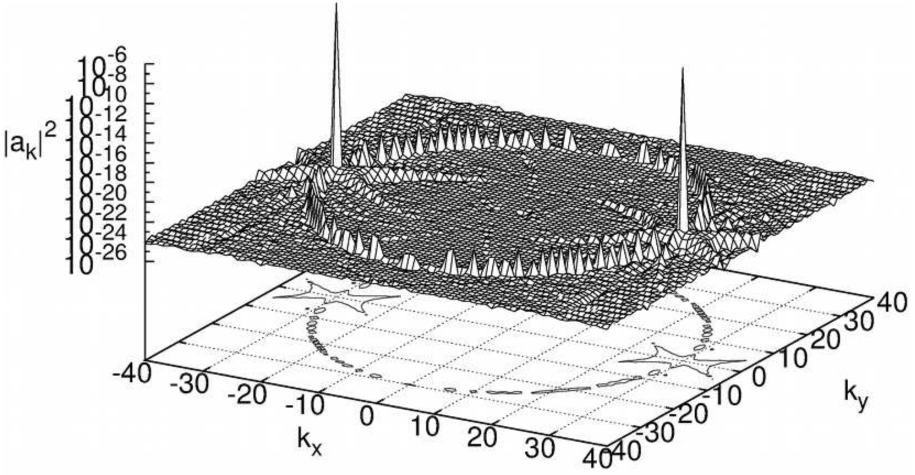


$k_0 = 30.\ T = 0.$

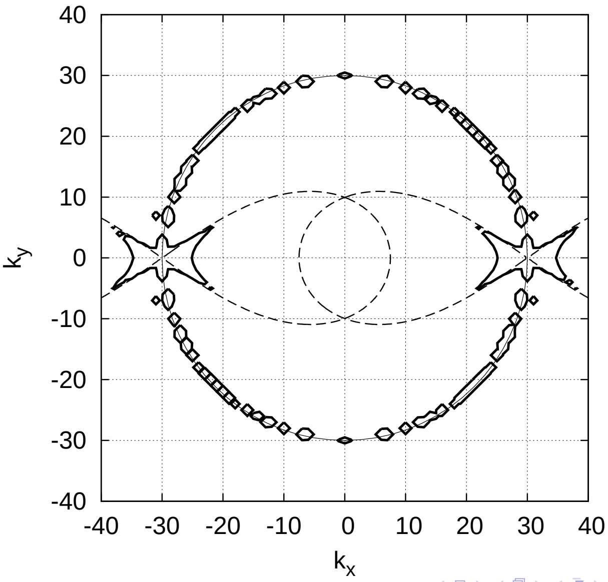


|a_k|^2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-18
10^-20
10^-22
10^-24
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

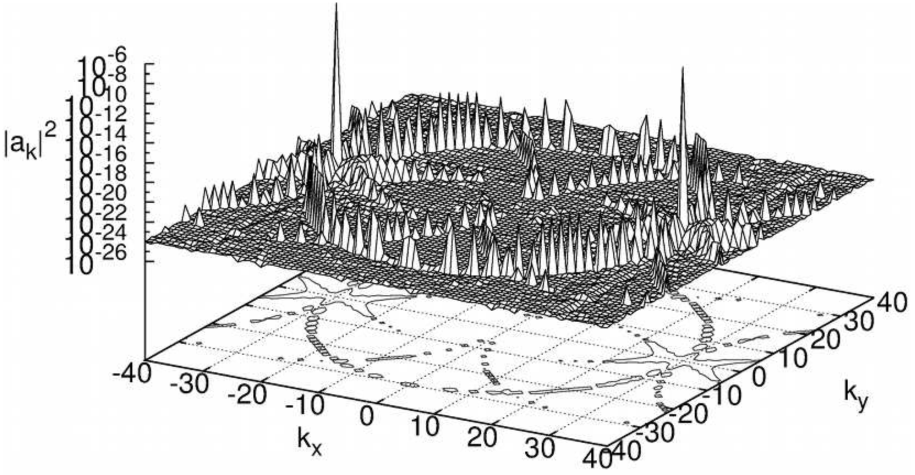
$k_0 = 30.\ T = 116\, T_0.$


$|a_k|^2$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

$k_0 = 30.\ \ T = 116\,T_0.$

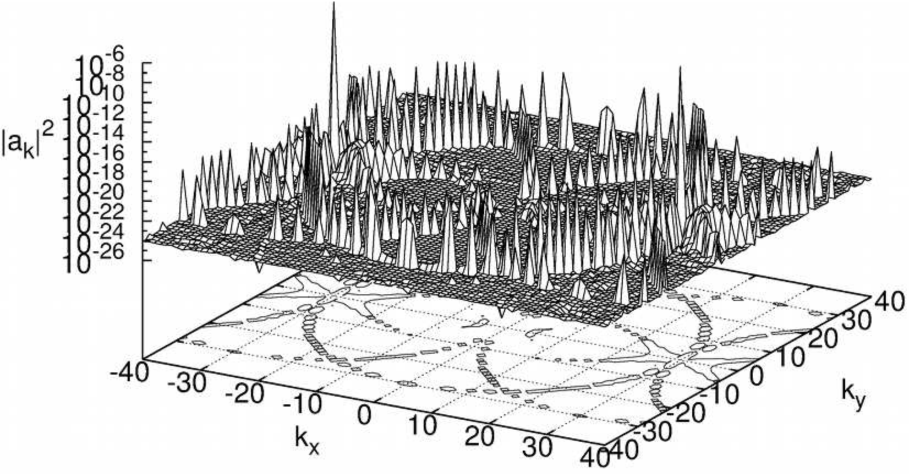


$k_0 = 30.\ \ T = 232\,T_0.$

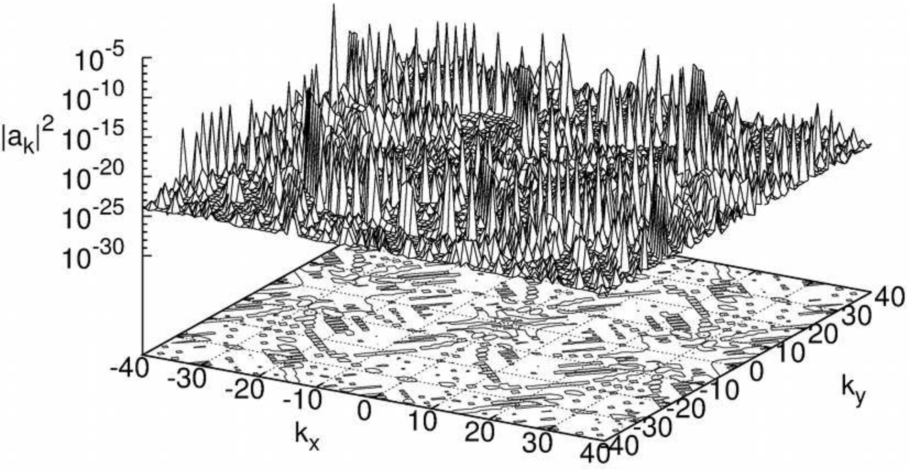


$|a_k|^2$
10-6
10-8
10-10
10-12
10-14
10-16
10-18
10-20
10-22
10-24
10-26
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

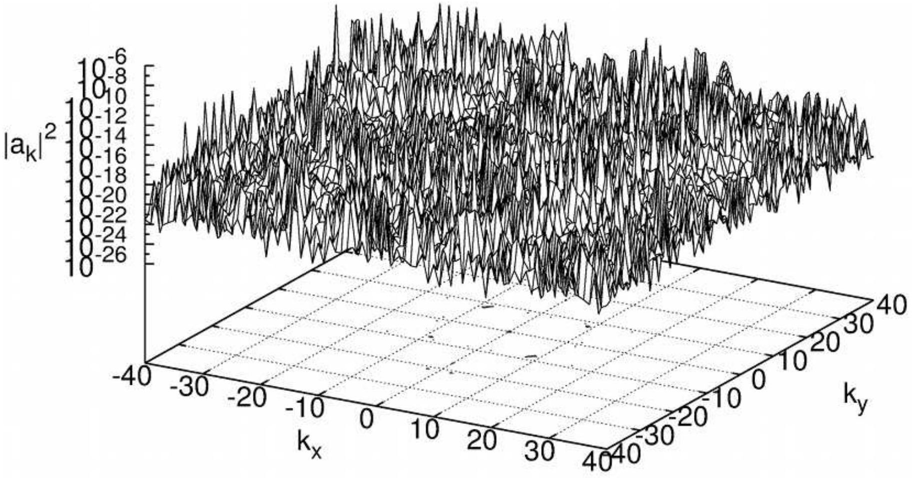
$k_0 = 30.\ \ T = 348\,T_0.$


$|a_k|^2$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

$k_0 = 30.\ \ T = 463\,T_0.$


$|a_k|^2$
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

$k_0 = 30.\ \ T = 580\,T_0.$


$|a_k|^2$
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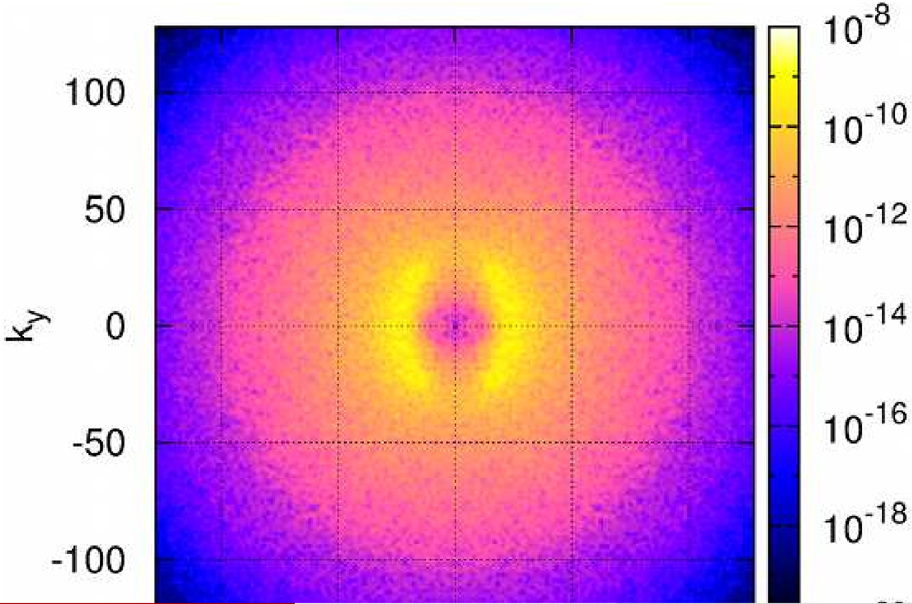
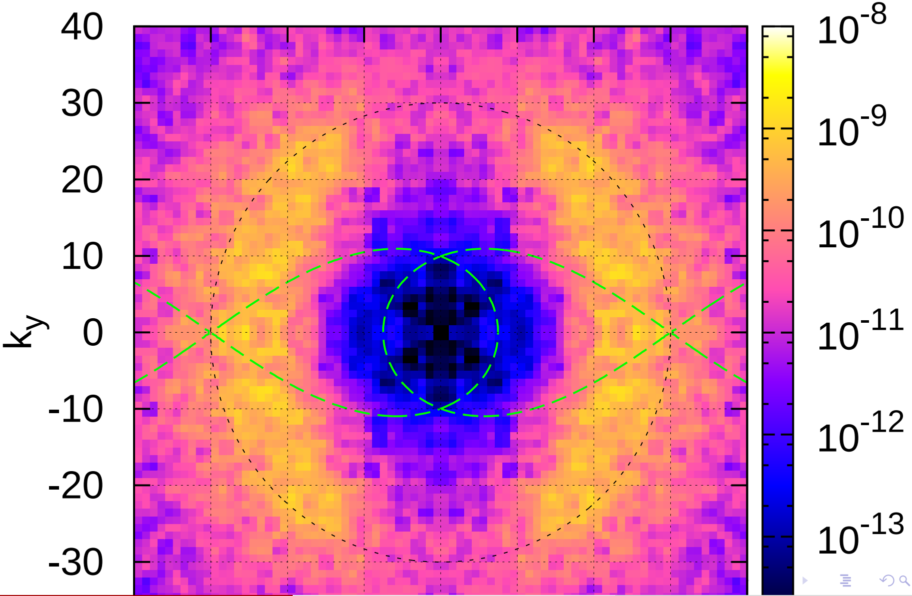
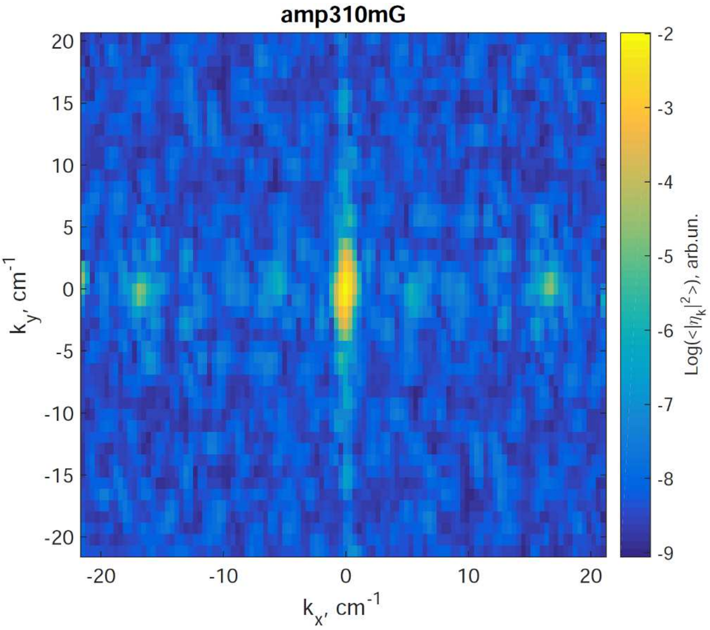
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
-40 -30 -20 -10 0 10 20 30 40
k_x
-40 -30 -20 -10 0 10 20 30 40
k_y

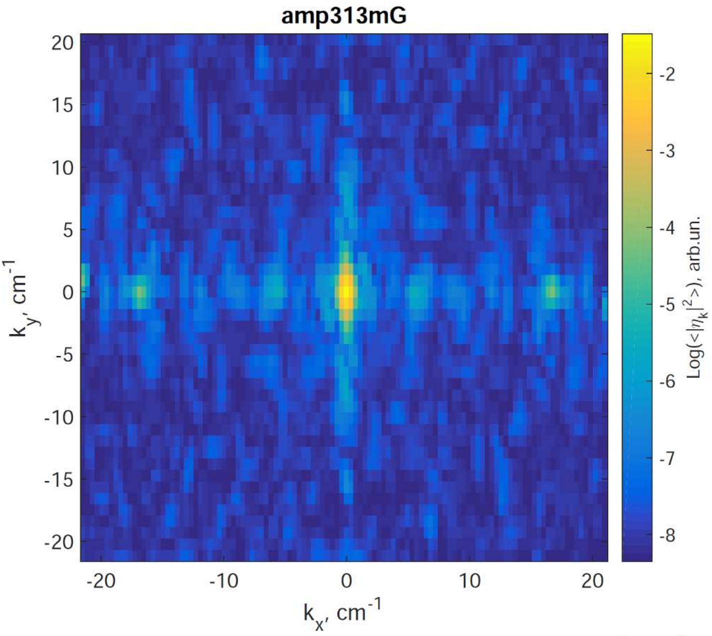
$k_0 = 30.$ $T = 3068\,T_0.$



$k_0 = 30.\ \ T = 3068\,T_0.$




amp310mG
20
15
10
5
0
-5
-10
-15
-20
k$_y$, cm^{-1}
-20
-10
0
10
20
k$_x$, cm^{-1}
-2
-3
-4
-5
-6
-7
-8
-9
Log(<|η$_k$|^2>), arb.un.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k_y, cm^{-1}
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k_x, cm^{-1}
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-6
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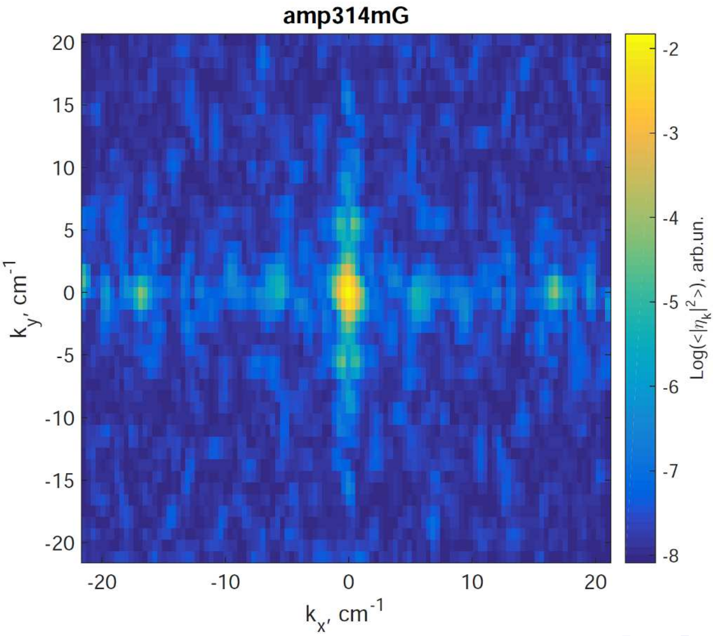
Log(<|η_k|^2>), arb.un.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k_y, cm^{-1}
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k_x, cm^{-1}
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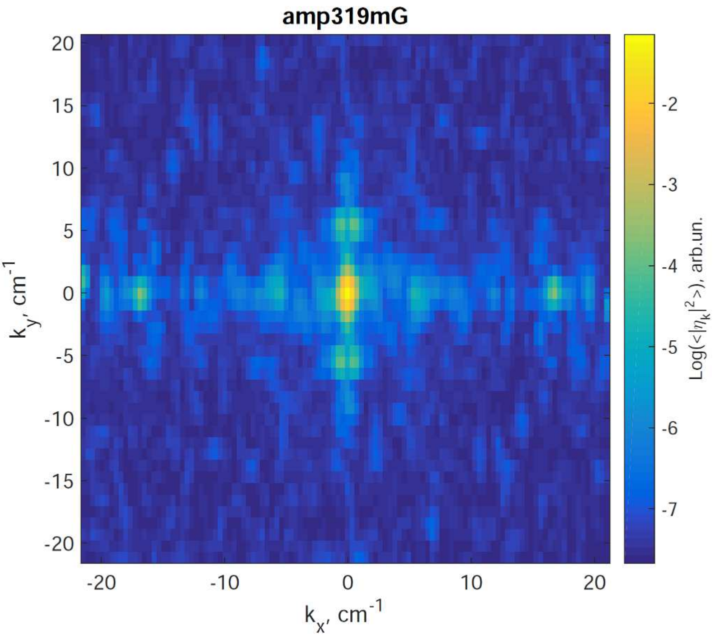
-5
-6
-7
-8
Log(<|η_k|^2>), arb.un.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-6
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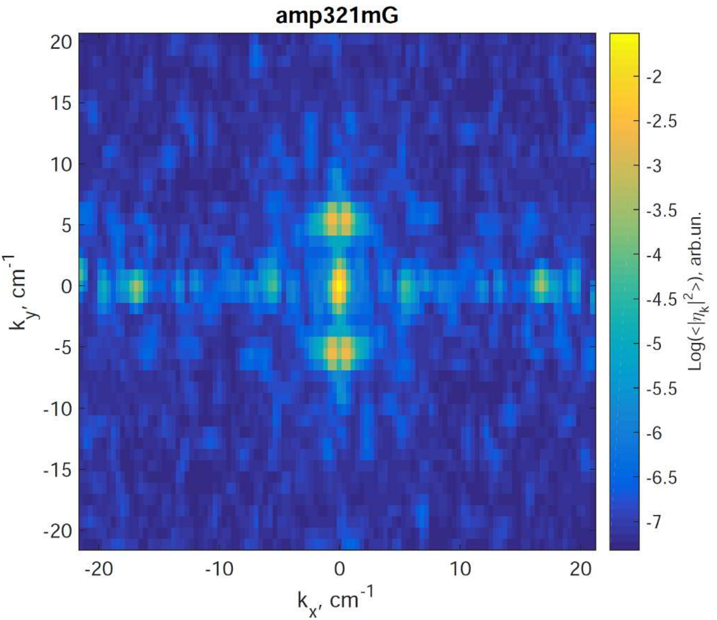
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10
20
k_x, cm^{-1}
-2
-2.5
-3
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-5.5
-6
-6.5
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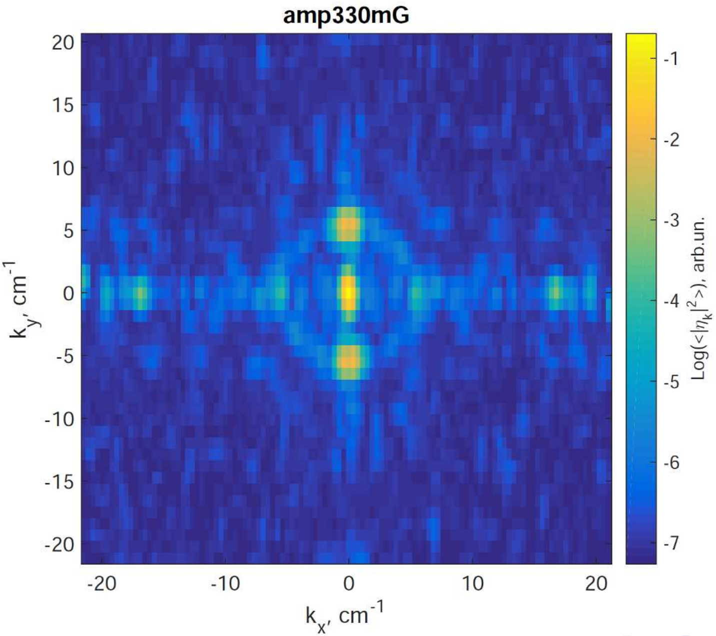
Log(<$|\eta_k|^2$>), arb.un.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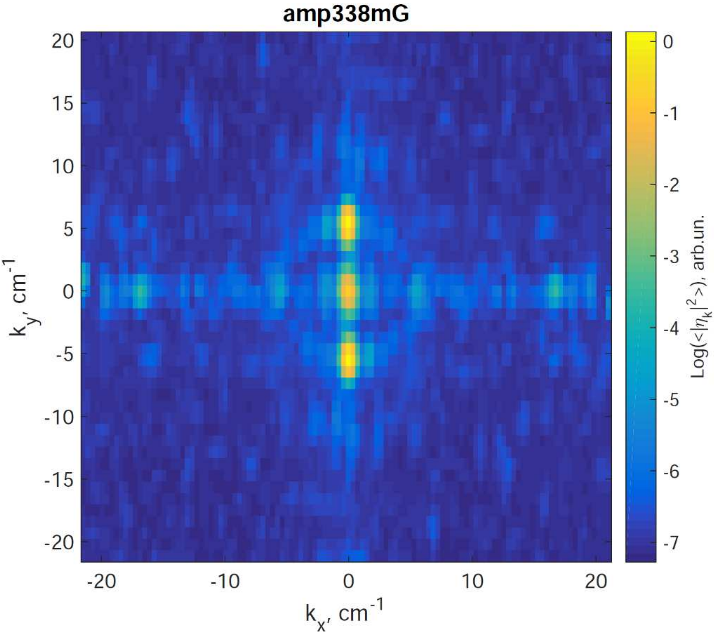
Log(<$|\eta_k|^2$>), arb.un.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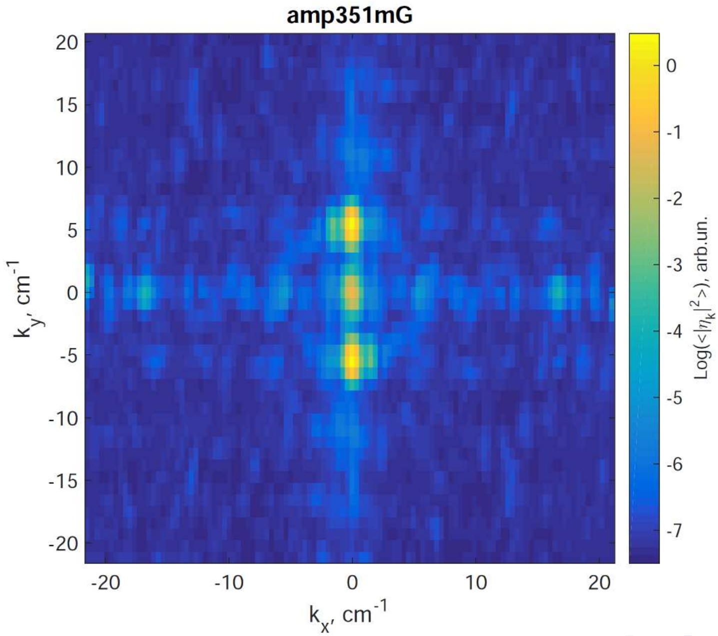
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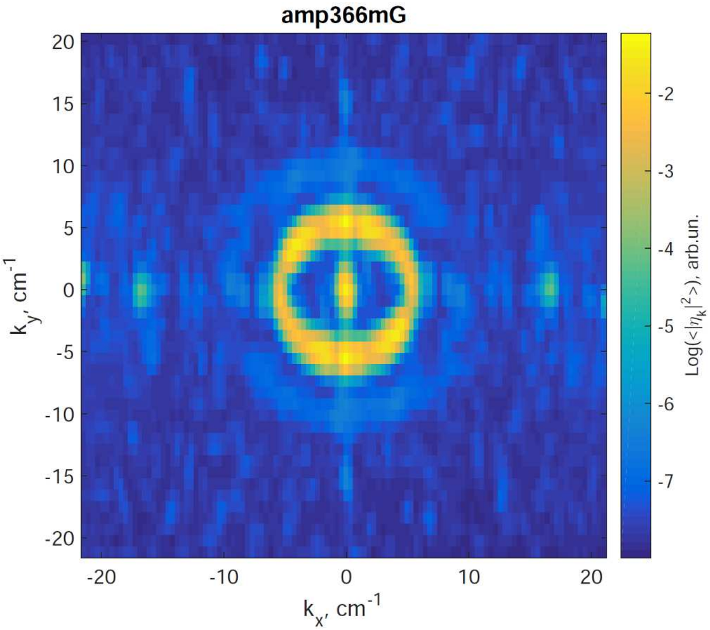
Log(<$|\eta_k|^2$>), arb.un.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k_x, cm^{-1}
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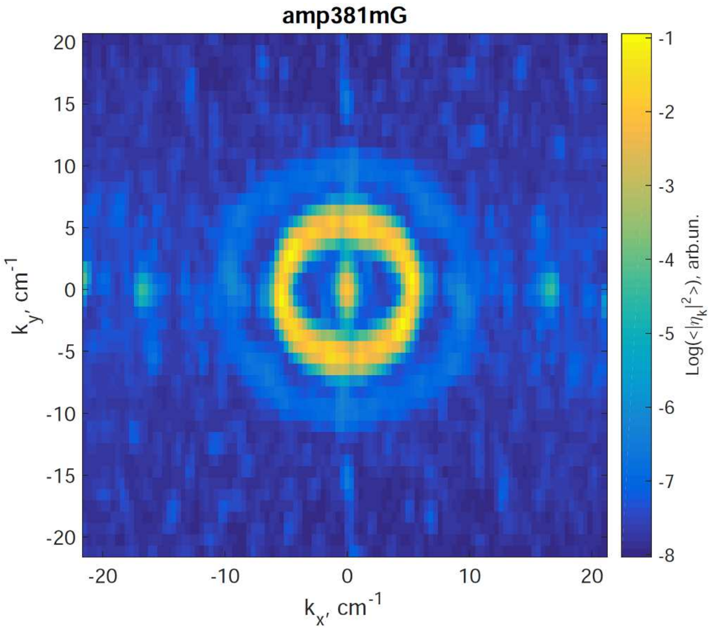
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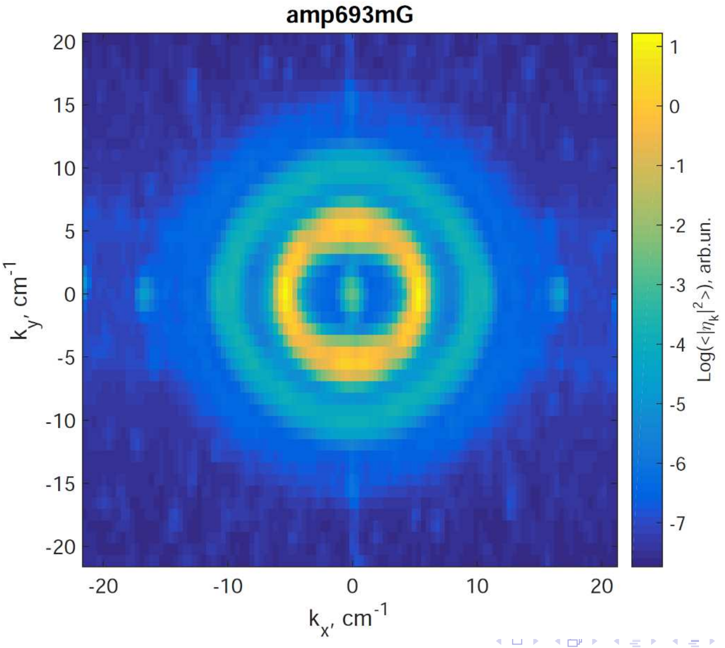
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k_x, cm^{-1}
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-4
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-6
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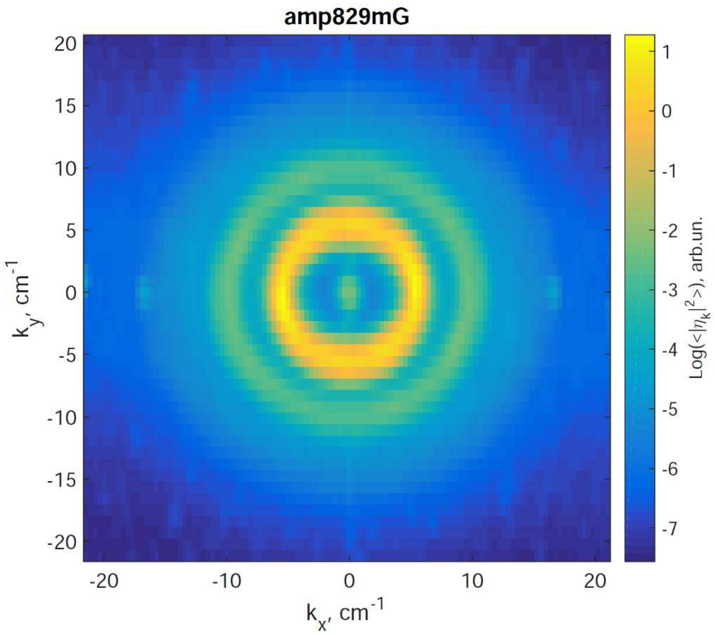
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k_x, cm^{-1}
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0
-1
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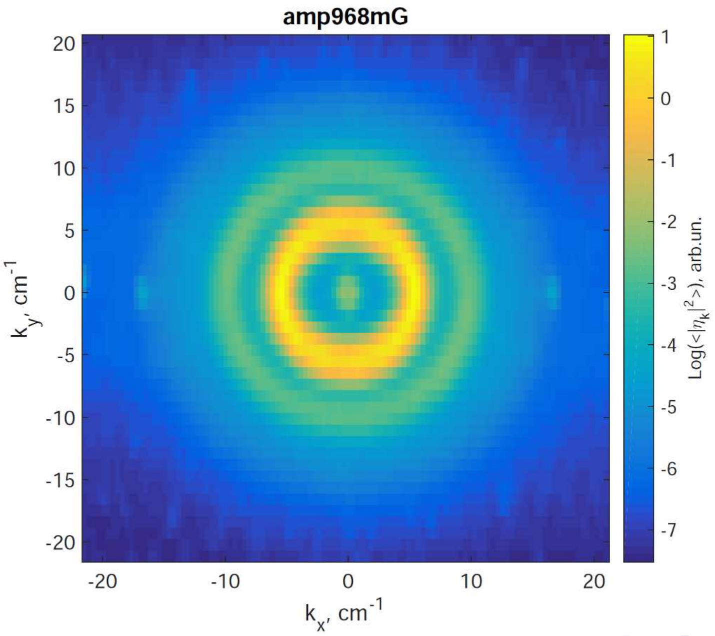
-3
-4
-5
-6
-7
Log(<$|\eta_k|^2$>), arb.un.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20
k_x, cm^{-1}
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0
-1
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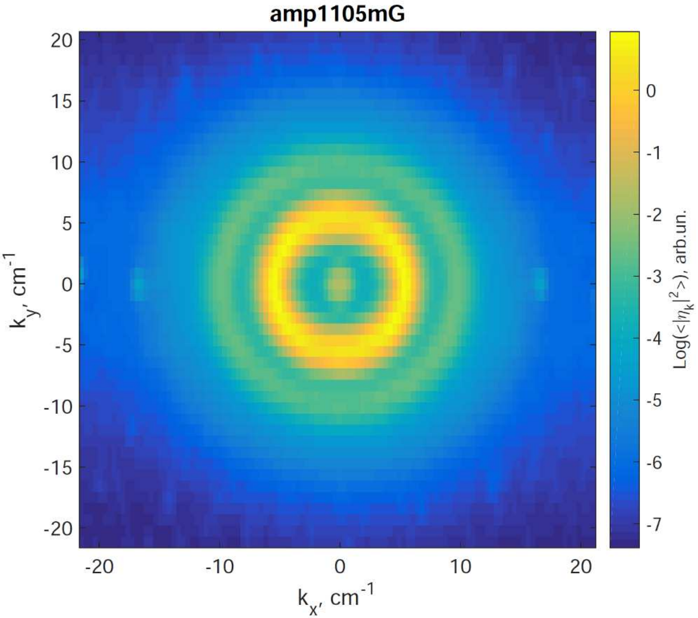
-3
-4
-5
-6
-7
$Log(<|\eta_k|^2>)$, arb.un.


amp1105mG
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k_y, cm^-1
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10
20
k_x, cm^-1
0
-1
-2
-3
-4
-5
-6
-7
Log(<|η_k|^2>), arb.un.

Results.

- Formulated conditions for standing wave instability.
- Performed simulation of standing wave instability for gravity and capillary case.
- Observation of the instability in a wave tank experiment.
- A natural way of isotropic excitation in laboratory experiments.

KAO, *"Numerical Simulation of Weak Turbulence of Surface Waves"*, PhD thesis, Landau Institute, (2003)
KAO, Dyachenko, Zakharov, *"Numerical simulation of surface waves instability on a discrete grid"*, Physica D **321-322**, 51–66 (2016).
All these texts can be found at `http://math.unm.edu/~alexkor/`
Work was supported by NSF OCE & DMS 1130450 grant.