Two proofs of the existance of Ramanujan graphs

Spectral Algorithms Workshop, Banff

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Acknowledgements:

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Outline

Motivation

In spectral graph theory, we are interested in eigenvalues of some matrix (Laplacian, adjacency, etc).

Since graphs are a union of edges, there is a natural way to think of these matrices as a sum of smaller matrices.

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In this talk, I will discuss the idea of adding randomly chosen matrices to other matrices and then show how the idea can be used to show existance of Ramanujan graphs in two ways.

Well-known techniques exist for bounding the eigenvalues of random sums of matrices.

Theorem (Matrix Chernoff, for example)

Let $\widehat{v}_1, \dots, \widehat{v}_n$ *be independent random vectors with* $\|\widehat{v}_i\| \leq 1$ *and* $\sum_i \widehat{v}_i \widehat{v}_i^T = \widehat{V}$*. Then*

$$\mathbb{P}\left[\lambda_{max}(\widehat{V}) \leq \theta\right] \geq 1 - d \cdot e^{-nD(\theta \| \lambda_{max}(\mathbb{E}\widehat{V}))}$$

Similar inequalities by Rudelson (1999), Ahlswede–Winter (2002).

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Similar inequalities by Rudelson (1999), Ahlswede–Winter (2002).

All such inequalities have two things in common:

1. They give results with *high probability*
2. The bounds depend on the dimension

This will *always* be true — tight concentration (in this respect) depends on the dimension (consider n/d copies of basis vectors).

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Furthemore, I want to keep the "probabilistic" nature:

Theorem

If $\widehat{\theta}$ is a random variable with finite support, then

$$\mathbb{P}\left[\widehat{\theta} \geq \mathbb{E}\widehat{\theta}\right] > 0 \quad \textit{and} \quad \mathbb{P}\left[\widehat{\theta} \leq \mathbb{E}\widehat{\theta}\right] > 0$$

In other words, I want to study one object (here $\mathbb{E}\widehat{\theta}$) and then be able to assert the existence of something at least as good (in both directions).

In fairy-tale land

So given a random frame $\widehat{V} = \sum_i \widehat{v}_i \widehat{v}_i^T$, I would like to say:

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So instead, we make an observation:

Observation

The eigenvalues of matrix are the roots of its characteristic polynomial. That is, if A is a $d \times d$ real, symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_d$, then

$$\chi_A(x) := \det[xI - A] = \prod_{i=1}^{d}(x - \lambda_i).$$

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Certainly this is nonsense, but let's play along with a toy problem:

Let A be a matrix and $\widehat{w}$ a random vector (taking values u or v uniformly).

What can we say about the eigenvalues of $A + \widehat{w}\widehat{w}^T$?

Still playing along

We would (naively) start by looking at the expected polynomial

$$p(x) = \frac{1}{2}\chi_{A+uu^T}(x) + \frac{1}{2}\chi_{A+vv^T}(x)$$

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Example: $p(x) = (x-2)^2 - 1$ (has double root at 1) and $q(x) = (x+2)^2 - 1$ (has double root at -1).

$$p(x) + q(x) = x^2 + 6$$

does not have any real roots (roots are $\pm\sqrt{-6}$).

Unless...

Lemma (Separation Lemma)

Let $p_1, \ldots, p_k$ *be polynomials and* $[s, t]$ *an interval such that*

- *Each* $p_i(s)$ *has the same sign (or is* 0*)*
- *Each* $p_i(t)$ *has the same sign (or is* 0*)*
- *each* p_i *has exactly one real root in* $[s, t]$*.*

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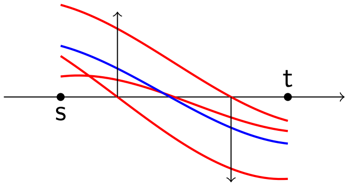
Proof.

By picture:



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What do I mean by "polynomial techniques"?

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Both inherit from recent work in polynomial geometry:

- Hyperbolic polynomials
- Stable polynomials

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In exchange for requiring extra structure, we are hoping to get some new "polynomial techniques" that we can use.

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Let p be a real rooted polynomial of degree d and q a real rooted polynomial of degree $d-1$

$$p(x) = \prod_{i=1}^{d}(x-\alpha_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{d-1}(x-\beta_i)$$

with $\alpha_1 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \cdots \leq \beta_{d-1}$.

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We say q *interlaces* p if $\alpha_1 \leq \beta_1 \leq \alpha_2 \cdots \leq \alpha_{d-1} \leq \beta_{d-1} \leq \alpha_d$.

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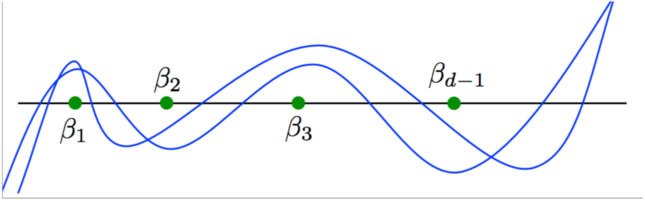
Example: $p'(x)$ interlaces $p(x)$.

Common Interlacer

We say that degree d real rooted polynomials $p_1, \ldots, p_k$ have a *common interlacer* if there exists a q such that q interlaces *every* p_i simultaneously.

Think: the roots of q split up $\mathbb{R}$ into d intervals, each of which contains exactly one root of each p_i.



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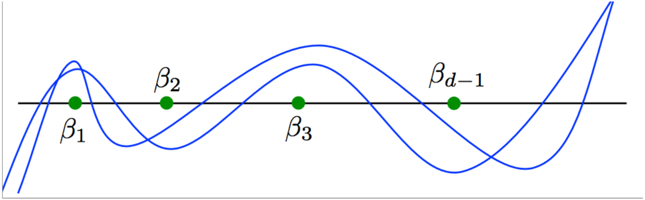

Note: if the p_i have a common interlacer (say q), then the intervals defined by the β_i can serve as separators for the lemma!

Back to the toy problem

Recall our goal was to understand the roots of

$$\begin{aligned} p(x) &= \frac{1}{2}\chi_{A+uu^T}(x) + \frac{1}{2}\chi_{A+vv^T}(x) \\ &= \frac{1}{2}q_0(x) + \frac{1}{2}q_1(x) \end{aligned}$$

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We will say that p forms an *interlacing star* with $\{q_i\}$ if

1. p and $\{q_i\}$ have the same degree and are all real rooted
2. The leading coefficients of the $\{q_i\}$ have the same sign
3. The collection of polynomials $\{q_i\}$ has a common interlacer
4. p is a convex combination of the $\{q_i\}$

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Corollary

If p forms an interlacing star with $\{q_i\}$, then there exist i, j such that

$$\mathrm{k}^{\mathrm{th}}\mathrm{root}\,(q_i) \leq \mathrm{k}^{\mathrm{th}}\mathrm{root}\,(p) \leq \mathrm{k}^{\mathrm{th}}\mathrm{root}\,(q_j)$$

More help from polynomials

Polynomial theory gives us a nice characterization of interlacing:

Lemma (Chudnovsky–Seymour, among others)

Let $\{p_i\}$ *be a collection of degree* d *polynomials. The following are equivalent:*

- *Every polynomial in the convex hull of* $\{p_i\}$ *has* d *real roots.*
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If we could show that

$$p(x) = \lambda\chi_{A+vv^T}(x) + (1-\lambda)\chi_{A+uu^T}(x)$$

was real rooted for all $\lambda \in [0, 1]$, then we would get the interlacing for free.

Back to reality

But remember we are interested in graphs — that is, sums of possibly *multiple* random vectors.

p_{00} p_{01} p_{10} p_{11}

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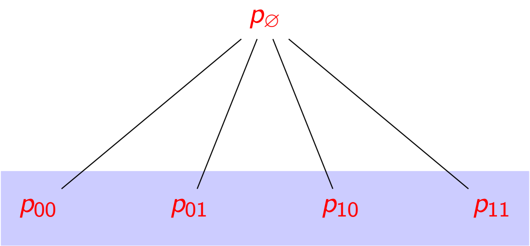

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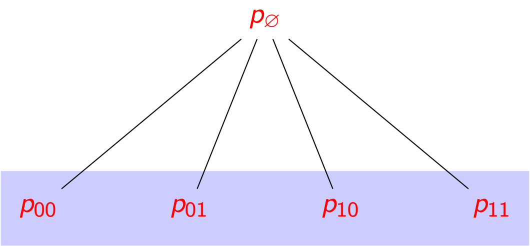

But in general they don't have a common interlacer...

Instead...

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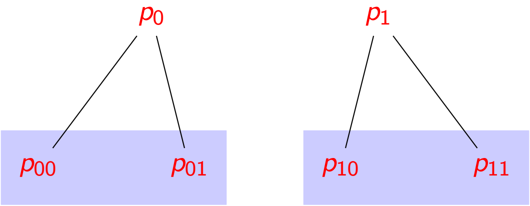
We can try to group them into smaller stars.



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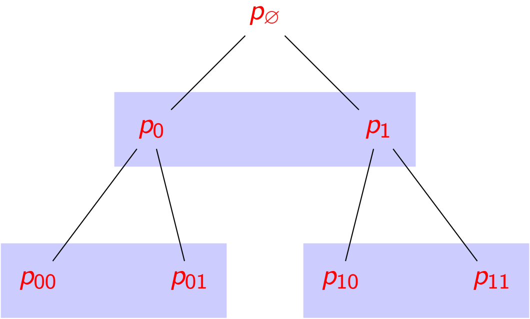

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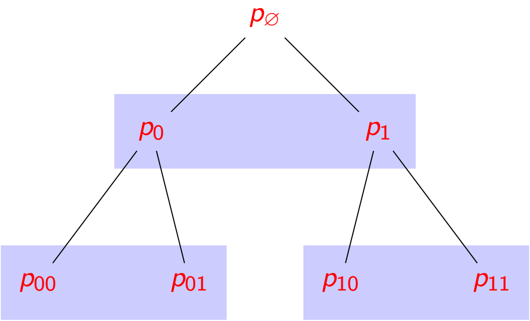

We will call a rooted, connected tree where each node forms an interlacing star with its children an *interlacing family*.

The punchline

Corollary

Every interlacing family contains leaf nodes p_{leaf_1} *and* p_{leaf_2} *such that*

$$\mathrm{k}^{\mathrm{th}}\mathrm{root}\left(p_{leaf_1}\right) \leq \mathrm{k}^{\mathrm{th}}\mathrm{root}\left(p_{root}\right) \leq \mathrm{k}^{\mathrm{th}}\mathrm{root}\left(p_{leaf_2}\right).$$

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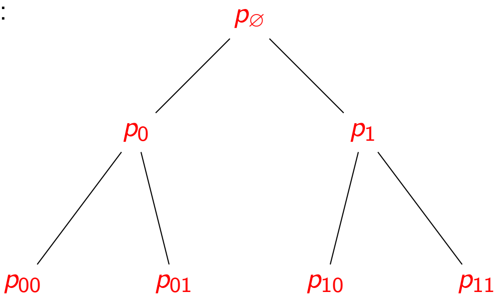

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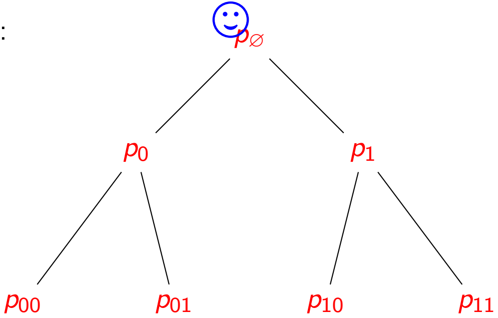

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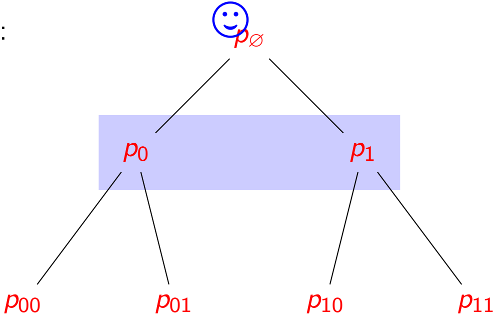

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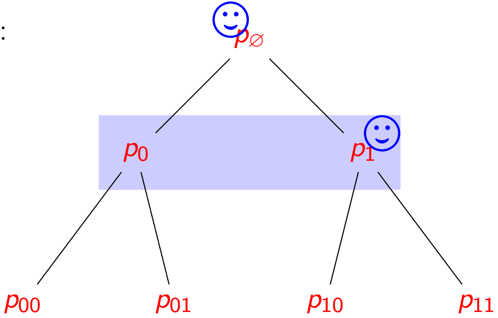

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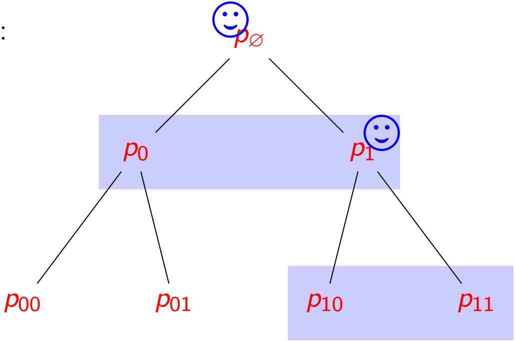

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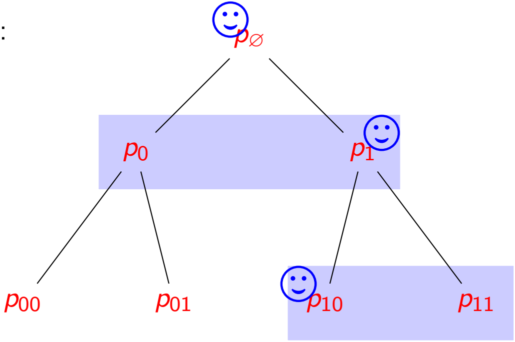

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where the $\widehat{v_i}$ have support size at most n.

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We will define a *choice vector* $\sigma \in [n]^m$ where σ_i is the index of a vector in the support of $\widehat{v}_i$. Then the characteristic polynomial of a fixed frame V in the support $\widehat{V}$ can be denoted

$$p_\sigma(x) = \chi_V(x).$$

Building an interlacing family

Consider the sum of random vectors

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We then define *partial choice* vectors $\sigma' \in [n]^k$ for $k < m$; the corresponding polynomial will be the conditional expectation:

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This forms an n-ary tree with fixed assignments at the leaves and $p_\varnothing = \mathbb{E}\left[\chi_{\widehat{V}}(x)\right]$ at the root.

Mixed characteristic polynomials

If we choose the vectors independently, the polynomials take a special form:

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Theorem

Let $\widehat{v}_1, \dots \widehat{v}_m$ be independent random vectors such that $\mathbb{E}\left[\widehat{v}_i \widehat{v}_i^T\right] = A_i$. Then

$$\mathbb{E}\left[\chi_{\widehat{V}}(x)\right] = \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right) \det\left[xI + \sum_{i=1}^{m} z_i A_i\right]\Bigg|_{z_1 = \dots = z_m = 0}$$

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We call this a *mixed characteristic polynomial* and denote it $\mu[A_1, \ldots, A_m]$.

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$$\begin{aligned} p_{\sigma'} &= \mathbb{E}_{\widehat{v}_{k+1}, \ldots, \widehat{v}_d}\left[\chi_{\widehat{V}}(x) \mid \widehat{v}_i = v_i^{\sigma'_i} \text{ for } 1 \le i \le k\right] \\ &= \mu[v_1 v_1^T, \ldots, v_k v_k^T, A_{k+1}, \ldots, A_m] \end{aligned}$$

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The advantage of having a multivariate formula is that we can utilize the theory of *real stable polynomials*, a multivariate extension of real rooted polynomials. Let
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An n-variate polynomial p is called *stable* if it is never 0 in $\mathbb{H}^n$. (i.e. if $p(z_1, \dots, z_n) = 0$, then some z_i has nonnegative imaginary part). If, in addition, all coefficients of p are real, it is called *real stable*.

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Two important properties:

- Univariate polynomials are real rooted if and only if they are real stable.
- Real stable polynomials are closed under substitution of reals $(z_1, z_2, \dots, z_n) \to (a, z_2, \dots, z_n)$ for $a \in \mathbb{R}$.

Similar to *hyperbolic polynomials*.

Real stable techniques

Real stability has been well studied in recent years. In particular,

Lemma

Let $A_1, \dots, A_m$ be Hermitian positive semidefinite matrices and $x_1 \dots x_m$ variables. Then

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Lemma

If $p(x_1, \ldots, x_m)$ is a multiaffine polynomial with real coefficients, then the following are equivalent:

1. *p is real stable*
2. $$\Delta_{ij}[p](x_1, \ldots, x_m) := \frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j} - \frac{\partial^2 p}{\partial x_i \partial x_j} p \geq 0$$

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Follows directly from the formula:

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□

This provides an easy way to generate interlacing families.

Corollary

Any tree of polynomials resulting from choosing independent random vectors forms an interlacing family.

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Hence we have a "probabilistic" way to deal with eigenvalues. That is, for any given k, let R be the k^{th} root of the *expected characteristic polynomial* (under whatever product distribution you want). Then there exists

1. an assignment of the random vectors that has $\lambda_k \geq R$
2. an assignment of the random vectors that has $\lambda_k \leq R$

Outline

Expanders

Let's switch gears to talk about expander graphs.

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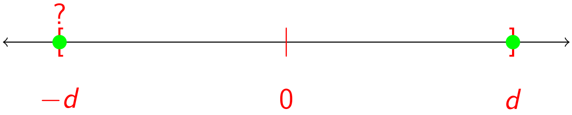
Can we get a finite graph to have all eigenvalues inside this interval?

Ramaujan graphs

No — the adjacency matrix of a d-regular graph has either 1 or 2 (so-called) *trivial* eigenvalues

1. d is always the largest eigenvalue
2. G is bipartite if and only if $-d$ is an eigenvalue



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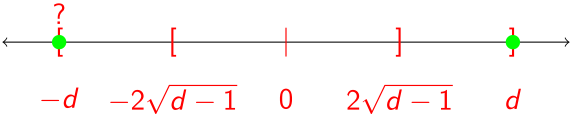

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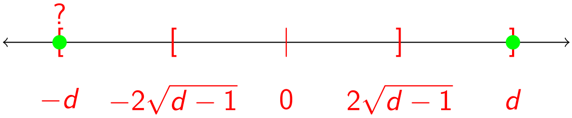

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Theorem (Alon, Boppana (1996))

No smaller interval can contain all nontrivial eigenvalues of an infinite collection of d-regular graphs.

Previous results

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On the other hand, almost everything is almost Ramanujan:

Theorem (Friedman (2008))

For fixed d, there is a large enough n such that a randomly chosen d-regular graph on n vertices have a nontrivial spectrum inside the interval

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Obvious question: are Ramanujan families really that special?

Idea 1: Covering space

It is easy to see that small graphs are Ramanujan.

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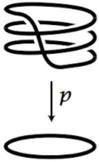
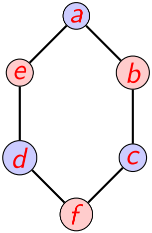
Their idea — construct a sequence of graphs from K_d (or $K_{d,d}$) to the d-infinite tree using a series of *lifts* (a common construction in algebraic topology).



Lifting

Start with a (Ramanujan) graph.



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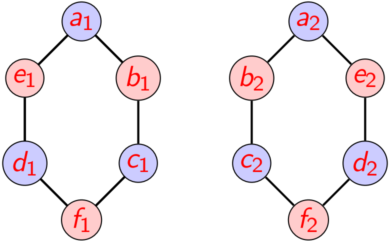

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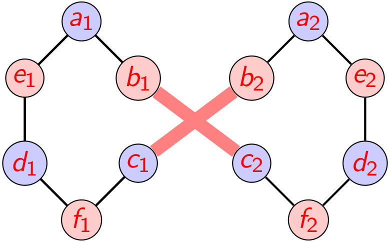

And make a copy (copies?) of it. And perturb it.

Want to find perturbations that cause new graph to be good.

2-lifts

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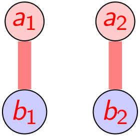

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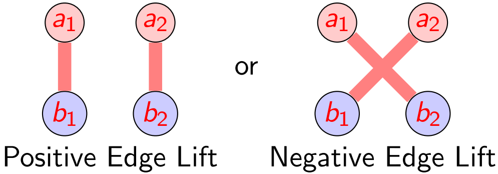

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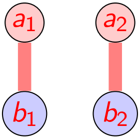
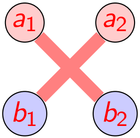


or



Positive Edge Lift Negative Edge Lift

The action of each 2-lift can be described by its *signed adjacency matrix* A_s:

$$\begin{matrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{matrix} \quad \rightarrow \quad \begin{matrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{matrix}$$

Main Eigenvalue lemma

Theorem (Bilu–Linial (2006))

Let G be a d-regular Ramanujan graph with n vertices and let s be a signing of G. If all eigenvalues of A_s lie in the interval

$$[-2\sqrt{d-1}, 2\sqrt{d-1}]$$

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We prove the conjecture for every *bipartite* graph G.

Bipartite Adjacency Matrices

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In this case, the signed adjacency matrix can be written in block form

$$\left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)$$

causing eigenvalues/vectors to come in pairs

$$v_i = [u_i \mid u_i] \quad \text{and} \quad v_{n-i} = [u_i \mid -u_i]$$

for $1 \leq i \leq n/2$ and so the eigenvalues satisfy $\lambda_i = -\lambda_{n-i}$.

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Corollary

A bipartite signed adjacency matrix A_s has all of its eigenvalues in the interval

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if and only if all of its eigenvalues are at most $2\sqrt{d-1}$.

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Hence it suffices to bound the largest root of the expected characteristic polynomial.

The expected characteristic polynomial

Theorem (Godsil–Gutman (1981))

For any graph G,

$$\mathbb{E}_{s \in \{\pm\}^m} \chi_{A_s}(x) = \mu_G(x),$$

the matching polynomial of G.

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In the original paper, Heilmann and Lieb also proved the following bound:

Theorem (Heilmann–Lieb (1972))

Let G be a graph with maximum degree Δ. Then

$$\mathrm{maxroot}\left(\mu_G\right) \leq 2\sqrt{\Delta - 1}$$

Piecing things together

Theorem

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Set G_{i+1} to be the 2-lift given by s^* — this is bipartite, d-regular, and (by Bilu and Linial) Ramanujan — and continue.

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Set $G_0 = K_{d,d}$ (which is bipartite, d-regular, and Ramanujan for any d). Given G_i, form G_{i+1} as follows:

For each possible signing $s \in \{\pm 1\}^{|E_i|}$, form the polynomials p_s. By our theorem, this is an interlacing family.

Combining this with Godsil–Gutman and Heilmann–Lieb ensures some p_{s^*} such that $\mathrm{maxroot}(p_{s^*}) \leq \mathrm{maxroot}(p_\varnothing) \leq 2\sqrt{d-1}$

Set G_{i+1} to be the 2-lift given by s^* — this is bipartite, d-regular, and (by Bilu and Linial) Ramanujan — and continue.

Extended to d-lifts by Hall, Puder, and Sawin (2015).

$\square$

Idea 2: Free group

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Can we build a Ramanujan graph this way?

What's happening

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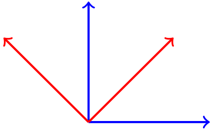
A union of graphs is a sum of matrices... and the eigenvalues of $A + B$ depend on

1. the eigenvalues of A
2. the eigenvalues of B
3. the dot product of the corresponding eigenvectors

To keep eigenvalues low, you want the eigenvectors to be as "orthogonal as possible"



Free probability

As always, we can try to add A and B randomly.

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In the infinite limit, "orthogonal as possible" happens with probability one — the corresponding eigenvalue distributions are known as the *free convolution*.

The free convolution of d copies of a Bernoulli random variable is known as the *Kesten–McKay law*:

$$d\mu(x) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} \mathbf{1}_{[\lambda_-, \lambda_+]} \, dx$$

where $\lambda_\pm = \pm 2\sqrt{d-1}$.

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The *finite free convolution* of p and q is

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where R is a Haar–distributed orthogonal matrix.

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The finite free convolution of real rooted polynomials always has real roots! (Borcea, Brändén)

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Theorem (Quadrature for Laplacian matrices)

Let A and B be $n \times n$ matrices such that $A\mathbb{1} = a\mathbb{1}$ and $B\mathbb{1} = b\mathbb{1}$ (where $\mathbb{1}$ is the all-1 vector) such that

$$\det[xI - A] = (x - a)p(x) \quad \text{and} \quad \det[xI - B] = (x - b)q(x).$$

Then

$$\mathbb{E}_P\left[\det\left[xI - A - PBP^T\right]\right] = (x - a - b)[p \boxplus_{n-1} q]$$

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Since this is also an expected characteristic polynomial, we can use interlacing families!

Interlacing Families

Theorem (Quadrature for Laplacian matrices, extended)

Let A and B be $n \times n$ matrices with $A\mathbb{1} = a\mathbb{1}$ and $B\mathbb{1} = b\mathbb{1}$ and

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Then

$$\mathbb{E}_P\left[\det\left[xI - A - PBP^T\right]\right] = (x - a - b)[p \boxplus_{n-1} q](x) := r(x)$$

and there exists a P_0 such that the largest root of

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is smaller than the largest root of $r(x)$.

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Proof uses random “swaps” to build an interlacing family.

Putting everything together

Lastly, we need to compare the free and finite free convolutions:

Theorem

The largest root of the finite free convolution of two eigenvalue distributions always lies inside the spectrum of the free convolution of those distributions.

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The largest root of the finite free convolution of two eigenvalue distributions always lies inside the spectrum of the free convolution of those distributions.

Putting everything together, gives

Theorem

For all d and all even n, there exists a d-regular bipartite Ramanujan graph on n vertices.

Disclosure: because of the bipartiteness, the actual proof requires a more complicated convolution, but the proof idea is similar.

Outline

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Typically, this is when I beg the audience to think about trying to make these polynomial techniques constructive.

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The Ramanujan graphs provided by the second proof are constructible in polynomial time.

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Theorem (Cohen, 2015)

The Ramanujan graphs provided by the second proof are constructible in polynomial time.

Still open:

1. Can the method of interlacing polynomials be extended to bound the largest root from above and the smallest root from below *simultaneously*?

2. Other connections to free probability and/or random matrix theory?

3. Find more interlacing families!!

Thanks

Thank you for inviting me to speak today.

Thanks

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And thank you for your attention!