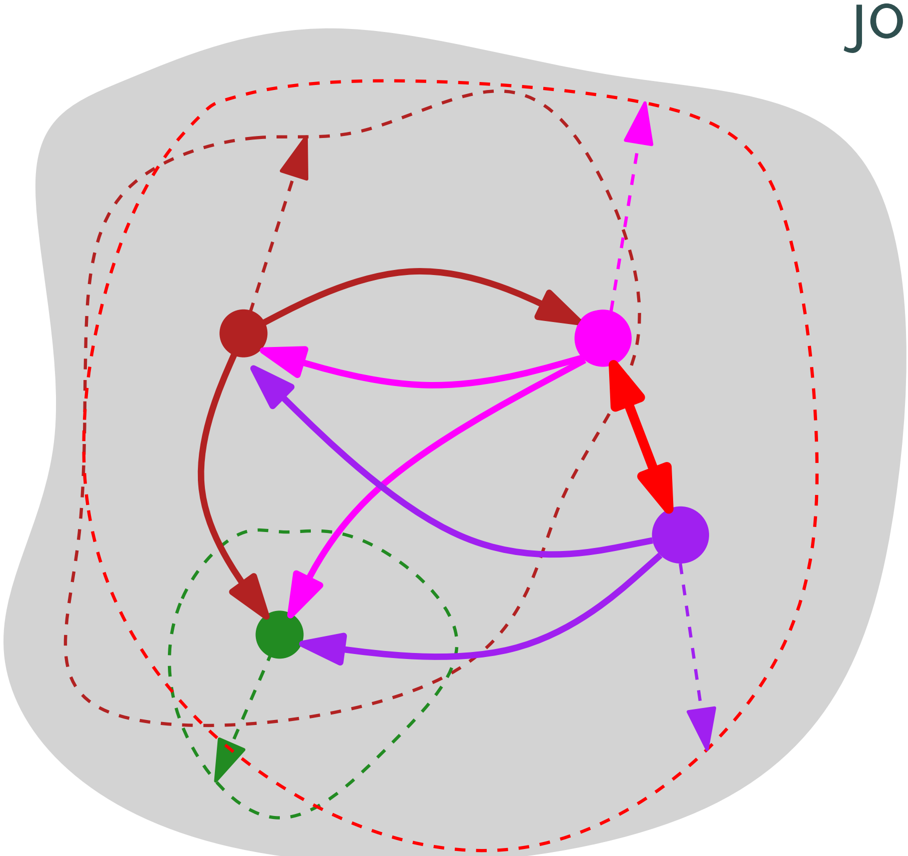
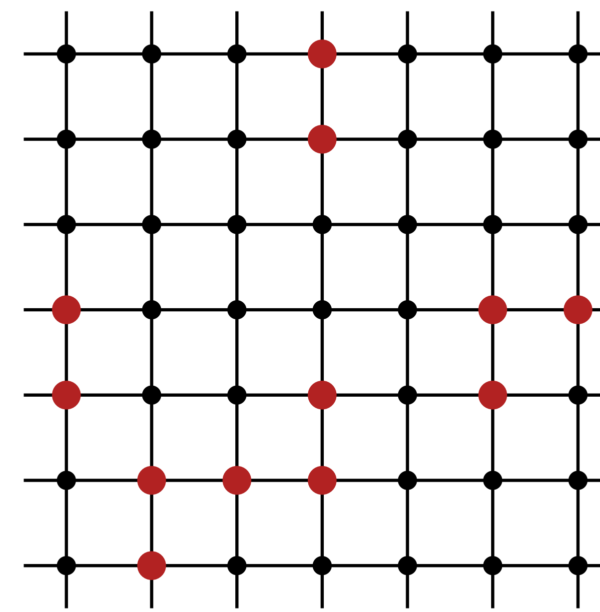
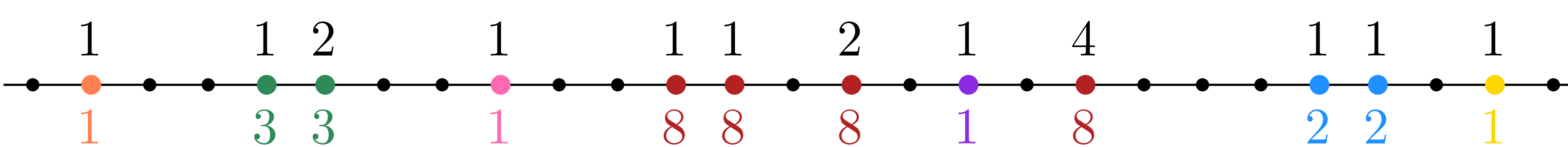
Percolation by cumulative merging and phase transition of the contact process

Laurent Ménard (Paris Nanterre)

joint work with **Arvind Singh** (Paris Orsay)





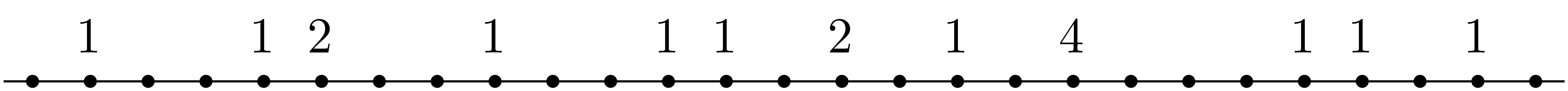


Outline

1. Cumulative merging
2. Phase transition for cumulative merging percolation
3. The contact process
4. Heuristics for the contact process
5. Link with cumulative merging

Cumulative Merging?

Take $G = (V, E, r)$ any locally finite connected weighted graph with $r : V \to [0, +\infty]$



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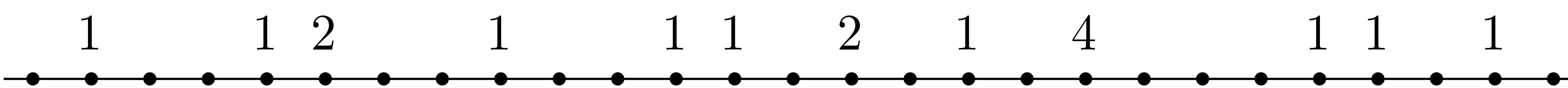

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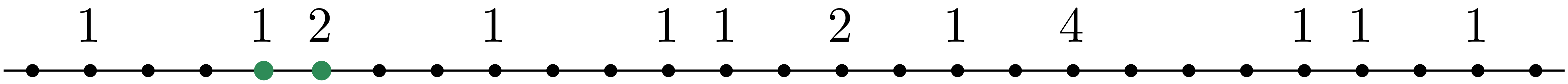

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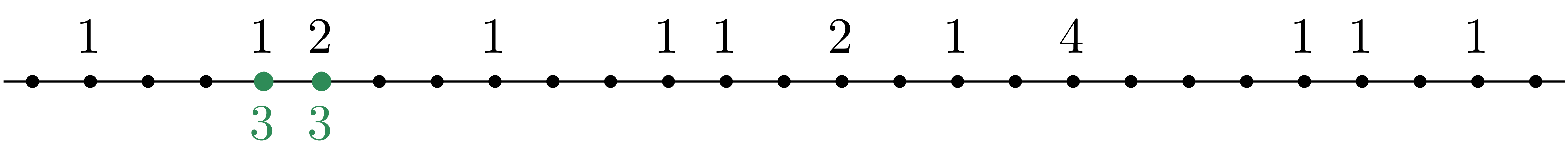

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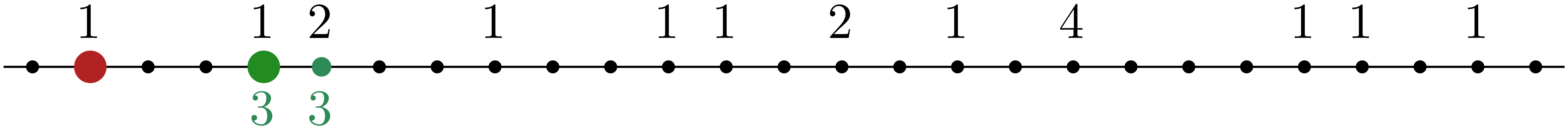

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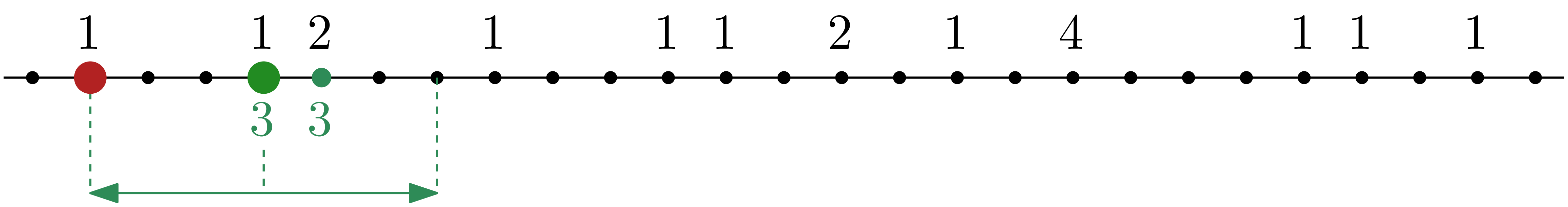

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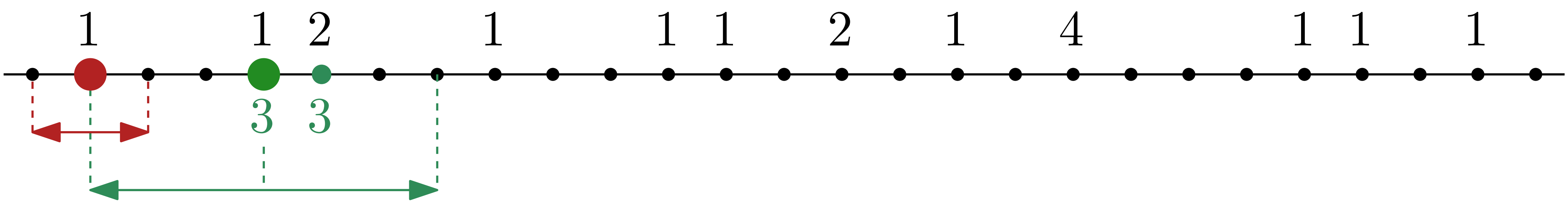

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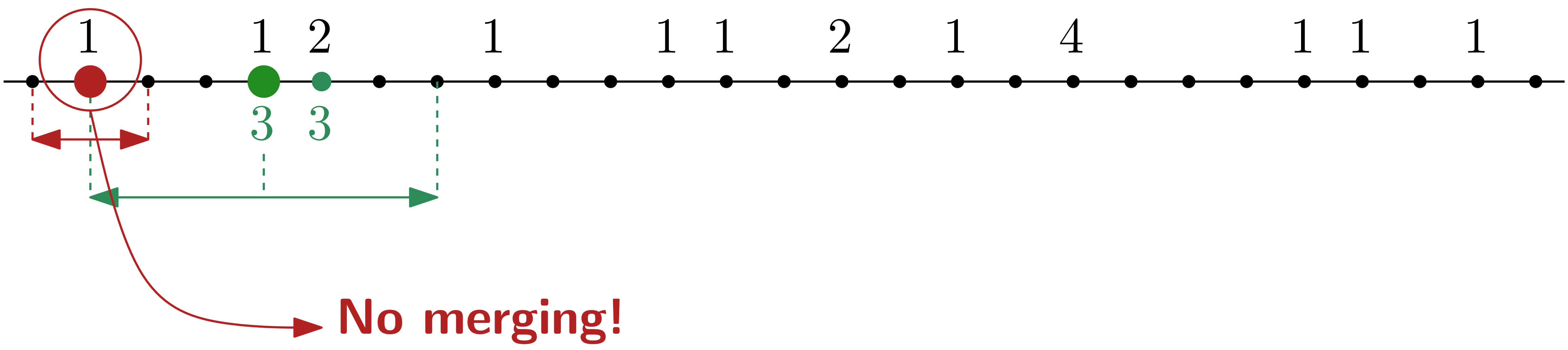

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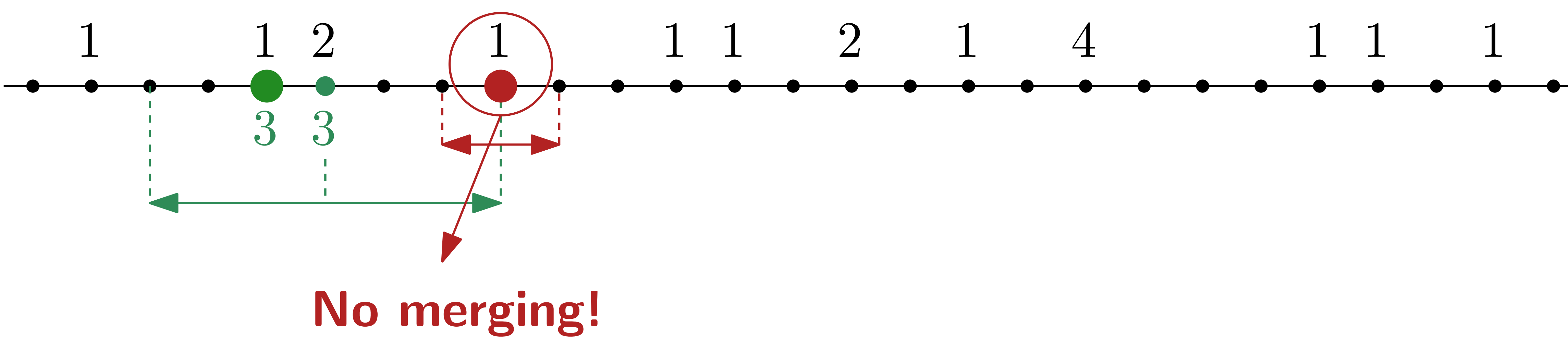

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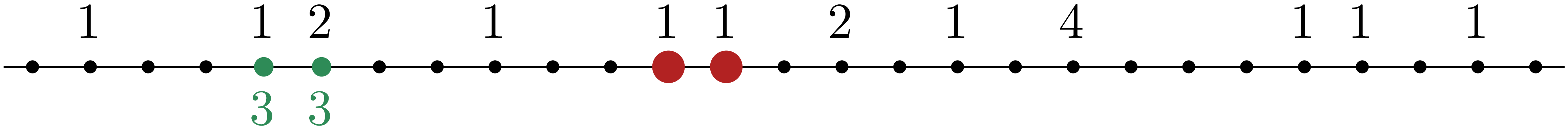

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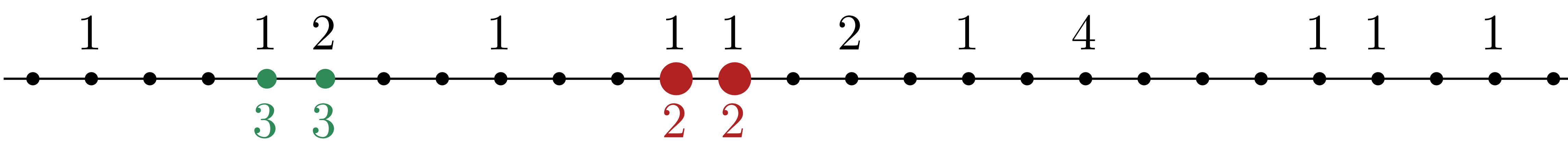

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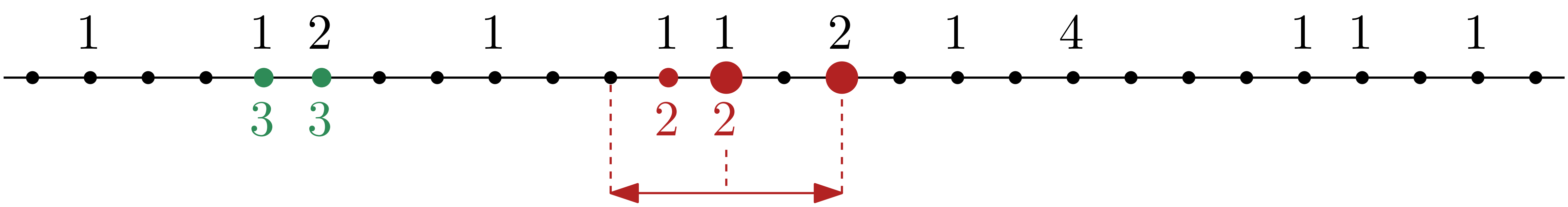

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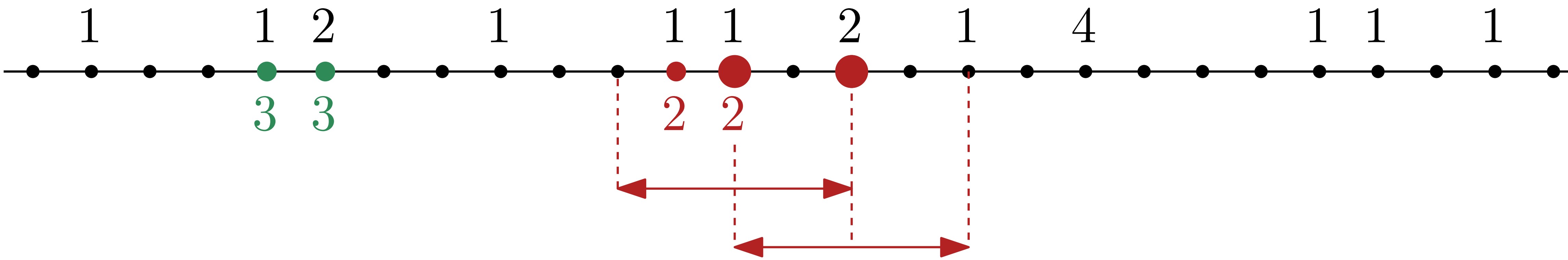

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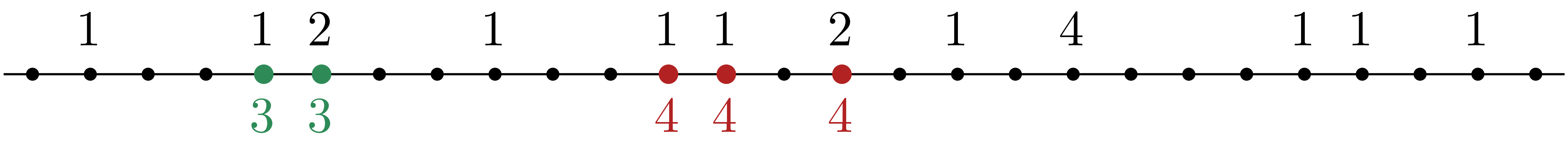

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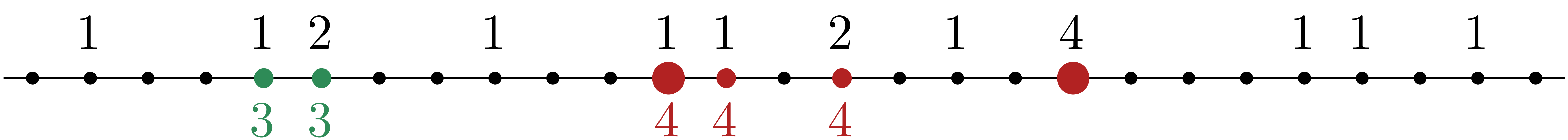

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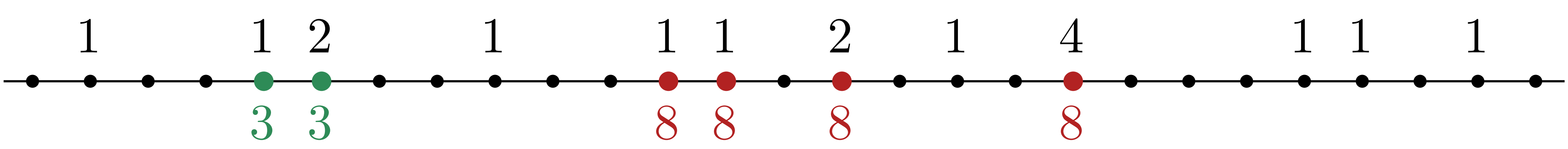

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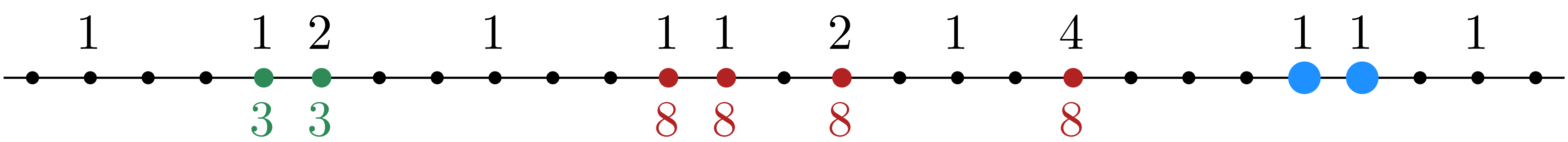

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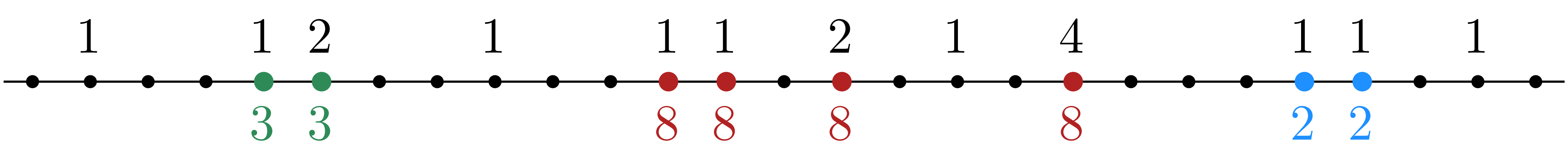

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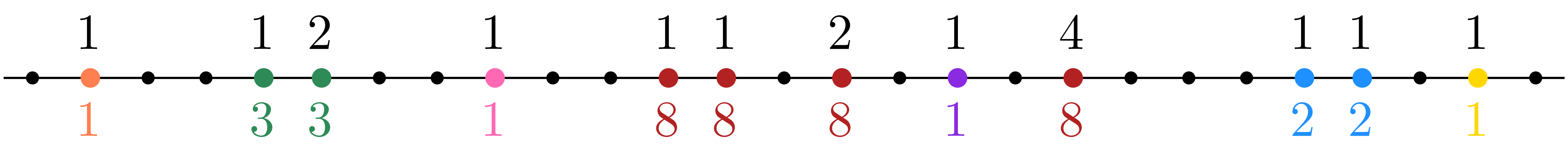

Cumulative Merging: Admissible partitions

Consider a weighted graph $G = (V, E, r)$ with $r : V \to [0, \infty]$.

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a partition $\mathscr{P}$ of V is **admissible** *iff* $\forall A \neq B \in \mathscr{P}$:

$$d_G(A, B) > r(A) \wedge r(B).$$

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- $\{V\}$ is admissible.
- If $\mathscr{P}_1$ and $\mathscr{P}_2$ are admissible, then

$$\mathscr{P}_1 \wedge \mathscr{P}_2 := \{C_1 \cap C_2 : C_1 \in \mathscr{P}_1, C_2 \in \mathscr{P}_2\}$$

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Definition

$$\mathscr{C}(G, r) := \bigwedge_{\text{admissible } \mathscr{P}} \mathscr{P} \qquad \text{(finest admissible partition)}$$

Merging operators

Definition: Merging operators

For $x \neq y \in V$, $M_{x,y}$: {partitions of V} $\to$ {partitions of V} defined by

$$M_{x,y}(\mathscr{P}) := \begin{cases} (\mathscr{P} \setminus \{\mathscr{P}_x, \mathscr{P}_y\}) \cup \{\mathscr{P}_x \cup \mathscr{P}_y\} & \text{if } \mathscr{P}_x \neq \mathscr{P}_y \text{ and} \\ & d(x,y) \leq r(\mathscr{P}_x) \wedge r(\mathscr{P}_y), \\ \mathscr{P} & \text{otherwise.} \end{cases}$$

for every partition $\mathscr{P}$, where $\mathscr{P}_x$ is the **cluster** of x in $\mathscr{P}$.

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Proposition:

The merging operators are monotone: for every $x \neq y \in V$ and every partitions $\mathscr{P}$ and $\mathscr{P}'$

- $\mathscr{P}$ is finer than $M_{x,y}(\mathscr{P})$;
- If $\mathscr{P}$ is finer than $\mathscr{P}'$, then $M_{x,y}(\mathscr{P})$ is finer than $M_{x,y}(\mathscr{P}')$.

Cumulative Merging

Proposition:
Take $(x_n, y_n) \in V^{\mathbb{N}} \times V^{\mathbb{N}}$ such that for every $x \neq y \in V$:

$$\{x_n, y_n\} = \{x, y\} \text{ for infinitely many } n.$$

Then

$$\mathscr{C}(V, E, r) = \lim_{n\to\infty} \uparrow \; M_{x_n, y_n} \circ \cdots \circ M_{x_1, y_1}(\bar{V})$$

where $\bar{V} = \{\{x\}\}_{x \in V}$ is the finest partition of V.

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Cumulatively Merged Partition (CMP)

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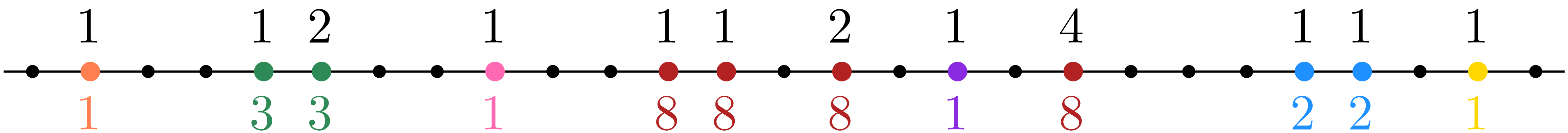
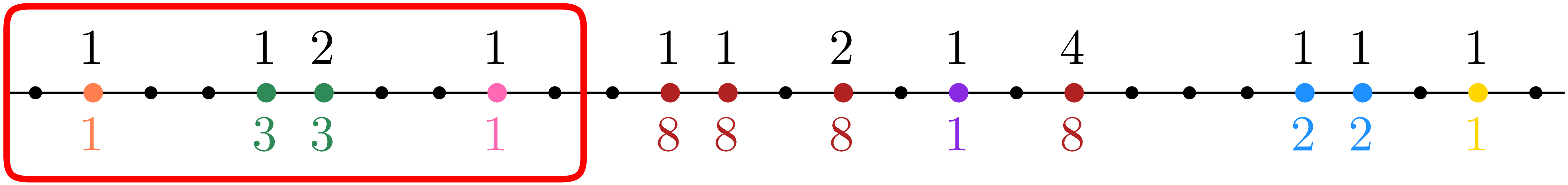
Proof: Monotonicity of the merging operators.

Cumulative Merging: Basic observations



- Clusters in $\mathscr{C}$ are not necessarily connected sets!
- If $r(x) < 1$, then $\{x\} \in \mathscr{C}$.
- If $\mathscr{C}$ has an infinite cluster, it has infinite weight and is unique.
- For any $C \in \mathscr{C}$, one has $|C| \leq \max\{1, r(C)\}$.

CMP: Stable sets



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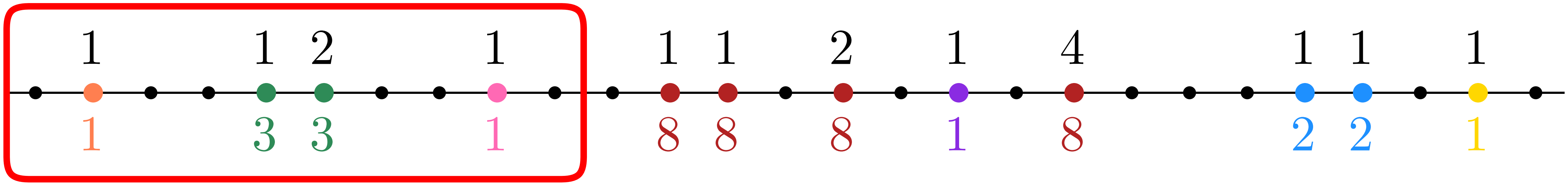

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Definition:
Fix $H \subset V$. We say that H is a **stable set** *iff*:

$$\forall C \in \mathscr{C}(H, E_H, r) \text{ one has } B\left(C, r(C)\right) \subset H.$$

Remark:

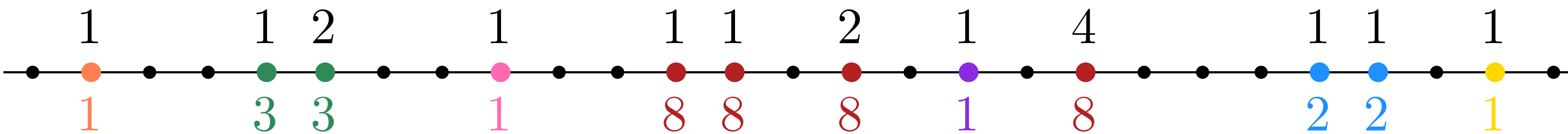
- Unions and intersections of stable sets are stable.
- Being stable is a **local** porperty.
- If H is stable, then

$$\mathscr{C}(G) = \mathscr{C}(H) \sqcup \mathscr{C}(G \setminus H).$$

CMP: Stabilisers

Definition

Let $H \subset V$ the **stabiliser** of H, $\mathcal{S}_H$, is the smallest stable set of vertices containing H.



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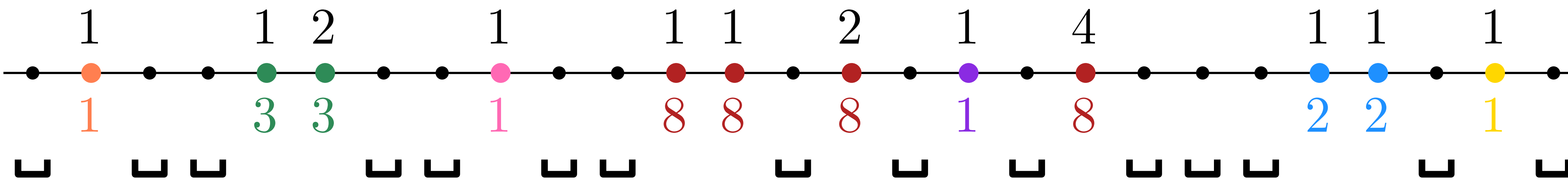

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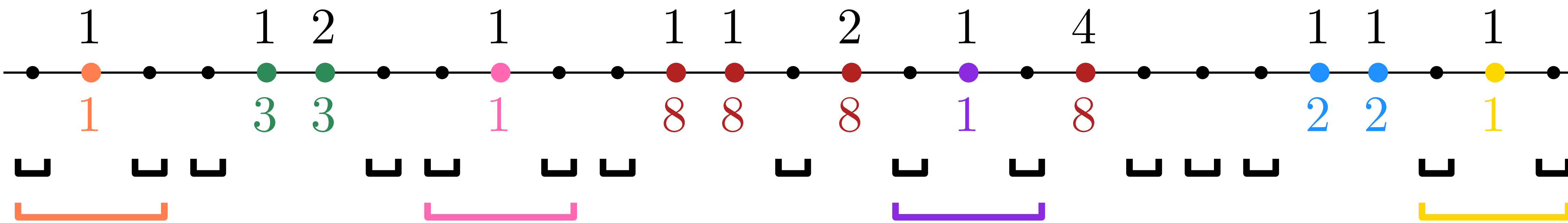

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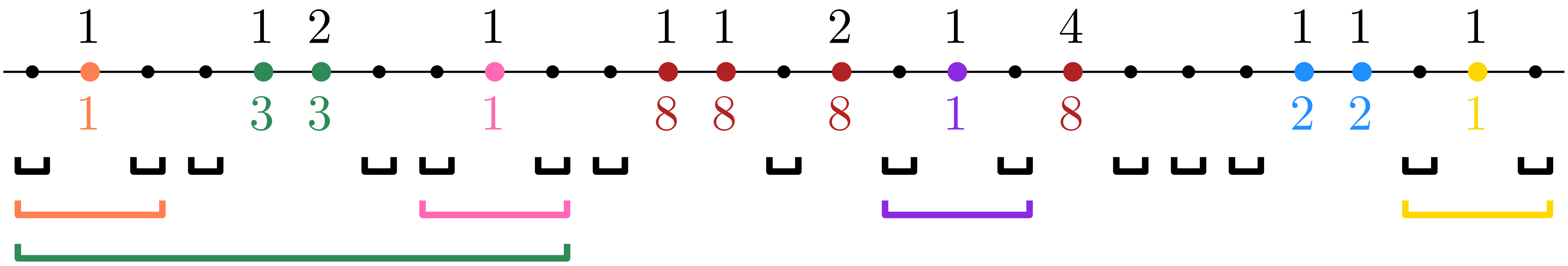

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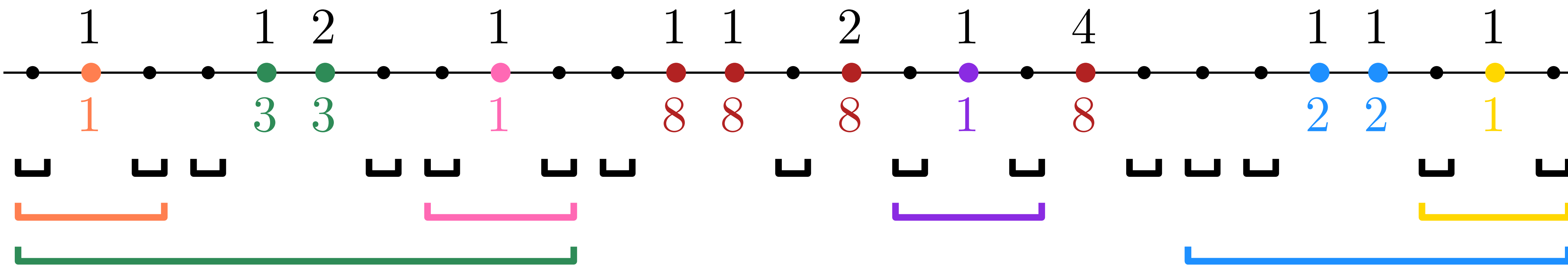

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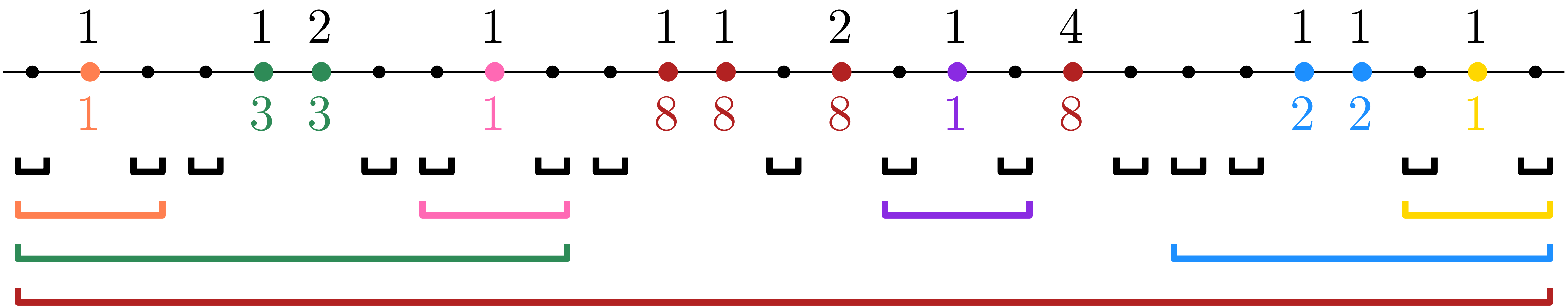

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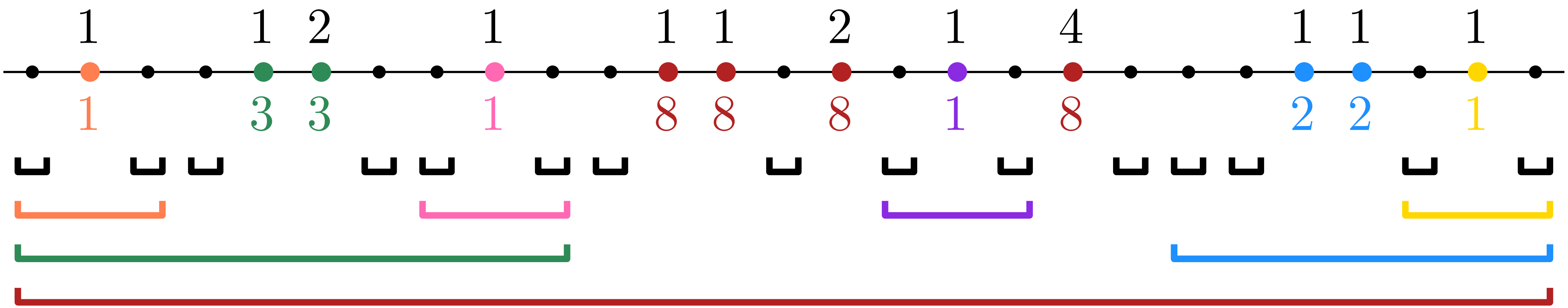

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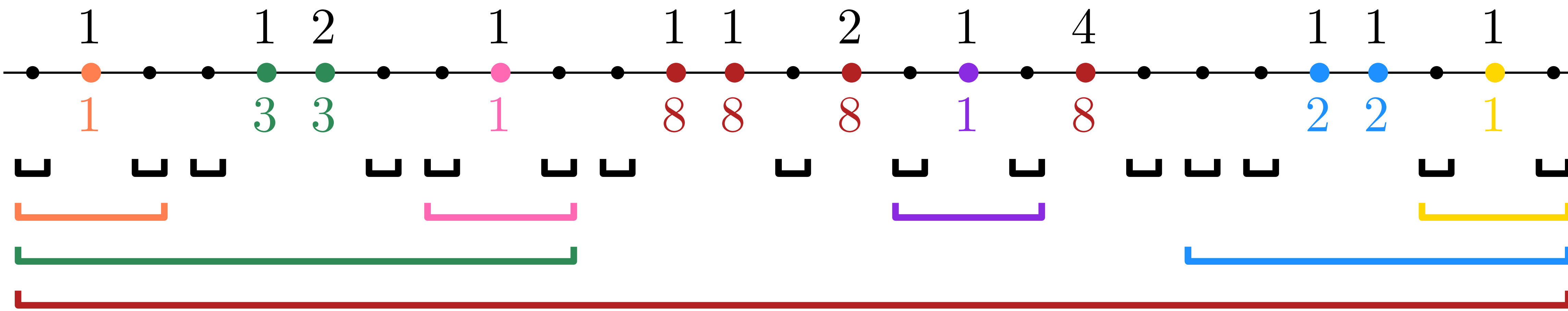

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Theorem

Suppose G infinite:

1. $\forall x \in V : |\mathscr{C}_x| = \infty \Leftrightarrow |\mathcal{S}_x| = \infty \Leftrightarrow \mathcal{S}_x = V$.
2. $\mathscr{C}$ has no infinite cluster *iff* there exists an increasing sequence of stable sets S_n *s.t.* $\lim \uparrow S_n = V$.

Cumulative Merging: Phase transitions

Several natural choices of weights for a locally finite connected graph (V,E):

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1. CMP on $\mathbb{Z}^d$: $p_c \in (0, 1)$.
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Proofs: Multiscale analysis

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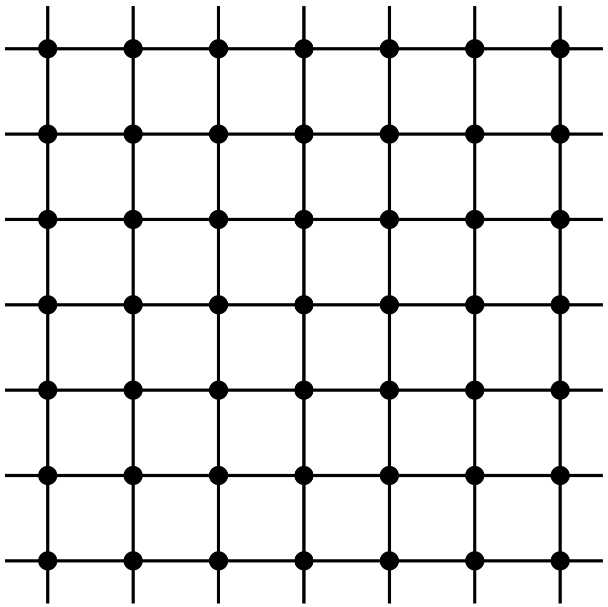
- Phase transition for CMP on a **tree**?

  1. Binary tree;
  2. Galton-Watson tree with light tails.

The contact process (Susceptible-Infected-Susceptible)

Epidemic model on graphs introduced by [Harris 74]



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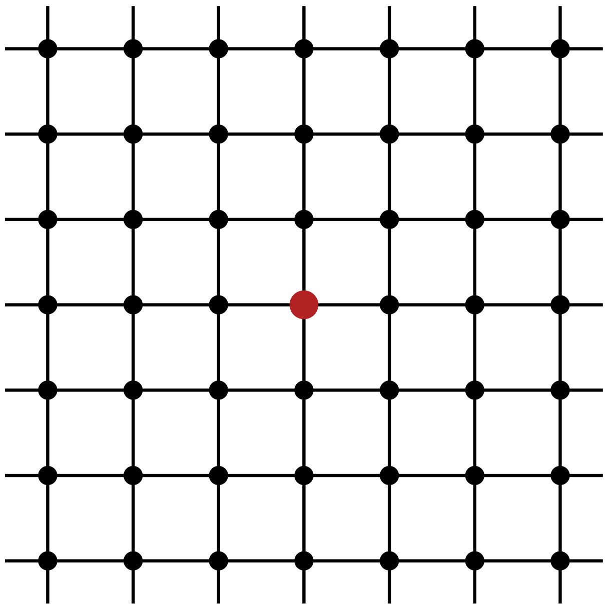

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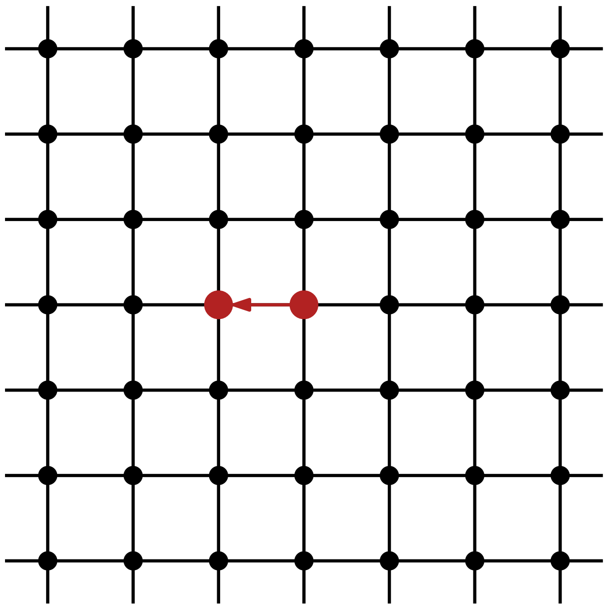

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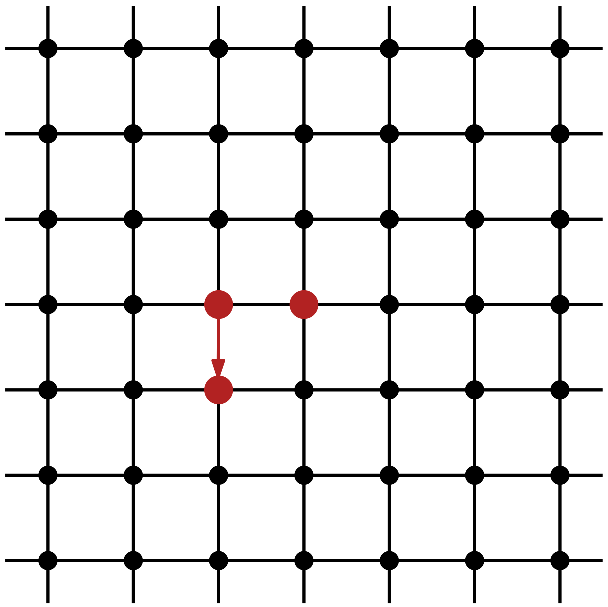

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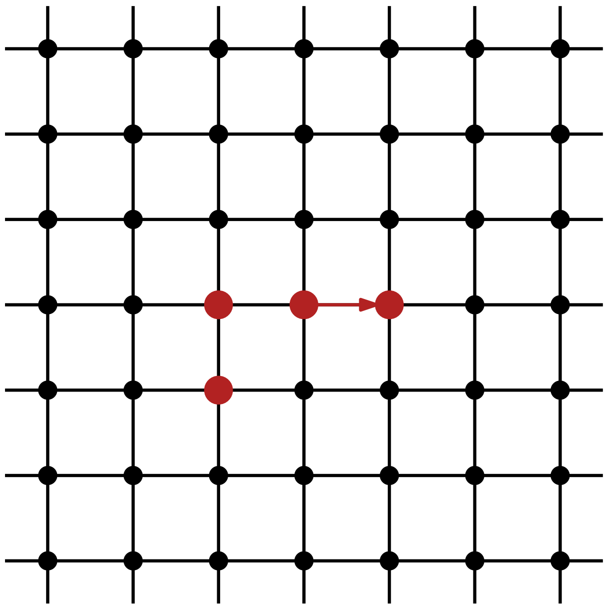

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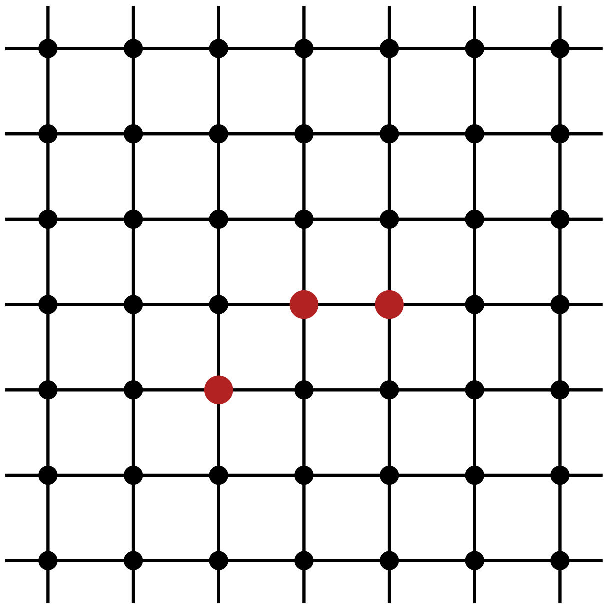

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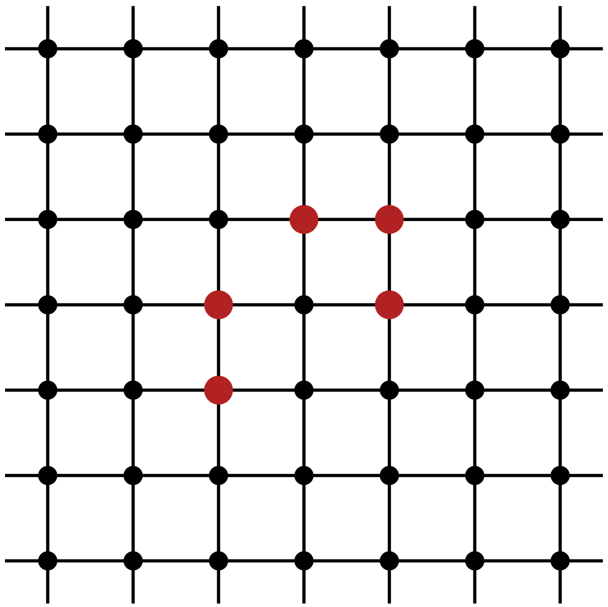

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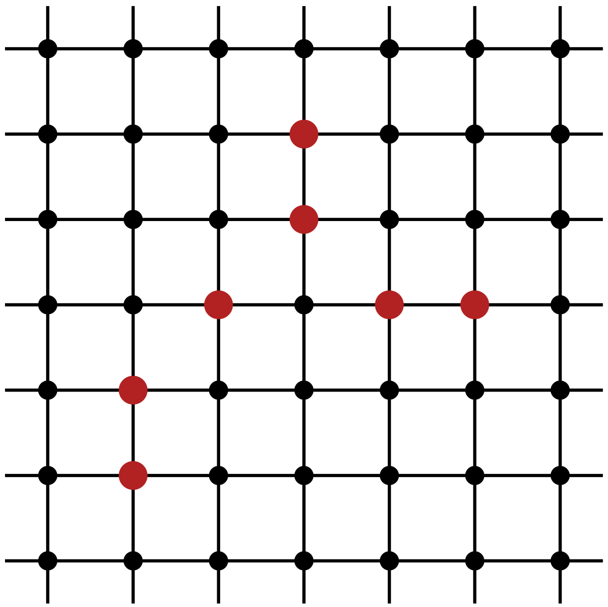

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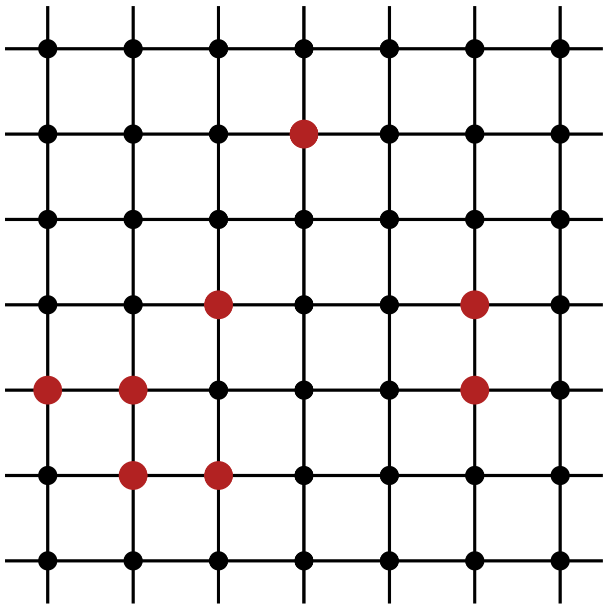

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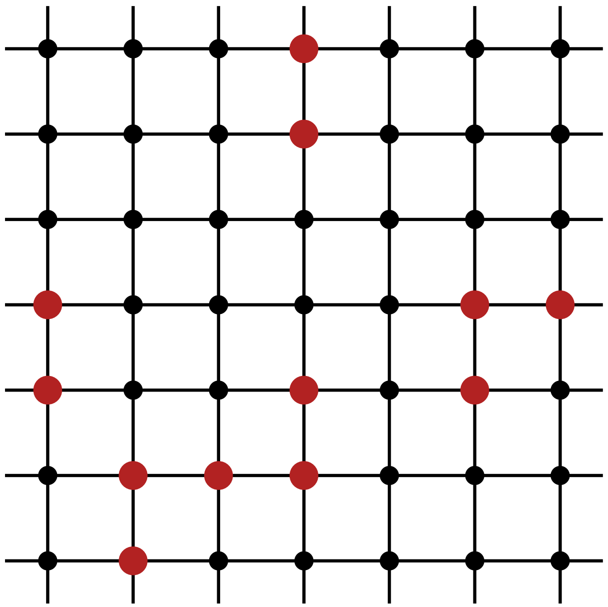

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Question: condition on G to ensure $\lambda_c > 0$?

The contact process on a graph with bounded degrees

If G has **bounded** degrees, then $\lambda_c > 0$.

Compare with **branching random walk**:

- **No interaction** between particles;
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- No other method to prove that contact process dies out.
- No example of graph with unbounded degrees for which we know $\lambda_c > 0$.

Contact process on geometric graphs

Theorem

Let G be either a

- (supercritical) random geometric graph
- Delaunay triangulation

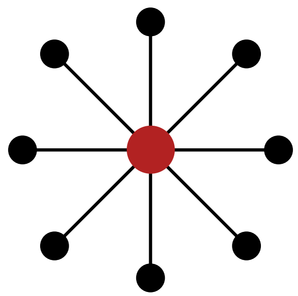
constructed from a Poisson point process on $\mathbb{R}^d$ with Lebesgue intensity. Then one has $\lambda_c > 0$.

Proof:

Criterion on G for $\lambda_c > 0$ in terms of Cumulative Merging Percolation.

Heuristics for the contact process

Contact process on a star graph of large degree d:



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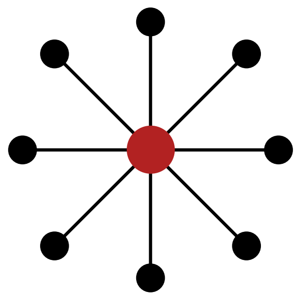

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- One vertex (●) has large degree d_0 with $\lambda > \lambda_c(d_0)$;
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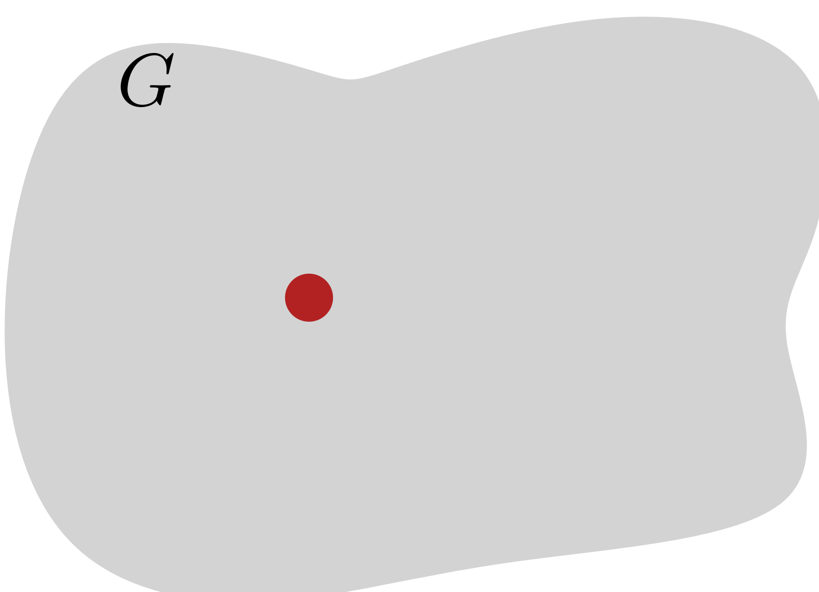

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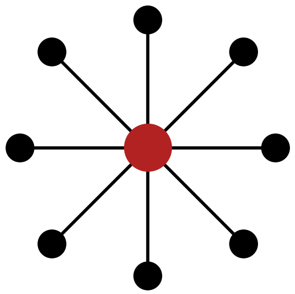

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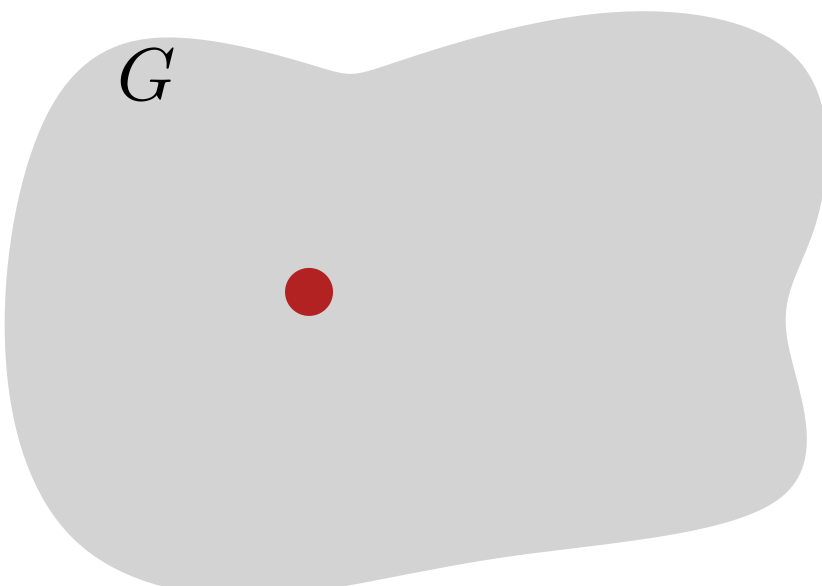

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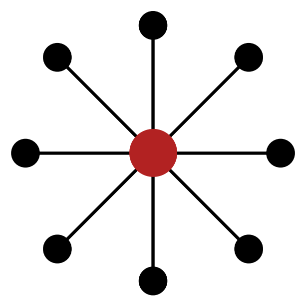

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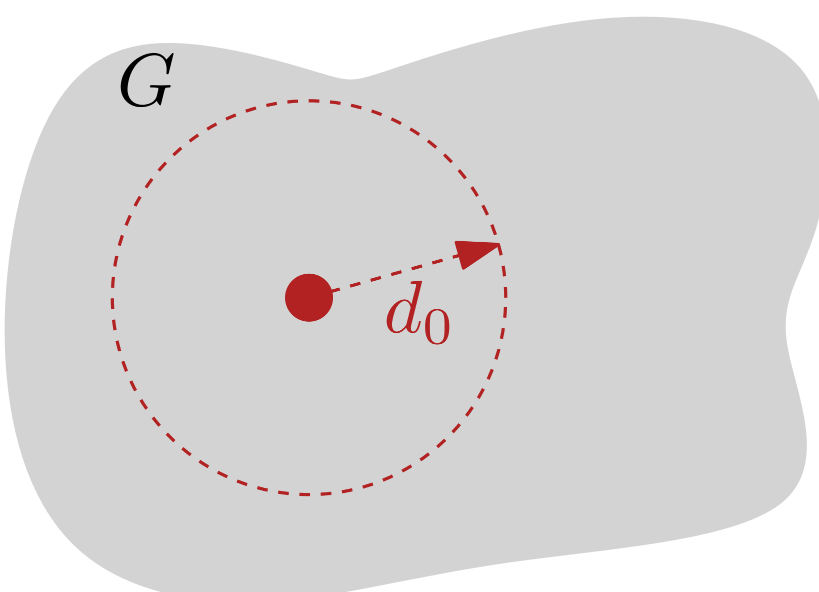

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Maximal distance reached by infection is $\approx d_0$.

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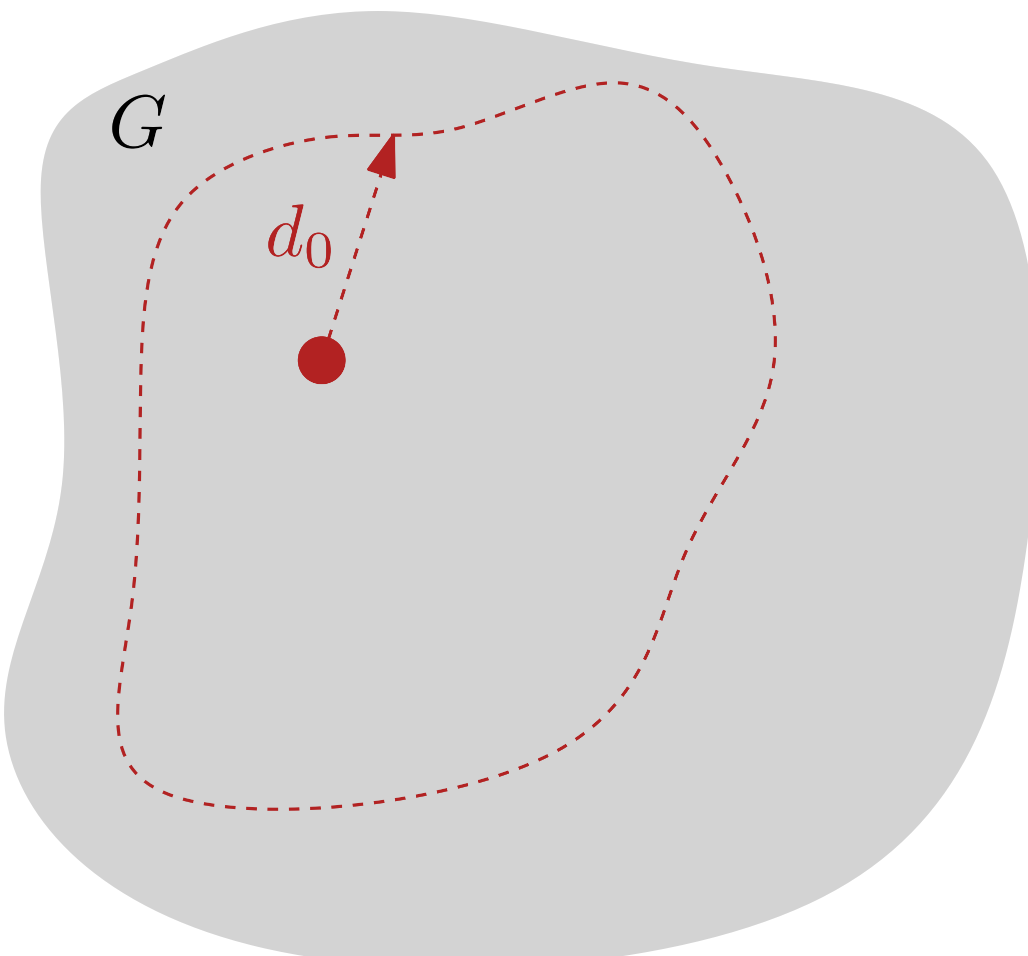
Same as before for ● and start of the infection.



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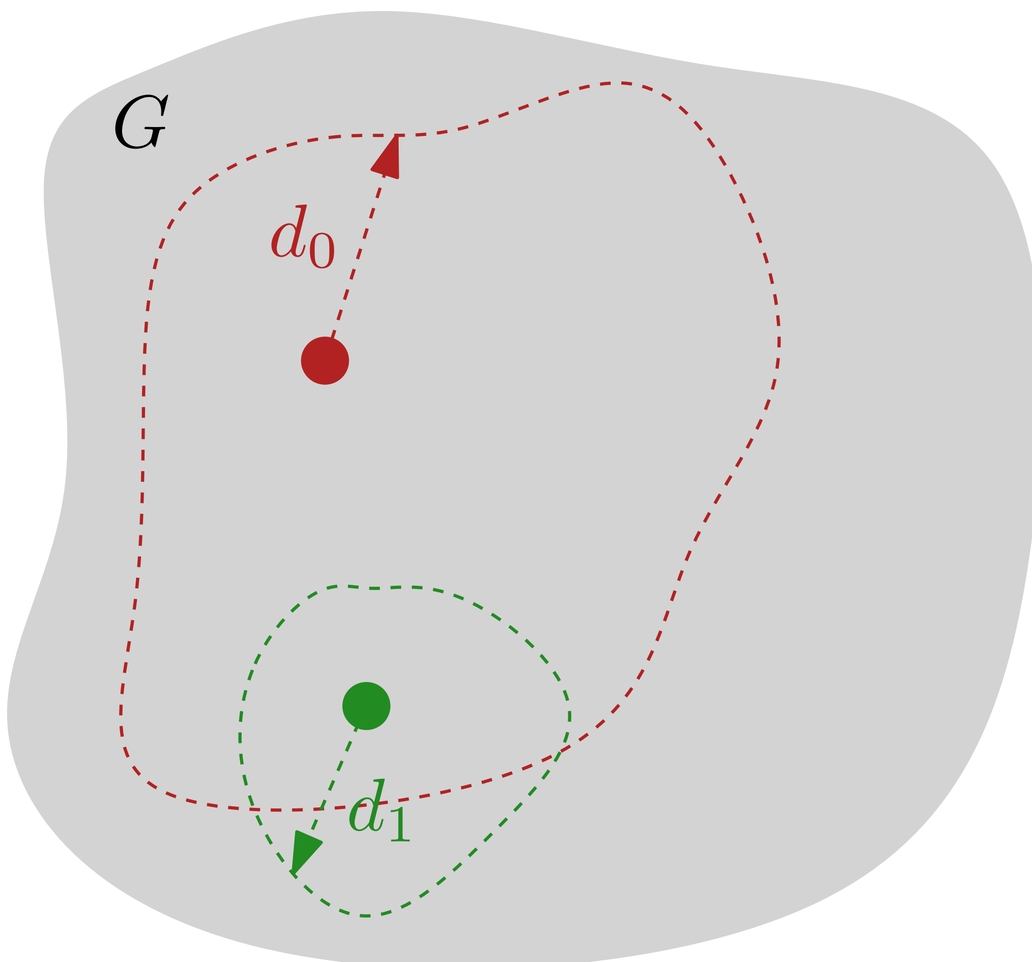

In addition suppose:

- At distance $< d_0$ from ● , there is another vertex (●) with large degree d_1 *s.t.* $\lambda > \lambda_c(d_1)$.
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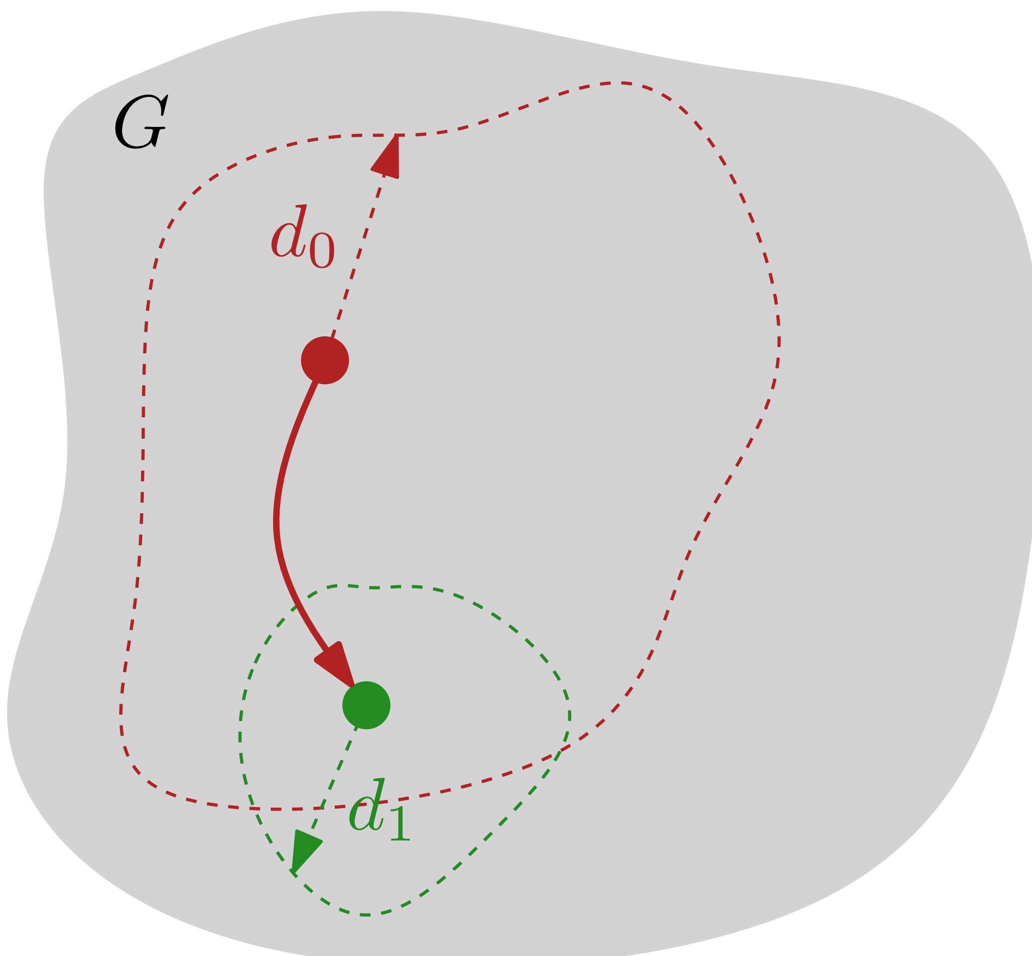

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● cannot send infections to ● and the survival time of the process is $\approx \exp(d_0) + \exp(d_1) \approx \exp(d_0)$.

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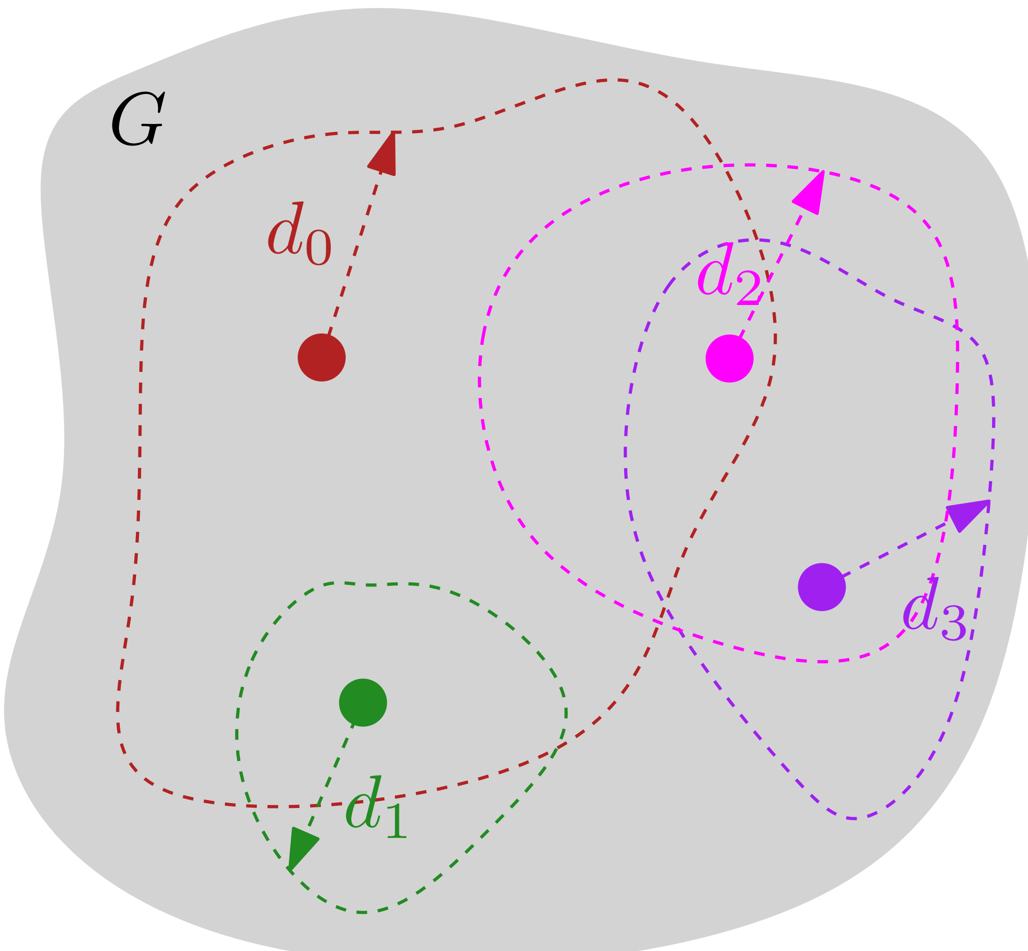

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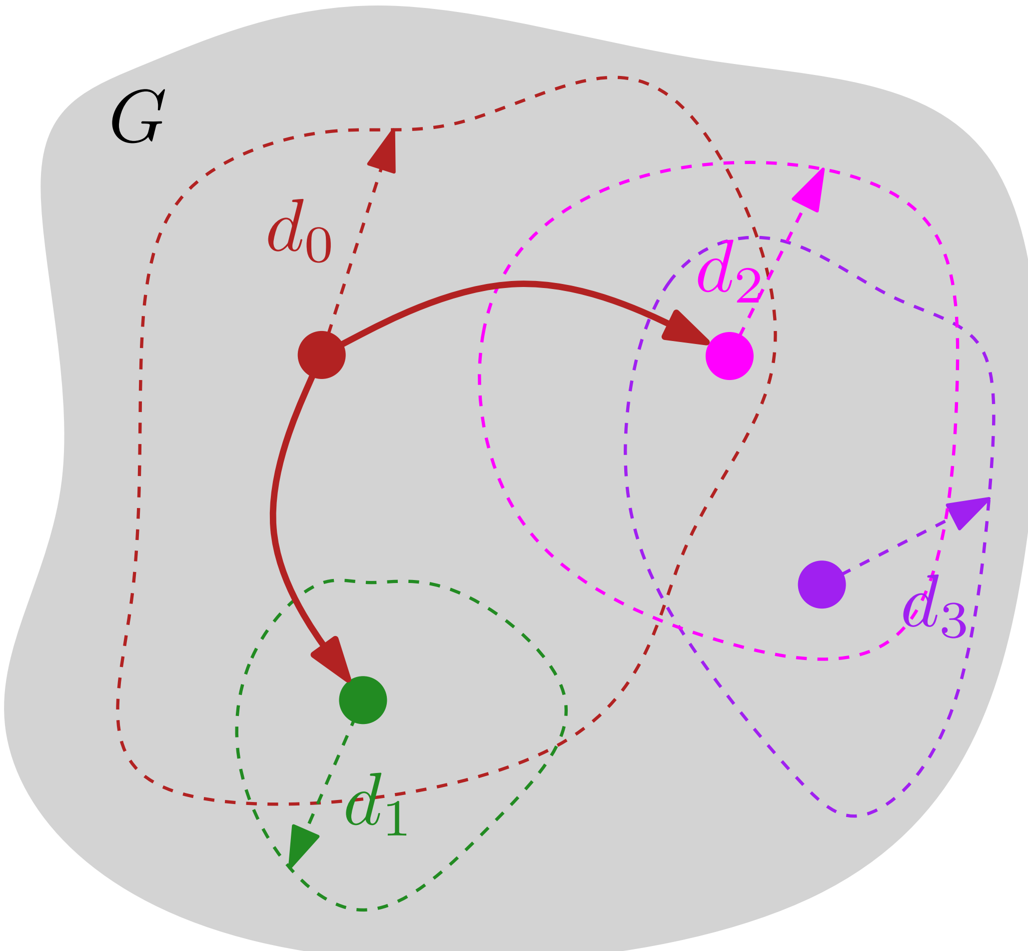

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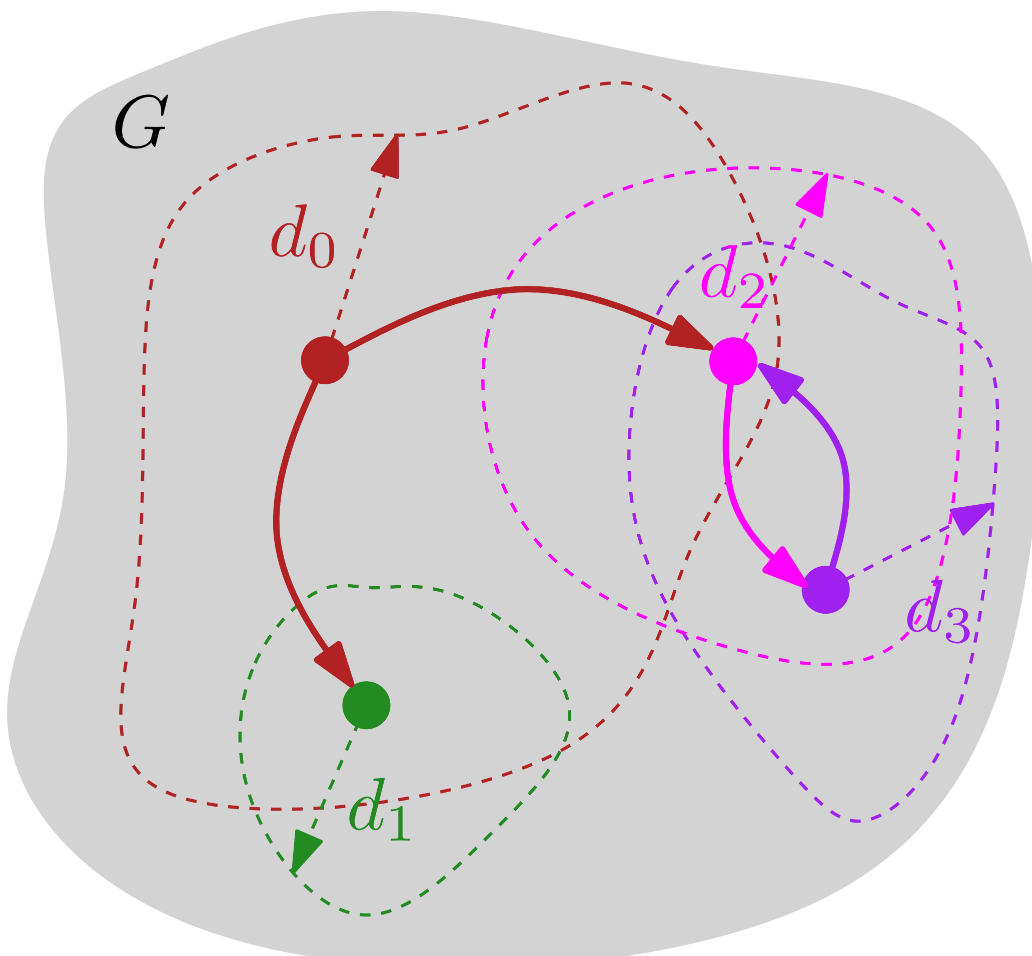

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● and ● interact and their combined survival time is $\approx \exp(d_2) \times \exp(d_3) = \exp(d_2 + d_3)$

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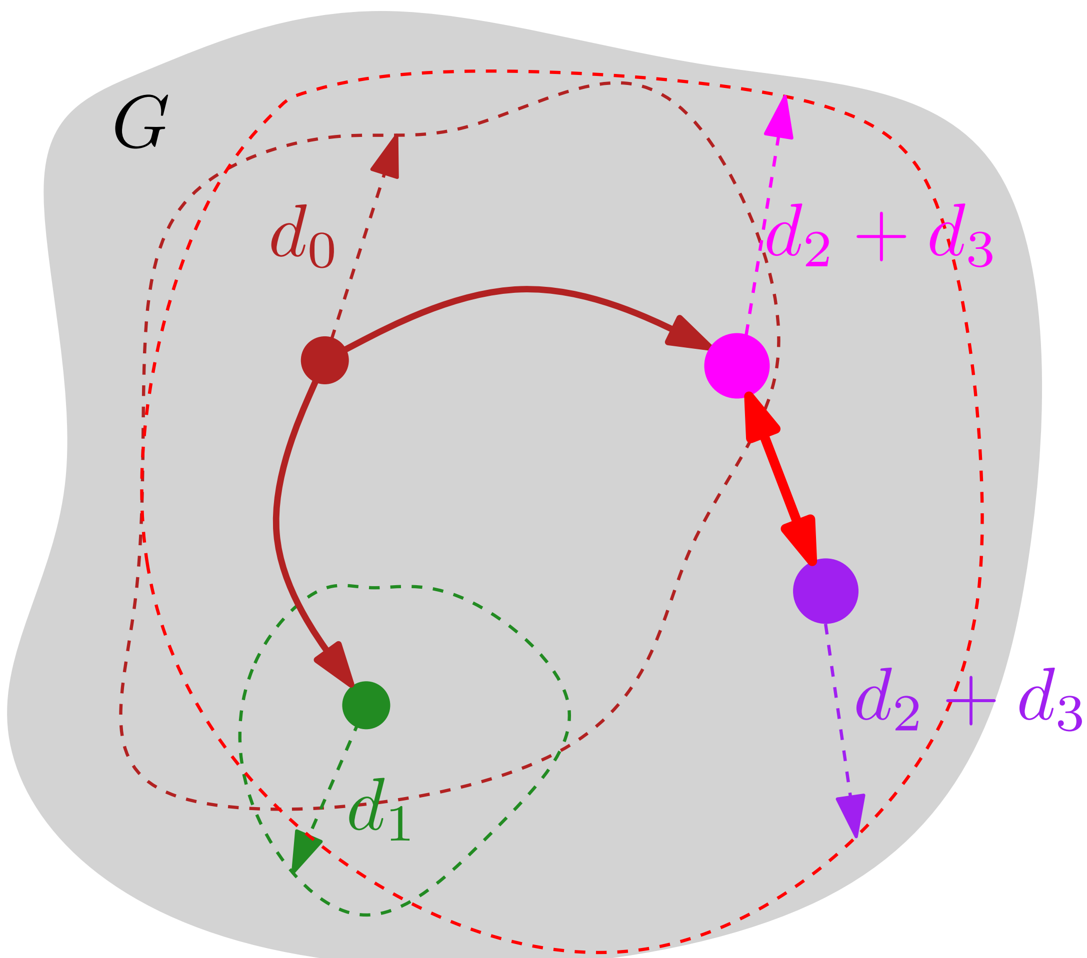

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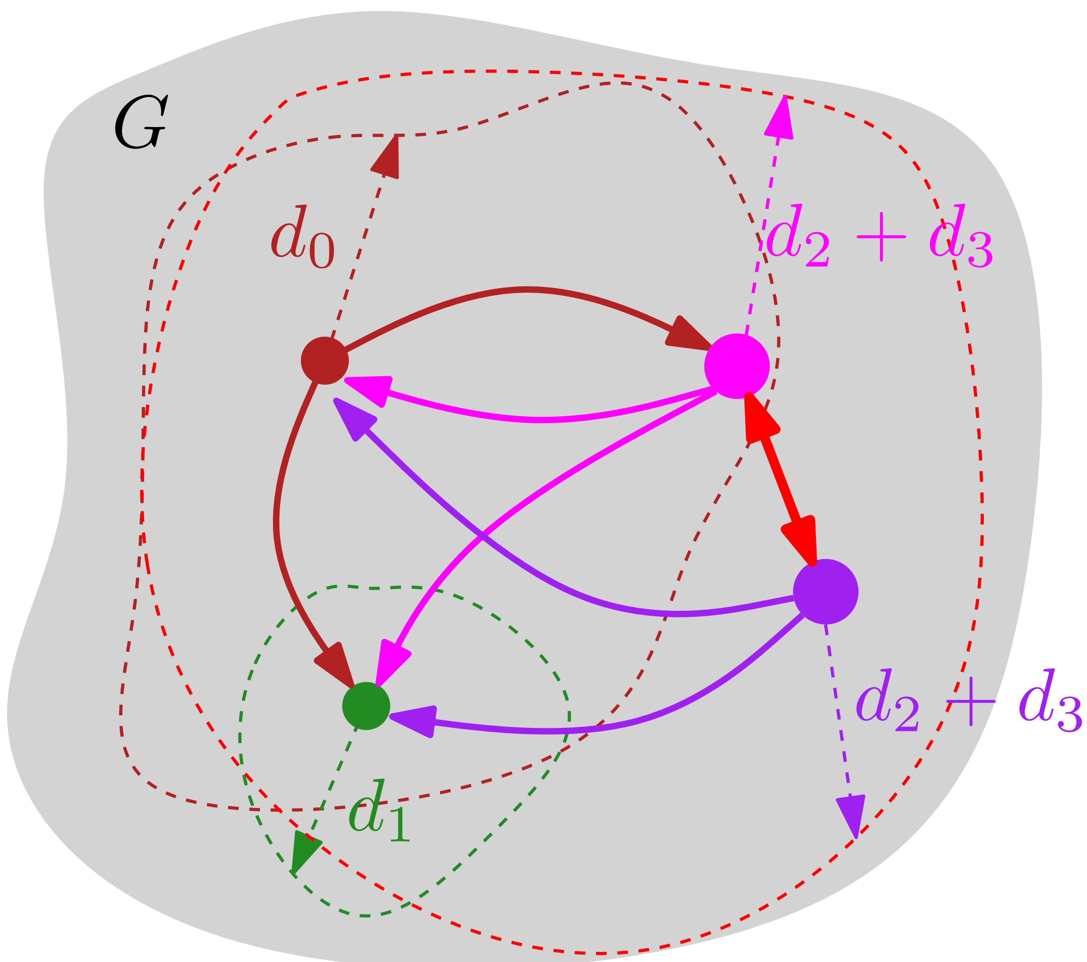
● and ● interact and their combined survival time is $\approx \exp(d_2) \times \exp(d_3) = \exp(d_2 + d_3)$

Heuristics for the contact process

Same as before for ● and start of the infection.



Now suppose:

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⟶ Now ● and ● can reach ● or ● !

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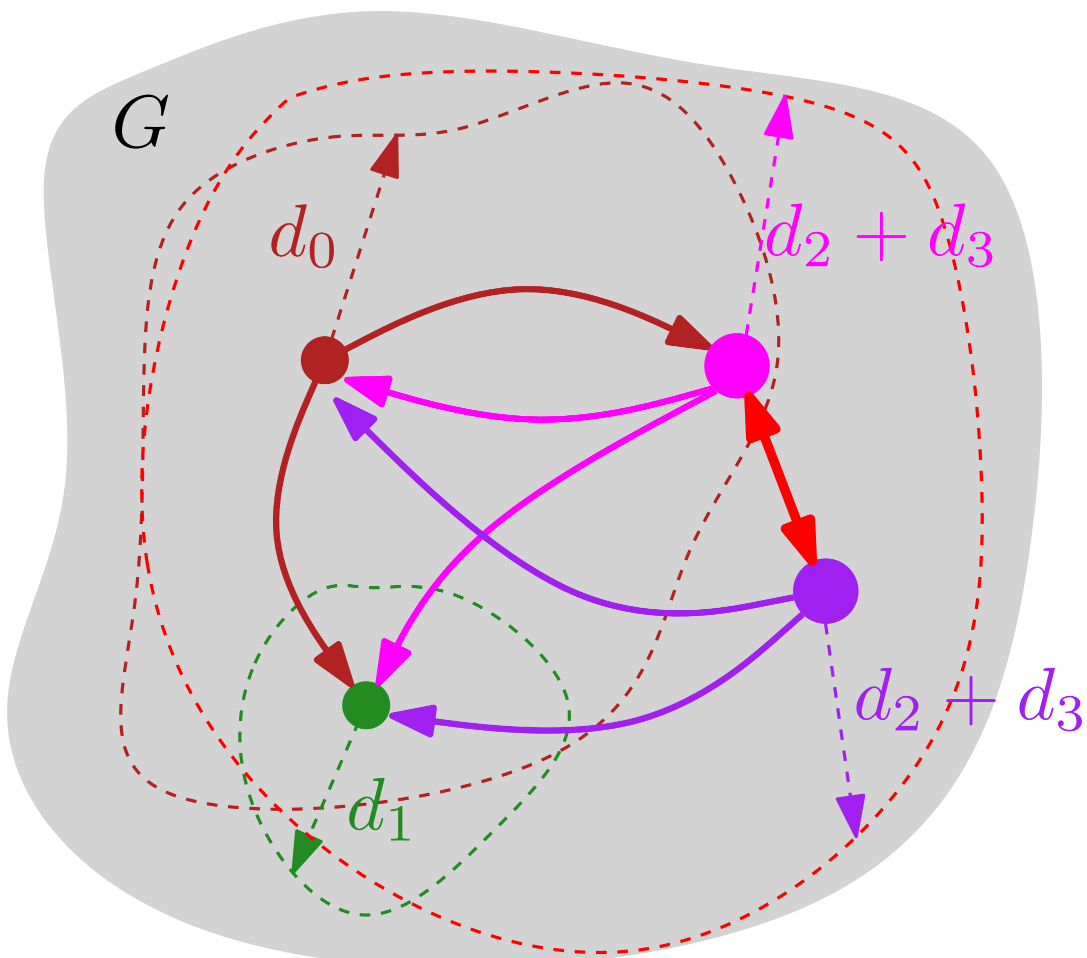

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→ ●, ● and ● are all in interaction.

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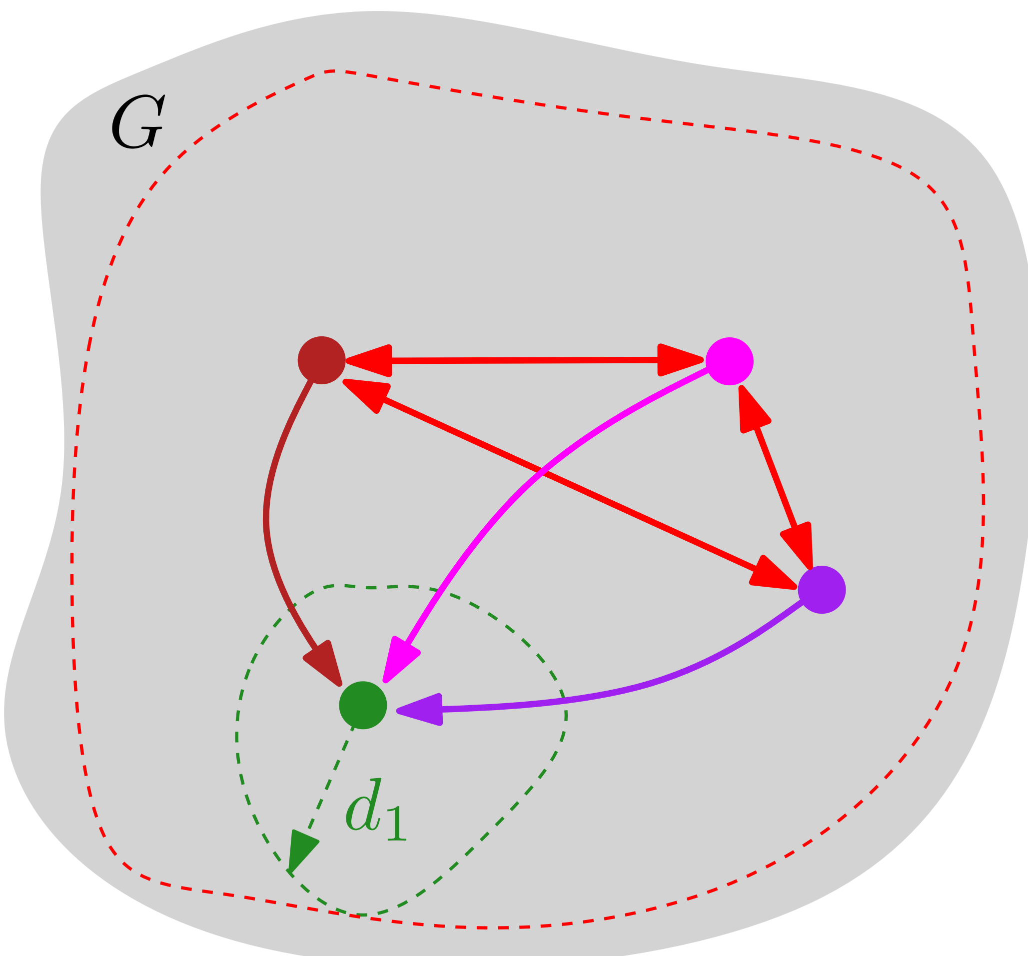

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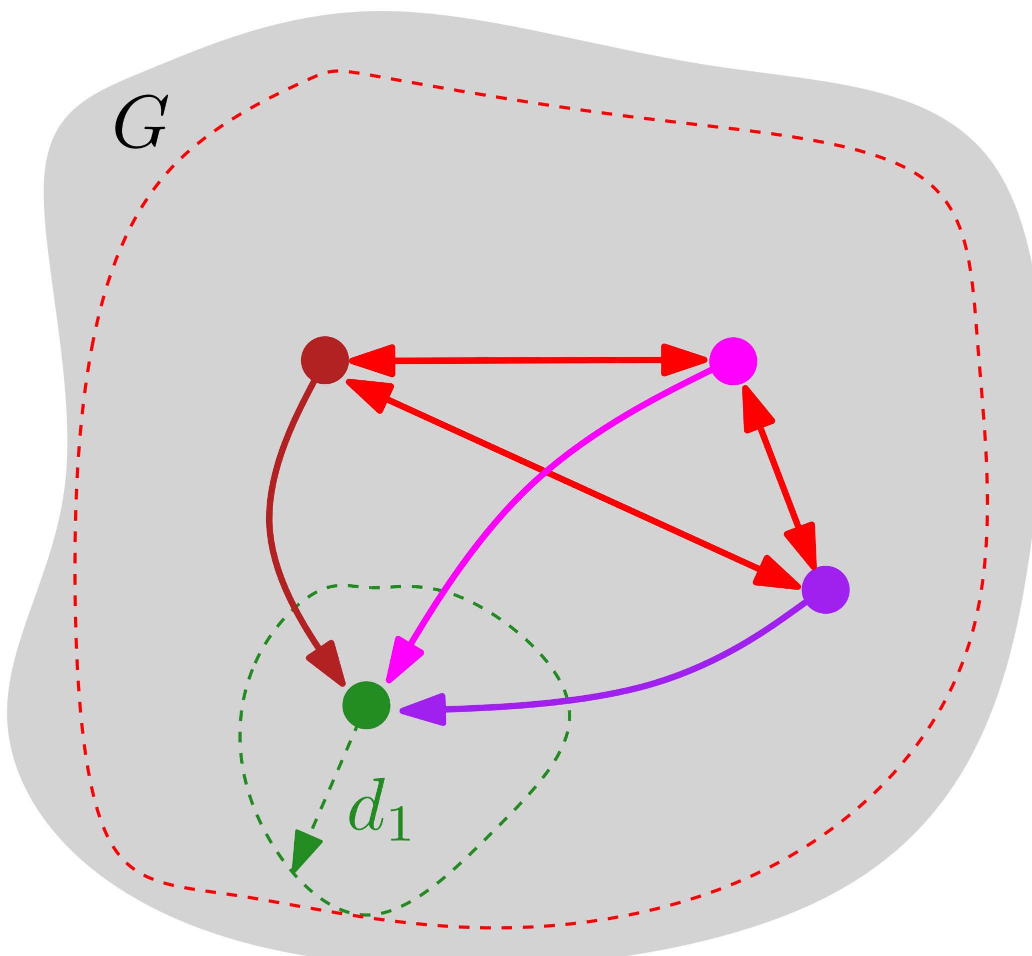

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⟶ ● still cannot reach the other 3 vertices to interact.

Cumulative Merging: Link with the contact process

Need to change the definition of admissible partitions:

$\mathcal{P}$ is **admissible** *iff* $\forall A, B \in \mathcal{P}$

$$d_G(A, B) > \quad r(A) \wedge r(B) \qquad .$$

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Theorem:
Let $G = (V, E)$ be **any** locally finite graph. Suppose that, for $\alpha > 2.5$, CMP on G with weights given by:

$$\forall x \in V,\ r(x) = \begin{cases} \deg(x) & \text{if } \deg(x) > \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

has a non-trivial phase transition (*i.e.* $\Delta_c < \infty$).

Then the contact process on G has a non trivial phase transition (*i.e.* it dies out for small infection rates).

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Question: true for $\alpha = 1$?

Conference **Dynamics on random graphs and random planar maps**

October 23 to 27, 2017 in Marseille France

Org. LM, Pierre Nolin, Bruno Schapira and Arvind Singh

Thank you for your attention!