Singular Solutions for Systems of Conservation Laws

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Joint work with Darko Mitrovic and Vincent Teyekpiti

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Study conservation law

$$\left.\begin{array}{rcl} \partial_t u + \partial_x f(u,v) & = & 0 \\ \partial_t v + \partial_x g(u,v) & = & 0 \end{array}\right\} \qquad (\star)$$

Study conservation law

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Riemann problem:

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

$$v(x,0) = v_0(x) = \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases}$$

Weak solutions:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u\partial_t\varphi + f(u,v)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = 0$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(v\partial_t\varphi + g(u,v)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} v_0(x)\varphi(x,0)\, dx = 0$$

For some initial data, the Riemann problem may not have a solution.

Delta shocks: early work

- Korchinski (PhD thesis, Adelphi University, 1978)

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$
$$v_t + \left(\tfrac{1}{2}uv\right)_x = 0$$

- Keyfitz & Kranzer (JDE, 1995)

$$u_t + (u^2 - v)_x = 0$$
$$v_t + (\tfrac{1}{3}u^3 - u)_x = 0$$

- Hayes & Le Floch (Nonlinearity, 1996)

$$u_t + \tfrac{1}{2}\left(u^2 + v^2\right)_x = 0$$
$$v_t + (uv - u)_x = 0$$

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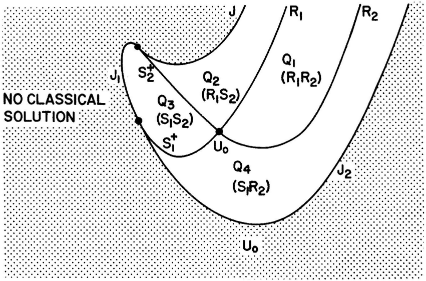

Introduce Rankine-Hugoniot deficit:

$$c[u] - [u^2 - v] = 0, \quad (1)$$
$$c[v] + \left[\tfrac{1}{3}u^3 - u\right] = \alpha'(t), \quad (2)$$

Delta shocks: early work

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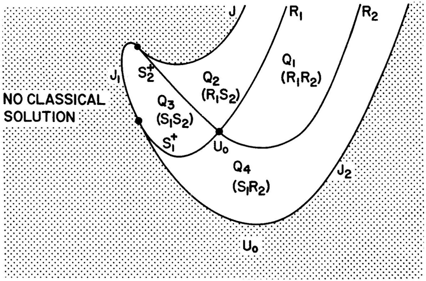

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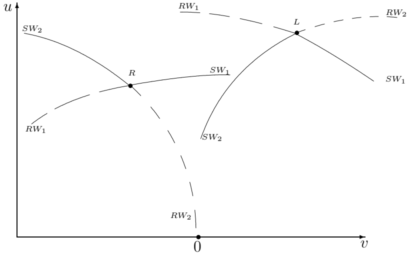
Delta shocks: early work

- Hayes & Le Floch (Nonlinearity, 1996): Brio system

$$u_t + \frac{1}{2}\left(u^2 + v^2\right)_x = 0$$
$$v_t + \left(uv - u\right)_x = 0$$

Not genuinly nonlinear at $v = 0$.



Delta shocks: early work

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Delta shocks: early work

- Hayes & Le Floch (Nonlinearity, 1996): Brio system

$$u_t + \tfrac{1}{2}\left(u^2 + v^2\right)_x = 0$$
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Similar system:

$$u_t + (\tfrac{1}{2}u^2)_x = 0$$
$$v_t + (uv - u)_x = 0$$

If $u_l > u_r + 2$, the solutions contains singular shocks with Dirac delta distributions:

$$u(x,t) = u_l + (u_r - u_l)H(x - ct)$$
$$v(x,t) = v_l + (v_r - v_l)H(x - ct) + \alpha(t)\delta(x - ct)$$

To understand the term uv, use theory of

Dal Maso et al. (J. Math. Pure Appl. 1995)

$$\left.\begin{array}{rcl} \partial_t u + \partial_x f(u,v) & = & 0 \\ \partial_t v + \partial_x g(u,v) & = & 0 \end{array}\right\} \qquad (\star)$$

Riemann problem:

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

$$v(x,0) = v_0(x) = \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases}$$

Weak solutions: $u, V \in L^\infty$, $v = V + \alpha(t)\delta(x - ct)$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u\partial_t\varphi + f(u,V)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)dx = 0$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(V\partial_t\varphi + g(u,V)\partial_x\varphi\right)\, dxdt$$

$$+ \int_{\mathbb{R}} v_0(x)\varphi(x,0)\, dx + \int_{\{x=ct\}} \alpha(t)\frac{\partial\varphi}{\partial \mathbf{l}} + \alpha(0)\varphi(0,0) = 0$$

Generalized weak solution (Danilov & Shelkovich, JDE, 2005)

Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the upper half plane, containing arcs γ_i, $i \in I$. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of the arcs γ_k with $k \in I_0$.

Define the singular part by $\alpha(x,t)\delta(\Gamma) = \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i)$.

Let $u, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, and let $v(x,t) = V(x,t) + \alpha(x,t)\delta(\Gamma)$.

Definition 1

The pair of distributions u and $v = V + \alpha(x,t)\delta(\Gamma)$ are called a generalized δ-shock wave solution of system $(\star)$ with the initial data $u_0(x)$ and $V_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0)$ if the integral identities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u\partial_t\varphi + f(u,V)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = 0,$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(V\partial_t\varphi + g(u,V)\partial_x\varphi\right)\, dxdt + \sum_{i \in I}\int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} V^0(x)\varphi(x,0)\, dx + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0,$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Generalized weak solution (Danilov & Shelkovich, JDE, 2005)

Definition 1 is quite general, allowing a combination of initial steps and delta distributions; but its effectiveness is already demonstrated by considering the Riemann problem with a single jump. Indeed, for this configuration it can be shown that a δ-shock wave solution exists for any 2×2 system of conservation laws. Consider the Riemann problem for $(\star)$ with initial data $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, where

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}, \quad v_0(x) = \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases}.$$

$$\left.\begin{array}{rcl}\partial_t u + \partial_x f(u,v) & = & 0 \\ \partial_t v + \partial_x g(u,v) & = & 0\end{array}\right\} \qquad (\star)$$

Theorem 1 (K. and Mitrovic)

a) *If $u_l \neq u_r$ then the pair of distributions*

$$\begin{aligned} u(x,t) &= u_0(x-ct), \\ v(x,t) &= v_0(x-ct) + \alpha(t)\delta(x-ct), \end{aligned}$$

where

$$c = \frac{[f(u,V)]}{[u]} = \frac{f(u_r,v_r) - f(u_l,v_l)}{u_r - u_l}, \quad \text{and} \quad \alpha(t) = (c[V] - [g(u,V)])t,$$

represents the δ-shock wave solution of $(\star)$ with initial data $u_0(x)$ and $v_0(x)$ in the sense of Definition 1.

Weak asymptotic method

Weak asymptotic method

Definition 2

Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$, We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, we have

$$\langle f_\varepsilon, \phi \rangle = o(1), \quad \text{as} \quad \varepsilon \to 0$$

Link between RH-deficit and δ solutions?

Weak asymptotic method

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Definition 3 (K. and Mitrovic)

The collection of smooth **complex-valued** *distributions (u_ε) and (v_ε) represent a weak asymptotic solution to $(\star)$ if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$*

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as} \quad \varepsilon \to 0,$$

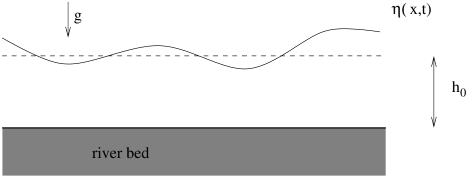
in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\begin{aligned} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1). \end{aligned}$$

In addition, we need

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \quad \text{and} \quad v_\varepsilon(x,0) \rightharpoonup v(x,0).$$

Example: shallow-water equations



Shallow-water equations:

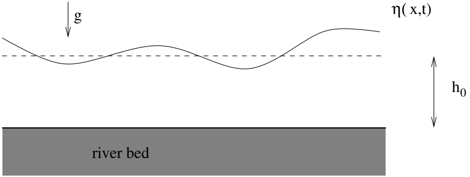
$$\eta_t + h_0 u_x + (\eta u)_x = 0 \tag{1}$$

$$u_t + \mathbf{g}\eta_x + u u_x = 0 \tag{2}$$

Assumptions:

- $p = \rho \mathbf{g}(\eta - z)$ (hydrostatic)

Example: shallow-water equations



Shallow-water equations:

$$\eta_t + h_0 u_x + (\eta u)_x = 0 \qquad (1)$$

$$u_t + \mathbf{g}\eta_x + u u_x = 0 \qquad (2)$$

Assumptions:

- $p = \rho \mathbf{g}(\eta - z)$ (hydrostatic)
- $u = u(x,t)$ (no vertical acceleration)

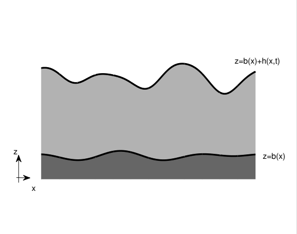
Momentum conservation:

$$\Big[(h_0+\eta)u\Big]_t \;+\; \Big[(h_0+\eta)u^2 + \frac{1}{2}\mathbf{g}(h_0+\eta)^2\Big]_x = 0$$

Energy conservation:

$$\frac{1}{2}\Big[(h_0+\eta)u^2 + (h_0+\eta)^2\Big]_t \;+\; \Big[\frac{1}{2}(h_0+\eta)u^3 + \mathbf{g}u(h_0+\eta)^2\Big]_x = 0$$

Shallow-water equations with bottom topography



Mass conservation:

$$\big[h\big]_t + \big[uh\big]_x = 0$$

Conservation of total head:

$$\big[u\big]_t + \big[\mathbf{g}(h+b) + \tfrac{u^2}{2}\big]_x = 0$$

Momentum balance:

$$\big[hu\big]_t + \Big[hu^2 + \tfrac{1}{2}\mathbf{g}h^2\Big]_x = -ghb_x \tag{3}$$

Energy conservation:

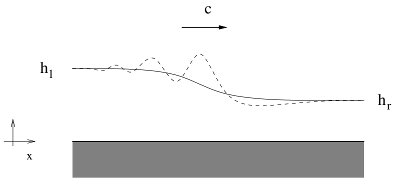
$$\Big[\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2 + bh\Big]_t + \Big[\tfrac{1}{2}u^3 + \mathbf{g}uh(h+b)\Big]_x = 0 \tag{4}$$

A traveling hydraulic jump over an even bottom must respect conservation of mass and momentum.
In shallow-water theory, it is useful to consider the jump as having a discontinuity at the bore front. Rankine-Hugoniot conditions

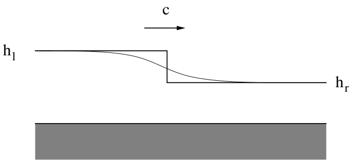
$$c[h] = [uh]$$
$$c[uh] = \left[u^2 h + \tfrac{1}{2} g h^2\right]$$

The velocity can be expressed as

$$c = \frac{u_r h_r - u_l h_l}{h_r - h_l} = \frac{\left(u_r^2 h_r + \frac{1}{2}\mathbf{g} h_r^2\right) - \left(u_l^2 h_l + \frac{1}{2}\mathbf{g} h_l^2\right)}{u_r h_r - u_l h_l}$$



Surface profile of a traveling hydraulic jump



Shallow-water approximation

Traveling hydraulic jump: induced Rankine-Hugoniot deficit

Relative veclocity:

$$m = h_r(u_r - c) = h_l(u_l - c) = \mp h_r h_l \sqrt{\frac{\mathbf{g}}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}$$

Energy loss:

$$\frac{1}{\rho Y}\Delta E = -\frac{m\mathbf{g}(h_r - h_l)^3}{4h_r h_l}$$

Head loss:

$$\Delta H = \frac{(h_l - h_r)^3}{4h_l h_r}$$

Traveling hydraulic jump over flat bottom

Mass conservation:

$$c\left[h\right] \ - \ \left[uh\right] = 0$$

Head loss:

$$c\left[u\right] \ - \ \left[\mathbf{g}(h+b) + \tfrac{u^2}{2}\right] = \mathbf{g}\Delta H$$

Momentum balance:

$$c\left[hu\right] \ - \ \left[hu^2 + \tfrac{1}{2}\mathbf{g}h^2\right] = 0$$

Energy loss:

$$c\left[\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2 + bh\right] \ - \ \left[\tfrac{1}{2}u^3 + \mathbf{g}uh(h+b)\right] = \frac{1}{\rho Y}\Delta E$$

Weak asymptotics for traveling hydraulic jump

Let $\rho \in C_c^\infty(\mathbb{R})$ be non-negative, smooth, compactly supported even function with $\text{supp}\rho \subset (-1,1)$ and $\int_{\mathbb{R}} \rho(z)dz = 1$
Define $C = \int_{\mathbb{R}} \rho^2(z)dz$, and

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x-ct-4\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x-ct+4\varepsilon}{\varepsilon}\right),$$

$$R_\varepsilon(x,t) = \frac{i}{2\varepsilon}\rho\left(\frac{x-ct-2\varepsilon}{\varepsilon}\right) - \frac{i}{2\varepsilon}\rho\left(\frac{x-ct+2\varepsilon}{\varepsilon}\right),$$

$$S_\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{C}}\rho\left(\frac{x-ct}{\varepsilon}\right)$$

$$U_\varepsilon(x,t) = \begin{cases} u_l, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon \le x \le ct + 10\varepsilon, \\ u_r, & x \ge ct + 20\varepsilon, \end{cases}$$

$$H_\varepsilon(x,t) = \begin{cases} h_l, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon \le x \le ct + 10\varepsilon, \\ h_r, & x \ge ct + 20\varepsilon. \end{cases}$$

Now make the ansatz

$$h_\varepsilon(x,t) = H_\varepsilon(x-ct),$$

$$u_\varepsilon(x,t) = U_\varepsilon(x-ct) + \alpha(t)(\delta_\varepsilon(x-ct) + R_\varepsilon(x-ct)) + \sqrt{c\alpha(t)}S_\varepsilon(x-ct)$$

We can show that

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \mathbf{g}\partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon \underbrace{- c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2} = o_{\mathcal{D}'}(1)$$

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \mathbf{g}\partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon = o_{\mathcal{D}'}(1)$$

Choosing

$$\alpha'(t) = (u_r - u_l)c + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l)$$

h_ε and u_ε are solutions of shallow water (1), (2) in the sense of Definition 3.

Note that the Rankine-Hugoniot deficit is nonzero:

$$\alpha'(t) = \mathbf{g}\Delta H \neq 0$$

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$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \mathbf{g}\partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon \underbrace{-c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2} = o_{\mathcal{D}'}(1)$$

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h_ε and u_ε are solutions of shallow water (1), (2) in the sense of Definition 3.

Note that the Rankine-Hugoniot deficit is nonzero:

$$\alpha'(t) = \mathbf{g}\Delta H \neq 0$$

Bottom step transition

For a bottom step, mass and energy need to be conserved.
Rankine-Hugoniot conditions are

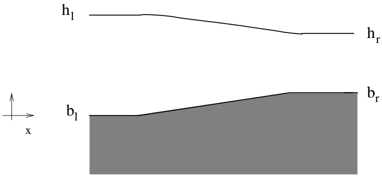
$$[uh] \quad = 0 \tag{5}$$

$$\left[guh(h+b) + h\frac{u_r^3}{2}\right] \quad = 0 \tag{6}$$

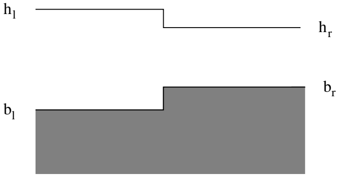
The second condition can be replaced by the simpler condition

$$\left[g(h+b) + \frac{u^2}{2}\right] = 0 \tag{7}$$

These are standard relations in hydraulic theory (Henderson, 1966).



Surface profile over a bottom transition



Shallow-water approximation

Bottom step transition

Mass conservation:

$$c\,[h] \;-\; [uh] = 0$$

Conservation of total head:

$$c\,[u] \;-\; \left[g(h+b) + \frac{u^2}{2}\right] = 0$$

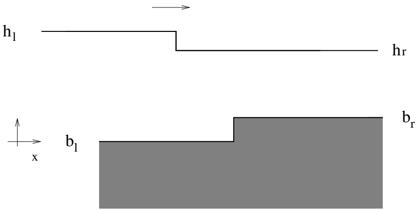
Momentum loss:

$$c\,[hu] \;-\; \left[hu^2 + \tfrac{1}{2}gh^2\right] \neq 0$$

Energy balance:

$$\left[\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2 + bh\right] \;-\; \left[\tfrac{1}{2}u^3 + guh(h+b)\right] = 0$$

Interaction Shock / Bottom step



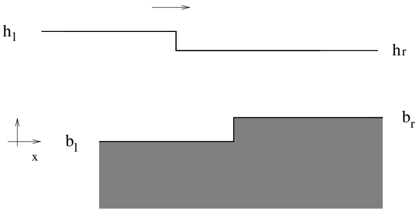
Initial configuration: Shock approaching a bottom step

Solve this using conservation of mass and total head:

$$h_t + (hu)_x = 0 \quad (1)$$

$$u_t + g(h+b)_x + uu_x = 0 \quad (2)$$

Interaction Shock / Bottom step



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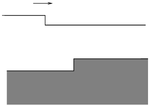
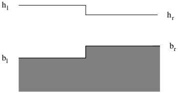
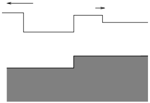


Initial configuration



Riemann problem over bottom step



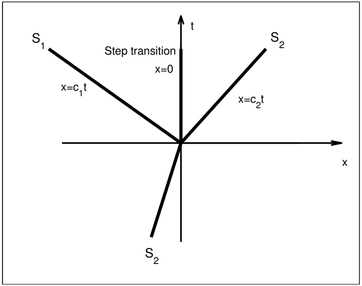
Solution

Interaction Shock / Bottom step

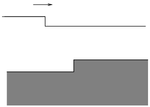
Solution of Riemann problem over bottom step can be solved by requiring

- Energy conservation across bottom step
- Momentum conservation across flat bottom

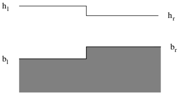
Alcrudo and Benkhaldoun, Computers and Fluids, 2001



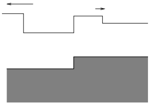
Shock trajectories



Initial configuration



Riemann problem over bottom step

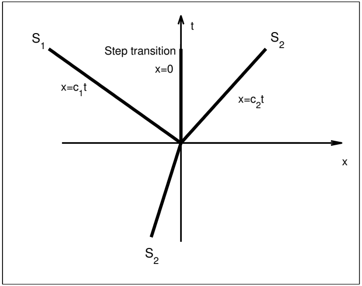


Solution

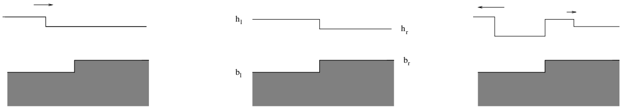
Different approaches due to

G. Rosatti and L. Begnudelli,
J. Comp. Physics, 2010

Fjordholm et al.
J. Comp. Physics, 2011



Shock trajectories



Initial configuration   Riemann problem over bottom step   Solution

Solution of Riemann problem for Brio systen

$$\left.\begin{array}{rl} u_t + \frac{1}{2}\left(u^2 + v^2\right)_x = 0 & = \\ v_t + (uv - u)_x = 0 & = \end{array}\right\} \quad (8)$$

Definition

A δ-shock solution of the Brio system (8), connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$ is i-admissible if

$$\lambda_i(u_2, v_2) \leq c \leq \lambda_i(u_1, v_1), \quad (9)$$

for $i = 1$ or $i = 2$. For such δ shock wave we say that it is compressive.

Lemma

Assume Riemann data are such that $u_1 = u_2 = \tilde{u}$, $v_1 = 0$ *and* $v_2 < 0$. *Then, the* δ*-shock solution*

$$\begin{aligned} u(x,t) &= \tilde{u} + \alpha(t)\delta(x - ct), \\ v(x,t) &= 0, \end{aligned} \quad (10)$$

where $\alpha(t)$ *and* c *are given by Thm 1, is a* 1*-admissible* δ*-shock solution of* (8).

Solution of Riemann problem for Brio systen

Proof.

The functions given by (10) represent δ shock solution to (8), (9) according to Theorem 1, b). In order to prove that the solution is 1-admissible, recall that $c = \frac{v_2(u_2-1)-v_1(u_1-1)}{v_2-v_1}$. Then, due to (9), we need to show:

$$\lambda_1(u_2, v_2) = u_2 - 1/2 - \sqrt{1/4 + v_2^2} \leq \frac{v_2(u_2-1) - v_1(u_1-1)}{v_2 - v_1}$$
$$\leq u_1 - 1/2 - \sqrt{1/4 + v_1^2} = \lambda_1(u_1, v_1).$$

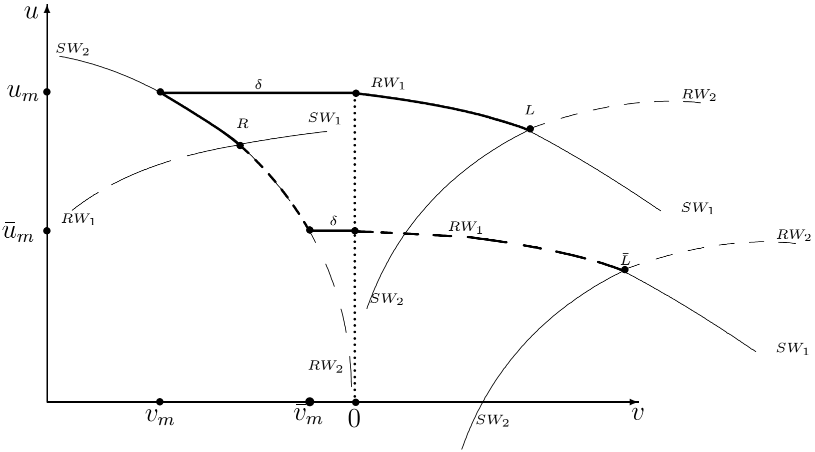
Since $u_1 = u_2 = u$ and $v_1 = 0$, the latter reduces to

$$u - 1/2 - \sqrt{1/4 + v_2^2} \leq u - 1 \quad \Leftarrow \quad 1/2 - \sqrt{1/4 + v_2^2} \leq 0,$$

which is clearly true. □

Solution of Riemann problem for Brio systen

H. K. and D. Mitrovic,
Proc. Edinburgh Math. Soc., 2012



Solution of Cauchy problem for $n \times n$ system

a) We split the x-axis into the intervals $(x^j_{\Delta x}, x^{j+1}_{\Delta x})$ of length Δx.

b) We approximate the functions U^0_1 and U^0_2 by the functions
$U_{01,\Delta x}(x) = \sum_{j=1}^{\infty} v^j_{1,\Delta x}\chi_{(x^j_{\Delta x}, x^{j+1}_{\Delta x})}(x)$ and $U^0_{2,\Delta x}(x) = \sum_{j=1}^{\infty} v^j_{2,\Delta x}\chi_{(x^j_{\Delta x}, x^{j+1}_{\Delta x})}(x)$,
respectively, which are piecewise constant on the intervals defined in item a).
For $j \in \mathbb{N}$, denote by

$$C^i_j = \frac{f_i(v^{j+1}_{1,\Delta x}, v^{j+1}_{2,\Delta x}) - f_i(v^j_{1,\Delta x}, v^j_{2,\Delta x})}{v^j_{i,\Delta x} - v^j_{i,\Delta x}} = \frac{[f_i]^j}{[v_i]^j}, \quad i = 1, 2$$

i.e. the speeds given by the Rankine-Hugoniot conditions of the first and second equations ($i = 1, 2$).

c) We thus obtain series of Riemann problems (at the edge of each interval from item a)) which we solve using Definition 1 such that c_j, satisfies the minimum amplitude condition

$$c_j = \begin{cases} C^1_j, & |C^1_j[v_2]^j - [f_2]^j| \leq |C^2_j[v_1]^j - [f_1]^j| \\ C^2_j, & else \end{cases}$$

The δ-distribution is adjoined to the function for which the minimum is not reached.

d) We obtain the family $(u^1_{\Delta x}, u^2_{\Delta x})$ of distributions which can be represented as a sum of bounded functions $U^i_{\Delta x}$ and δ-distributions, satisfying Definition 1