On symmetry and decay of traveling wave solutions to the Whitham equation

Gabriele Bruell

Norwegian University of Science and Technology, Trondheim

Joint work with

M. Ehrnström and L. Pei

BIRS Workshop on Theoretical and Computational Aspects of Nonlinear Surface Waves

Banff, November 2016

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right) u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Dispersion relation for the KdV equation:

$$c(\xi) = 1 - \frac{1}{6}\xi^2.$$

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Dispersion relation for the KdV equation:

$$c(\xi) = 1 - \frac{1}{6}\xi^2.$$

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Dispersion relation for the KdV equation:

$$c(\xi) = 1 - \frac{1}{6}\xi^2.$$

The KdV equation can be written more complicated as

$$u_t + 2uu_x + \mathcal{F}^{-1}(c(\xi)) * u_x = 0.$$

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Dispersion relation for the KdV equation:

$$c(\xi) = 1 - \frac{1}{6}\xi^2.$$

The KdV equation can be written more complicated as

$$u_t + 2uu_x + \mathcal{F}^{-1}(c(\xi)) * u_x = 0.$$

Exact dispersion relation for the linearized Euler equation:

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}} = 1 - \frac{1}{6}\xi^2 + \dots$$

Korteweg–de Vries (KdV) equation

$$u_t + 2uu_x + \left(1 + \frac{1}{6}\partial_x^2\right)u_x = 0$$

- Local, nonlinear equation, modeling shallow water waves.
- Valid approximation for long waves.
- Fails to be a valid approximation for relatively short waves.

Dispersion relation for the KdV equation:

$$c(\xi) = 1 - \frac{1}{6}\xi^2.$$

The KdV equation can be written more complicated as

$$u_t + 2uu_x + \mathcal{F}^{-1}(c(\xi)) * u_x = 0.$$

Exact dispersion relation for the linearized Euler equation:

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}} = 1 - \frac{1}{6}\xi^2 + \dots$$

Gerald B. Whitham introduced in 1967 the following equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

- Nonlocal, nonlinear equation.

Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

- Nonlocal, nonlinear equation.
- Valid approximation for modeling relatively short waves in shallow water. [Moldabayev, Kalisch, Dutykh '15]

Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

- Nonlocal, nonlinear equation.
- Valid approximation for modeling relatively short waves in shallow water. [Moldabayev, Kalisch, Dutykh '15]
- Existence of: periodic traveling waves [Ehrnström, Kalisch '09] , solitary waves [Ehrnström, Groves, Wahlén '12] , wave breaking [Naukin, Shishmarev '94, Constantin, Escher '98, Hur '15] , cusped waves [Ehrnström, Wahlén '16] .

Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

- Nonlocal, nonlinear equation.
- Valid approximation for modeling relatively short waves in shallow water. [Moldabayev, Kalisch, Dutykh '15]
- Existence of: periodic traveling waves [Ehrnström, Kalisch '09] , **solitary waves** [Ehrnström, Groves, Wahlén '12] , wave breaking [Naukin, Shishmarev '94, Constantin, Escher '98, Hur '15] , cusped waves [Ehrnström, Wahlén '16] .

Symmetry and decay of solitary waves:

Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

- Nonlocal, nonlinear equation.
- Valid approximation for modeling relatively short waves in shallow water. [Moldabayev, Kalisch, Dutykh '15]
- Existence of: periodic traveling waves [Ehrnström, Kalisch '09] , **solitary waves** [Ehrnström, Groves, Wahlén '12] , wave breaking [Naukin, Shishmarev '94, Constantin, Escher '98, Hur '15] , cusped waves [Ehrnström, Wahlén '16] .

<u>Symmetry and decay of solitary waves:</u>

Solitary solutions to the Euler equations [Craig, Sternberg '88] :
Any *supercritical* solitary solution satisfies:

- it decays exponentially fast at $\pm\infty$.
- it is symmetric and has exactly one crest.

Main findings for the Whitham equation:

1. Any solitary wave solution decays exponentially.

2. Any solitary wave solution is symmetric and has exactly one crest.

Main findings for the Whitham equation:

1. Any solitary wave solution decays exponentially.

2. Any solitary wave solution is symmetric and has exactly one crest.

3. Any symmetric, classical, unique solution is traveling.

1. Any solitary wave solution decays exponentially.

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Let $u(t,x) = \phi(x - ct)$ be a solitary solution, then

$$-c\phi + \phi^2 + \mathcal{F}^{-1}(m(\xi)) * \phi = 0,$$

where $c > 1$ is the wave speed.

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Let $u(t,x) = \phi(x - ct)$ be a solitary solution, then

$$-c\phi + \phi^2 + \mathcal{F}^{-1}(m(\xi)) * \phi = 0,$$

where $c > 1$ is the wave speed.

Theorem (Bona, Li '97)

If ϕ is a solitary solution of $\phi = H * G(\phi)$ and

- $|G(\phi)| \leq C|\phi|^r$ for some $r > 1$,
- $e^{\delta|\cdot|}H \in L_2(\mathbb{R})$ for some $\delta > 0$,

$$\implies \quad \phi \text{ has exponential decay.}$$

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Let $u(t,x) = \phi(x - ct)$ be a solitary solution, then

$$-c\phi + \phi^2 + \mathcal{F}^{-1}(m(\xi)) * \phi = 0,$$

where $c > 1$ is the wave speed.

Theorem (Bona, Li '97)

If ϕ is a solitary solution of $\phi = H * G(\phi)$ and

- $|G(\phi)| \leq C|\phi|^r$ for some $r > 1$,
- $e^{\delta|\cdot|}H \in L_2(\mathbb{R})$ for some $\delta > 0$,

$$\Longrightarrow \quad \phi \text{ has exponential decay.}$$

The steady Whitham equation can be written as

$$\phi\,(c - \phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c - m}\right).$$

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Let $u(t,x) = \phi(x - ct)$ be a solitary solution, then

$$-c\phi + \phi^2 + \mathcal{F}^{-1}(m(\xi)) * \phi = 0,$$

where $c > 1$ is the wave speed.

Theorem (Bona, Li '97)

If ϕ is a solitary solution of $\phi = H * G(\phi)$ and

- ✔ $|G(\phi)| \leq C|\phi|^r$ for some $r > 1$,
- ▶ $e^{\delta|\cdot|}H \in L_2(\mathbb{R})$ for some $\delta > 0$,

$\implies$ ϕ has exponential decay.

The steady Whitham equation can be written as

$$\phi\,(c - \phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c - m}\right).$$

Exponential decay

Whitham equation:

$$u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$$

Let $u(t,x) = \phi(x - ct)$ be a solitary solution, then

$$-c\phi + \phi^2 + \mathcal{F}^{-1}(m(\xi)) * \phi = 0,$$

where $c > 1$ is the wave speed.

Theorem (Bona, Li '97)

If ϕ is a solitary solution of $\phi = H * G(\phi)$ and

✔ $|G(\phi)| \leq C|\phi|^r$ for some $r > 1$,

✘ $e^{\delta|\cdot|}H \in L_2(\mathbb{R})$ for some $\delta > 0$,

$$\overset{?}{\Longrightarrow} \quad \phi \text{ has exponential decay.}$$

The steady Whitham equation can be written as

$$\phi\,(c - \phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c - m}\right).$$

Exponential decay

$$\phi(c-\phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c-m}\right).$$

Exponential decay

$$\phi(c-\phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c-m}\right).$$

1. $0 < \phi < c \quad \Rightarrow \quad (c-\phi)$ is bounded!

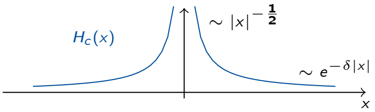
Exponential decay

$$\phi(c-\phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c-m}\right).$$

1. $0 < \phi < c \quad \Rightarrow \quad (c-\phi)$ is bounded!
2. The kernel H_c satisfies the following properties:

Theorem (Properties of H_c)

i) *H_c is even, smooth away from zero, monotonically decreasing on* $(0, \infty)$.
ii) $H_c(x) = O(|x|^{-\frac{1}{2}})$ *as* $|x| \to 0$.
iii) $x \mapsto xe^{\delta|\cdot|}H_c \in L_2(\mathbb{R})$ *for some* $\delta_c \in (0, \frac{\pi}{2})$.



Idea of Proof

$$\phi(c-\phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c-m}\right).$$

i) Set $h(x) = \frac{m(\sqrt{x})}{c-m(\sqrt{x})}$ and $f(x) = h(x^2)$.

Lemma (Ehrnström, Wahlén '16)

If h is completely monotone with $\lim_{x\to 0} h(x) < \infty$ and $\lim_{x\to\infty} h(x) = 0$, then $\mathcal{F}(f)$ is a positive, integrable function, which is smooth and monotone outside of the origin.

iii) Use a Paley–Wiener type theorem for $g = \partial_x\left(\frac{m}{c-m}\right)$.

Paley–Wiener type theorem

If $g \in L_2(\mathbb{R})$ and there exists $\delta_c > 0$ such that g is holomorphic in the complex strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < \delta_c\}$ and $\sup_{y<\delta_c} \|g(\cdot + iy)\|_{L_2(\mathbb{R})} < \infty$, then $e^{\delta_c|\cdot|}\mathcal{F}(g) \in L_2(\mathbb{R})$.

$\Rightarrow x \mapsto ixe^{\delta_c|x|}H_c(x)$ belongs to $L_2(\mathbb{R})$.

Exponential decay

$$\phi(c-\phi) = H_c * \phi^2, \qquad H_c := \mathcal{F}^{-1}\left(\frac{m}{c-m}\right),$$

Corollary

The convolution kernel H_c satisfies:

- $|\cdot|^\alpha H_c \in L_p(\mathbb{R})$ *iff* $\alpha > \frac{1}{2} - \frac{1}{p}$. *In particular,* $H_c \in L_p(\mathbb{R})$ *for* $p \in [1,2)$.
- $e^{\delta|\cdot|} H_c \in L_p(\mathbb{R})$ *for* $p \in [1,2)$ *and* $\delta < \delta_c$.

Following the approach of Bona and Li (1997), we prove the main result.[1]

Theorem (Exponential Decay)

Let δ_c be the exponential decay rate of H_c. Then any solitary solution to the Whitham equation decays exponentially fast:

$$e^{\eta|\cdot|}\phi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \qquad \textit{for some} \quad \eta \geq \delta.$$

[1] Bona, Li: $e^{\delta|\cdot|} H \in L_2(\mathbb{R})$

2. Any solitary solution is symmetric and has exactly one crest.

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

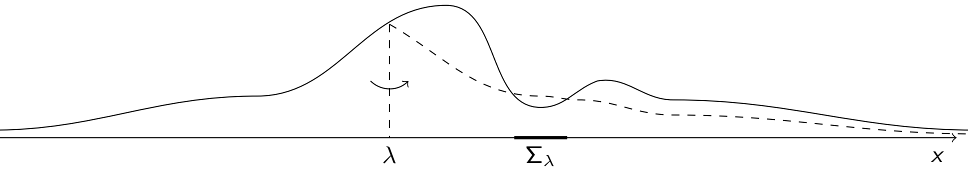
Sketch of proof: Method of moving planes [Chen, Li, Ou '05].

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Sketch of proof: Method of moving planes [Chen, Li, Ou '05].



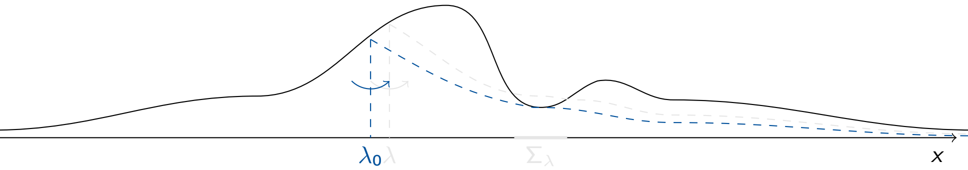
$$\Sigma_\lambda := \{x \geq \lambda \mid \phi(x) - \phi(2\lambda - x) < 0\}$$

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Sketch of proof: Method of moving planes [Chen, Li, Ou '05].



$$\Sigma_\lambda := \{x \geq \lambda \mid \phi(x) - \phi(2\lambda - x) < 0\}$$

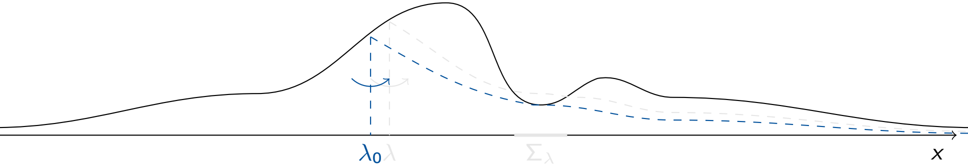
- Using decay result: There exists λ_* such that $\Sigma_\lambda = \emptyset$ for all $\lambda \leq \lambda_*$.

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Sketch of proof: Method of moving planes [Chen, Li, Ou '05].



$$\Sigma_\lambda := \{x \geq \lambda \mid \phi(x) - \phi(2\lambda - x) < 0\}$$

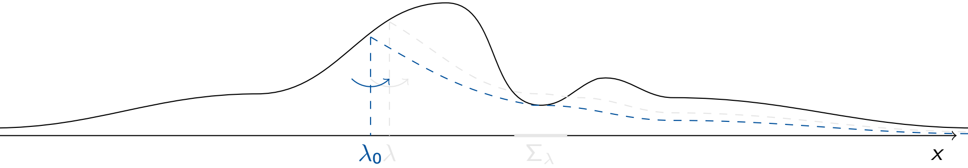
- Using decay result: There exists λ_* such that $\Sigma_\lambda = \emptyset$ for all $\lambda \leq \lambda_*$.
- Move the plane $x = \lambda$ to the right as long as $\phi(x) \geq \phi(2\lambda - x)$ on $[\lambda, \infty)$.

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Sketch of proof: Method of moving planes [Chen, Li, Ou '05].



$$\Sigma_\lambda := \{x \geq \lambda \mid \phi(x) - \phi(2\lambda - x) < 0\}$$

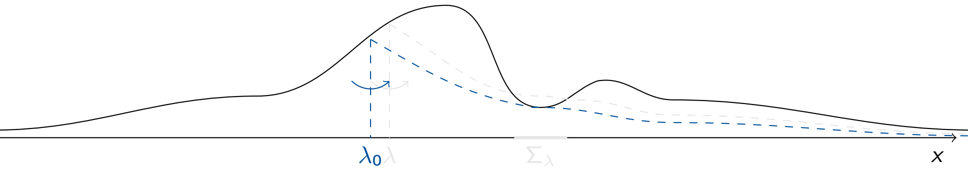
- Using decay result: There exists λ_* such that $\Sigma_\lambda = \emptyset$ for all $\lambda \leq \lambda_*$.
- Move the plane $x = \lambda$ to the right as long as $\phi(x) \geq \phi(2\lambda - x)$ on $[\lambda, \infty)$.
- This process stops at some point λ_0.

Symmetry of solitary waves

Theorem (Symmetry of solitary waves)

If ϕ is a solitary wave solution to the Whitham equation, then it is symmetric and has exactly one crest.

Sketch of proof: Method of moving planes [Chen, Li, Ou '05].



$$\Sigma_\lambda := \{x \geq \lambda \mid \phi(x) - \phi(2\lambda - x) < 0\}$$

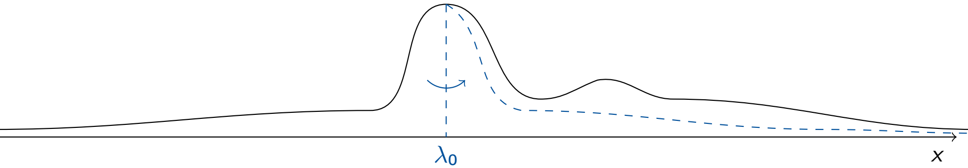
- Using decay result: There exists λ_* such that $\Sigma_\lambda = \emptyset$ for all $\lambda \leq \lambda_*$.
- Move the plane $x = \lambda$ to the right as long as $\phi(x) \geq \phi(2\lambda - x)$ on $[\lambda, \infty)$.
- This process stops at some point λ_0.
- Touching lemma: There can not be a touching point unless ϕ is symmetric.

Symmetry of solitary waves

Hence, we have that

- ϕ is symmetric or
- $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).

Assume that $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).

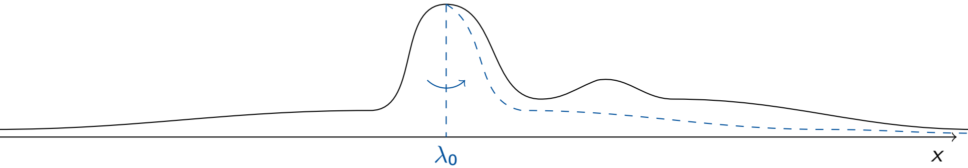


Symmetry of solitary waves

Hence, we have that

- ϕ is symmetric or
- $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).

Assume that $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).



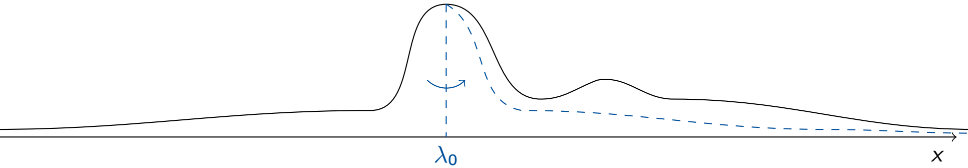
- We show that there exists $\varepsilon > 0$ such that $\Sigma_\lambda = \emptyset$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.

Symmetry of solitary waves

Hence, we have that

- ϕ is symmetric or
- $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).

Assume that $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).



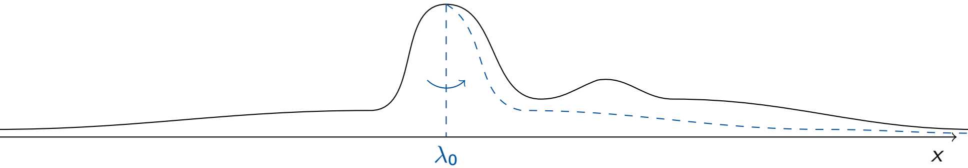
- We show that there exists $\varepsilon > 0$ such that $\Sigma_\lambda = \emptyset$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.
- Plane $x = \lambda$ can be moved further to the right. Contradition!

Symmetry of solitary waves

Hence, we have that

- ϕ is symmetric or
- $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).

Assume that $\phi(x) > \phi(2\lambda_0 - x)$ on (λ_0, ∞).



- We show that there exists $\varepsilon > 0$ such that $\Sigma_\lambda = \emptyset$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.
- Plane $x = \lambda$ can be moved further to the right. Contradition!
- ϕ is symmetric and has exactly one crest.

3. Any classical, symmetric, unique solution is traveling.

Symmetric solutions

We say that a function u is symmetric in x, if

$$u(t,x) = u(t, 2\lambda(t) - x),$$

where $\lambda \in C^1(\mathbb{R})$ the axis of symmetry.

Symmetric solutions

We say that a function u is symmetric in x, if

$$u(t,x) = u(t, 2\lambda(t) - x),$$

where $\lambda \in C^1(\mathbb{R})$ the axis of symmetry.

Theorem (Symmetric solutions are traveling)

Any classical, symmetric, unique solution of the Whitham equation is a traveling wave solution, that is

$$u(t,x) = u_0(x - ct).$$

- $u_0 = u(0,x)$*: initial datum.*
- c*: speed of the propagating wave.*

Symmetric solutions

We say that a function u is symmetric in x, if

$$u(t,x) = u(t, 2\lambda(t) - x),$$

where $\lambda \in C^1(\mathbb{R})$ the axis of symmetry.

Theorem (Symmetric solutions are traveling)

Any classical, symmetric, unique solution of the Whitham equation is a traveling wave solution, that is

$$u(t,x) = u_0(x - ct).$$

- $u_0 = u(0,x)$*: initial datum.*
- c*: speed of the propagating wave.*

The statement holds true for a large class of partial differential equations.

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

then u is a traveling wave solution.

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Whitham equation (nonlocal): $u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Whitham equation (nonlocal): $\quad u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$

- "odd number of x–derivatives in each term" .

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Whitham equation (nonlocal): $u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$

- "odd number of x–derivatives in each term" .
- On the Fourier side: each term is odd in ξ.

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Whitham equation (nonlocal): $u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$

- "odd number of x–derivatives in each term" .
- On the Fourier side: each term is odd in ξ.
- $\mathcal{F}(\mathcal{F}^{-1}(m) * u_x) = m(\xi)\mathcal{F}(u_x) = m(\xi)i\xi\hat{u}$ is odd in ξ (m even).

Theorem (Local Principle, M. Ehrnström, H. Holden, X. Raynaud '09)

If u is a classical, symmetric, unique solution of

$$P(\partial_x)u_t + F(u) = 0,$$

where $F(u) = F(u, u_x, u_{xx}, \ldots)$ is odd in the sense that

$$F(u, -u_x, u_{xx}, -u_{xxx} \ldots) = -F(u, u_x, u_{xx}, u_{xxx} \ldots),$$

and P is an even polynomial, then u is a traveling wave solution.

Whitham equation (nonlocal): $u_t + 2uu_x + \mathcal{F}^{-1}(m(\xi)) * u_x = 0.$

- "odd number of x–derivatives in each term" .
- On the Fourier side: each term is odd in ξ.
- $\mathcal{F}(\mathcal{F}^{-1}(m) * u_x) = m(\xi)\mathcal{F}(u_x) = m(\xi)i\xi\hat{u}$ is odd in ξ (m even).

Theorem (Nonlocal Principle, G.B., M. Ehrnström, A. Geyer, L. Pei)

If u is a symmetric, unique solution of $P(\partial_x)u_t + F(u) = 0$, P is even and $G(\hat{u}, \xi) = \mathcal{F}(F(u))$ satisfies

$$G(\hat{u}, -\xi) = -G(\hat{u}, \xi),$$

then u is a traveling wave solution.

Summary

For the Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m) * u_x = 0, \qquad m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}},$$

the following holds true:

Summary

For the Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m) * u_x = 0, \qquad m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}},$$

the following holds true:

1. u solitary wave solution $\Rightarrow$ u decays exponentially.
2. u solitary wave solution $\Rightarrow$ u is symmetric and has exactly one crest.

Summary

For the Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m) * u_x = 0, \qquad m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}},$$

the following holds true:

1. u solitary wave solution $\Rightarrow$ u decays exponentially.
2. u solitary wave solution $\Rightarrow$ u is symmetric and has exactly one crest.
3. u classical, symmetric, unique solution $\Rightarrow$ u is a traveling wave.

Summary

For the Whitham equation

$$u_t + 2uu_x + \mathcal{F}^{-1}(m) * u_x = 0, \qquad m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}},$$

the following holds true:

1. u solitary wave solution $\Rightarrow$ u decays exponentially.
2. u solitary wave solution $\Rightarrow$ u is symmetric and has exactly one crest.
3. u classical, symmetric, unique solution $\Rightarrow$ u is a traveling wave.

Thank you very much!