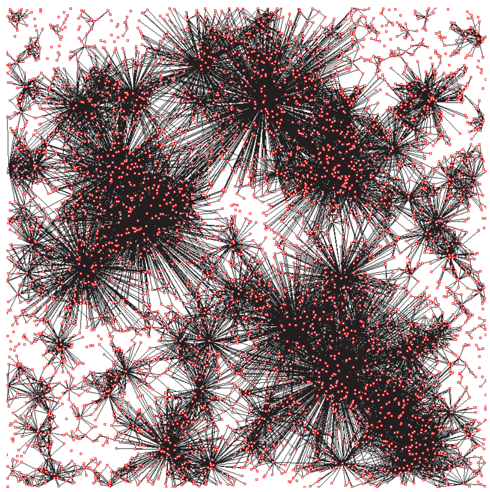
Random Geometric Graphs and Their Applications to Complex Networks



November 6-11, 2016

Banff International Research Station

Banff, AB, Canada

https://www.birs.ca/events/2016/5-day-workshops/16w5095

Banff International Research Station for Mathematical Innovation and Discovery

Program

Sunday, November 8

17:00 - 17:30	Check-in
17:30 - 19:30	Dinner
20:00 - 22:00	Informal gathering

Monday, November 7

7:00 - (9:00 - ε)	Breakfast ($\varepsilon \approx 15$ mins)
(9:00 - ε) - 9:00	Welcome by BIRS Station Manager and Organizers
9:00 - 10:00	**Mathew Penrose**, University of Bath, UK *Long talk: Random Bipartite geometric graphs*
10:00 - 10:30	Coffee Break
10:30 - 11:00	**Abbas Mehrabian**, University of British Columbia *Short talk: Rumour spreading in the SPA model*
11:00 - 11:30	**Jane Gao**, Monash University, Australia *Short talk: Packing edge-disjoint spanning trees in random geometric graphs*
11:30 - 13:00	Lunch
13:00 - 14:00	Guided Tour of The Banff Centre
14:00 - 14:20	Group Photo
14:20 - 15:00	Problem Session / Progress Report
15:00 - 15:30	Coffee Break
15:30 - 17:30	Hard Work
17:30 - 19:30	Dinner
19:30 - (19:30 + δ)	More Hard Work

Tuesday, November 8

7:00 - 9:00	Breakfast
9:00 - 10:00	**Dmitri Krioukov**, Northeastern University, USA *Long talk: Clustering Implies Geometry in Networks*
10:00 - 10:30	Coffee Break
10:30 - 11:00	**Nikolaos Fountoulakis**, University of Birmingham, UK *Short talk: The emergence of the giant component in random graphs on the hyperbolic plane*
11:00 - 11:30	**Yuval Peres**, Microsoft Research, USA *Short talk: Random Geometric Graphs beyond the Poisson process*
11:30 - 12:00	**Jeannette Janssen**, Dalhousie University, Canada *Short talk: Recognizing graphs with linear random structure*
12:00 - 13:30	Lunch
13:30 - 15:00	Problem Session / Progress Report
15:00 - 15:30	Coffee Break
15:30 - 17:30	Hard Work
17:30 - 19:30	Dinner
19:30 - (19:30 + δ)	More Hard Work

Wednesday, November 9

7:00 - 9:00	Breakfast
9:00 - 10:00	**Dieter Mitsche**, Universite de Nice Sophia-Antipolis, France *Long talk: On the spectral gap of random hyperbolic graphs*
10:00 - 10:30	Coffee Break
10:30 - 11:00	**Carl Dettmann**, University of Bristol, UK *Short talk: Random connection models*
11:00 - 11:30	**Anthony Bonato**, Ryerson University, Canada *Short talk: Isomorphism results for infinite random geometric graphs*
11:30 - 12:00	Problem Session / Progress Report
12:00 - 13:30	Lunch
13:30 - 17:30	Excursion / Hard Work
17:30 - 19:30	Dinner
19:30 - (19:30 + δ)	More Hard Work

Thursday, November 10

7:00 - 9:00	Breakfast
9:00 - 10:00	**Joseph Yukich**, Lehigh University, USA *Long talk: Statistics of random graphs on clustering point sets*
10:00 - 10:30	Coffee Break
10:30 - 11:00	**Matthias Schulte**, University of Bern, Switzerland *Short talk: Limit theorems for edge length statistics of random geometric graphs*
11:00 - 11:30	**Guillem Perarnau**, McGill University, Canada *Short talk: Random graphs from bridge-addable classes*
11:30 - 12:00	Problem Session / Progress Report
12:00 - 13:30	Lunch
13:30 - 15:00	Hard Work
15:00 - 15:30	Coffee Break
15:30 - 17:30	Hard Work
17:30 - 19:30	Dinner
19:30 - (19:30 + δ)	More Hard Work

Friday, November 11

7:00 - 9:00	Breakfast
9:00 - 10:00	**Laurent Menard**, Universit Paris Ouest, France *Long talk: Percolation by cumulative merging and phase transition* *for the contact process on random graphs*
10:00 - 10:30	Coffee Break
10:30 - 11:00	**Guenter Last**, Karlsruhe Institute of Technology, Germany *Short talk: Second order properties and asymptotic normality of cluster sizes* *in the random connection model*
11:00 - 11:30	**Kiril Solovey**, Tel Aviv University, Israel *Short talk: Applications of Random Geometric Graphs in Robot Motion Planning*
11:30 - 12:00	Checking out
12:00 - 13:30	Lunch

Participants (* – organizers)

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44) Pim van der Hoorn (University of Twente, The Netherlands) w.l.f.vanderhoorn@utwente.nl

Abstracts of Talks

Mathew Penrose, University of Bath, UK — Mon 9:00
Random Bipartite geometric graphs — 45 min

Consider a bipartite random geometric graph (RGG) on the union of two independent homogeneous Poisson point processes in Euclidean d-space, with fixed distance parameter r and intensities λ, μ. Given $\lambda > 0$, let $\mu_c(\lambda)$ be the infimum of those μ for which this RGG percolates (or infinity if there is no such μ). Also, let λ_c be the critical value of λ for percolation of the one-type RGG with distance parameter $2r$. If $\lambda > \lambda_c$ then $\mu_c(\lambda) < \infty$. Conversely, $\lim_{\lambda \downarrow \lambda_c} \mu_c(\lambda) = \infty$, and hence $\mu_c(\lambda_c) = \infty$.

Consider also the restriction of this graph to points in the unit square. We describe a strong law of large numbers as $\lambda \to \infty$ with μ/λ fixed, for the connectvity threshold, i.e. the smallest value of r such that the graph is connected.

Abbas Mehrabian, University of British Columbia — Mon 10:30
Rumour spreading in the SPA model — 25 min

The Spatial Preferential Attachment model is a spatial random graph used to model social networks. Nodes live in a metric space, and edges are formed based on the metric distance and degree of the nodes. Rumour spreading is a protocol for the spread of information through a graph. In each time step nodes can pass the rumour to only one of their neighbours. The spread time is the expected time when all nodes have the rumour. We analyze rumour spreading on the SPA model, and show that the spread time differs substantially from the diameter. Joint work with Jeannette Janssen.

Jane Gao, Monash University, Australia — Mon 11:00
Packing edge-disjoint spanning trees in random geometric graphs — 25 min

It was recently proved that $G(n,p)$ contains exactly $\min(\lfloor m/(n-1) \rfloor, \delta)$ edge-disjoint spanning trees, where m is the number of edges in $G(n,p)$ and delta is the minimum degree of $G(n,p)$. This result holds for any $p \in [0,1]$. We investigate this problem in random geometric graphs $G(n,r)$ and prove similar results, except for r in a critical range, which is left as a problem for the open problem session in the BIRS workshop. This is collaborated work with Xavier Perez-Gimenez and Cristiane Sato.

Dmitri Krioukov, Northeastern University, USA — Tue 9:00
Clustering Implies Geometry in Networks — 45 min

Two common features of many large real networks are that they are sparse and that they have strong clustering, i.e., large number of triangles homogeneously distributed across all nodes. In many growing real networks for which historical data is available, the average degree and clustering are roughly independent of the growing network size. Recently, (soft) random geometric graphs, also known as latent-space network models, with hyperbolic and de Sitter latent geometries have been used successfully to model these features of real networks, to predict missing and future links in them, and to study their navigability, with applications ranging from designing optimal routing in the Internet, to identification of the information-transmission skeleton in the human brain. Yet it remains unclear if latent-space models are indeed adequate models of real networks, as random graphs in these models may have structural properties that real networks do not have, or vice versa.

We show that the canonical maximum-entropy ensemble of random graphs in which the expected numbers of edges and triangles at every node are fixed to constants, are approximately soft random geometric graphs on the real line. The approximation is exact in the limit of standard random geometric graphs with a sharp connectivity threshold and strongest clustering. This result implies that a large number of triangles homogeneously distributed across all vertices is not only necessary but also a sufficient condition for the presence of a latent/effective metric space in large sparse networks. Strong clustering, ubiquitously observed in real networks, is thus a reflection of their latent geometry.

Nikolaos Fountoulakis, University of Birmingham, UK — Tue 10:30
The emergence of the giant component in random graphs on the hyperbolic plane — 25 min

We consider a recent model of random geometric graphs on the hyperbolic plane developed by Krioukov et al. (Phys. Rev. E 2010). This may be also viewed as a geometric version of the well-known Chung-Lu model of inhomogeneous random graphs and turns out to have basic properties that are ubiquitous in complex networks. We consider the size of the largest component of this random graph and show that a giant component emerges when the basic parameters of the model cross certain values. We also show that the fraction of vertices that are contained there converges in probability to a certain constant, which is related to a continuum percolation model on the upper-half plane. This is joint work with Tobias Müller and Michel Bode.

Yuval Peres, Microsoft Research, USA — Tue 11:00
Random Geometric Graphs beyond the Poisson process — 25 min

Abstract: Random Geometric graphs have traditionally been considered on the nodes of a Poisson process, but recently there has been enhanced interest in more rigid point processes. We study continuum percolation for the Ginibre ensemble and the planar Gaussian zero process, which are the primary models of translation invariant point processes in the plane exhibiting local repulsion. For the Ginibre ensemble, we establish the uniqueness of infinite cluster in the supercritical phase. For the Gaussian zero process, we establish that a non-trivial critical radius exists, and we prove the uniqueness of the infinite cluster in the supercritical regime. Finding suitable replacements for insertion and deletion tolerance is a crucial step. Joint work with Manju Krishnapur and Subhro Ghosh.

Jeannette Janssen, Dalhousie University, Canada Tue 11:30
Recognizing graphs with linear random structure 25 min

Abstract: In many real life applications, network formation can be modelled using a spatial random graph model: vertices are embedded in a metric space S, and pairs of vertices are more likely to be connected if they are closer together in the space. A general geometric graph model that captures this concept is $G(n, w)$, where $w : S \times S \to [0, 1]$ is a symmetric "link probability" function with the property that, for fixed $x \in S, w(x, y)$ decreases as y is moved further away from x. he function w can be seen as the graph limit of the sequence $G(n, w)$ as $n \to \infty$.

We consider the question: given a large graph or sequence of graphs, how can we determine if they are likely the results of such a general geometric random graph process? Focusing on the one-dimensional (linear) case where $S = [0, 1]$, we define a graph parameter Γ and use the theory of graph limits to show that this parameter indeed measures the compatibility of the graph with a linear model.

Dieter Mitsche, Universite de Nice Sophia-Antipolis, France — Wed 9:00
On the spectral gap of random hyperbolic graphs — 45 min

Random hyperbolic graphs have been suggested as a promising model of social networks. A few of their fundamental parameters have been studied. However, none of them concerns their spectra. We consider the random hyperbolic graph model as formalized by Gugelmann et al. and essentially determine the spectral gap of their normalized Laplacian. Specifically, we establish that with high probability the second smallest eigenvalue of the normalized Laplacian of the giant component of an n-vertex random hyperbolic graph is at least $n^{-(2\alpha-1)}/(D\log n)^{1+o(1)}$, where $\frac{1}{2} < \alpha < 1$ is a model parameter and D is the network diameter (which is known to be at most polylogarithmic in n). We also show a matching (up to a polylogarithmic factor) upper bound of $n^{-(2\alpha-1)}(\log n)^{1+o(1)}$.

As a byproduct we conclude that the conductance upper bound on the eigenvalue gap obtained via Cheeger's inequality is essentially tight. We also provide a more detailed picture of the collection of vertices on which the bound on the conductance is attained, in particular showing that for all subsets whose volume is $O(n^{\varepsilon})$ for $0 < \varepsilon < 1$, the obtained conductance is with high probability $\Omega(n^{-(2\alpha-1)\varepsilon+o(1)})$. Finally, we also show consequences of our result for the minimum and maximum bisection of the giant component.

Joint work with Marcos Kiwi.

Carl Dettmann, University of Bristol, UK — Wed 10:30
Random connection models — 25 min

Recent work has considered a generalization of the random geometric graph, in which pairs of points are linked with a probability depending on their mutual distance through a "connection function." Such models arise in the study of wireless networks and many other spatial networks. Calculations show that the connection probability for the whole graph can be estimated from just a few moments of the connection function for a wide variety of domain geometries. Furthermore, there are qualitative differences as a result of the random connections, for example, the more realistic random connection model allows a more accurate estimation of k-connectivity than the original random geometric graph. Anisotropy can improve connectivity only for sufficiently slowly decaying connection functions. These results have practical application in the design of wireless ad-hoc networks.

Anthony Bonato, Ryerson University, Canada — Wed 11:00
Isomorphism results for infinite random geometric graphs — 25 min

Recent work with Jeannette Janssen proved the existence of a family of random geometric graphs with unique countable limits. These graphs arise in the normed space ℓ_∞^n, which consists of $\mathbb{R}^n$ equipped with the L_∞-norm. Using tools from functional analysis, Balister, Bollobás, Gunderson, Leader, and Walters proved that these unique limit graphs are deeply tied to the L_∞-norm. Precisely, a random geometric graph on any normed, finite-dimensional space not isometric ℓ_∞^n gives non-isomorphic limits with probability 1. We survey properties of these infinite random geometric graphs, and discuss new results for the infinite dimensional case.

Joseph Yukich, Lehigh University, USA — Thu 9:00
Statistics of random graphs on clustering point sets — 45 min

Statistics of graphs on vertex sets $\mathcal{X} \subset \mathbb{R}^d$ often consist of sums of spatially dependent terms admitting the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1}$$

where the $\mathbb{R}$-valued *score function* ξ, defined on pairs $(x, \mathcal{X})$, $x \in \mathcal{X}$, represents the *interaction* of x with respect to $\mathcal{X}$. Statistics having the representation (1) include number of components, clique counts, and total edge length. If the vertex set $\mathcal{X}$ is the realization of a clustering point process and if ξ is 'locally determined' (i.e., stabilizing), then we establish general expectation and variance asymptotics as well as central limit theorems for the suitably scaled and centered sums

$$\sum_{x \in \mathcal{X} \cap W_n} \xi(x, \mathcal{X} \cap W_n), \quad W_n \uparrow \mathbb{R}^d. \tag{2}$$

We deduce the limit theory for clique counts and for the total edge length of the random geometric graph as well as for general proximity graphs on clustering input, including determinantal and permanental point processes with a fast decreasing kernel (e.g. the Ginibre ensemble), the zero set of a Gaussian entire function, and rarified Gibbsian input. The talk is based on joint work with B. Błaszczyszyn and D. Yogeshwaran.

Matthias Schulte, University of Bern, Switzerland — Thu 10:30
Limit theorems for edge length statistics of random geometric graphs — 25 min

A random geometric graph is constructed by connecting two points of a Poisson process in a compact convex set whenever their distance does not exceed a prescribed distance. The aim of this talk is to investigate the asymptotic behaviour of the total edge length or, more general, sums of powers of the edge lengths of this random graph as the intensity of the underlying Poisson process is increased and the threshold for connecting points is adjusted. Depending on the interplay of these two parameters as well as the power of the edge lengths one obtains limit theorems where the limiting distribution can be Gaussian, compound Poisson or stable. This talk is based on joint work with Laurent Decreusefond, Matthias Reitzner and Christoph Thäle.

Guillem Perarnau, McGill University, Canada — Thu 11:00
Random graphs from bridge-addable classes — 25 min

A class of graphs is bridge-addable if given a graph G in the class, any graph obtained by adding an edge between two connected components of G is also in the class. Examples of bridge-addable classes are forests, planar graphs, triangle-free graphs or graphs with bounded treewidth. It has been recently proved that a uniform random graph in a bridge-addable class is connected with probability at least $(1 + o(1)) \exp(-1/2)$. The constant $\exp(-1/2)$ is best possible since it is reached for uniform random forests. Here, we will present a form of uniqueness in this statement: if a random graph in a bridge-addable class is connected with probability close to $\exp(-1/2)$, then it is asymptotically close to a random forest in some local sense. For example, such random graph converges in the sense of Benjamini-Schramm to the uniform infinite random forest. This is joint work with Guillaume Chapuy.

Laurent Menard, Universit Paris Ouest, France — Fri 9:00
Percolation by cumulative merging and phase transition for the contact process on random graphs — 45 min

Given a weighted graph, we introduce a partition of its vertex set such that the distance between any two clusters is bounded from below by the minimum weight of both clusters. This partition is obtained by recursively merging smaller clusters and cumulating their weights. For several classical random weighted graphs, we show that there exists a phase transition regarding the existence of an infinite cluster.

The motivation for introducing this partition arises from a connection with the contact process as it roughly describes the geometry of the sets where the process survives for a long time. We give a sufficient condition on a graph to ensure that the contact process has a non trivial phase transition in terms of the existence of an infinite cluster. As an application, we prove that the contact process admits a sub-critical phase on random geometric graphs and random Delaunay triangulations. (Joint work with Arvind Singh)

Guenter Last, Karlsruhe Institute of Technology, Germany — Fri 10:30
Second order properties and asymptotic normality of cluster sizes in the random connection model — 25 min

The random connection model is a random graph whose vertices are given by the points of a stationary Poisson process and whose edges are obtained by connecting pairs of Poisson points at random. The connection decisions are allowed to depend on the positions of the two involved vertices but are otherwise independent for different pairs and independent of the other Poisson points. We shall discuss first and second order properties of the number of clusters isomorphic to a given graph. We also present a multivariate central limit theorem whose proof is based on some new Berry-Esseen bounds for the normal approximation of functionals of a pairwise marked Poisson process. This is joint work with Franz Nestmann (Karlsruhe) and Matthias Schulte (Bern).

Kiril Solovey, Tel Aviv University, Israel — Fri 11:00
Applications of Random Geometric Graphs in Robot Motion Planning — 25 min

Robot motion planning is a fundamental research area in robotics with applications in diverse domains such as graphical animation, surgical planning, computational biology and computer games. In its basic form, motion planning is concerned with finding a collision-free path for a robot in a workspace cluttered with static obstacles. The high computational complexity of exact solutions to motion planning have led to the development of sampling-based planners. These algorithms aim to capture the connectivity of the free space—the set of collision-free robot configurations—in a graph data structure, whose vertices consist of randomly-sampled configurations. Interestingly, roadmaps constructed by many sampling-based planners coincide, in the absence of obstacles, with standard models of random geometric graphs (RGGs).

In this talk I will provide a brief introduction to sampling-based motion planning and survey several theoretical results concerning their behavior, including a recently-introduced framework that facilitates the extension of properties of RGGs to sampling-based techniques in motion planning.

Open Problems

Laurent Menard, Universit Paris Ouest, France
Does cumulative merging on trees have a phase transition?

Let T be an infinite (rooted) binary tree, or any infinite (random) tree. The vertices of T are assigned iid Bernoulli weights with parameter p. By recursively grouping vertices, one constructs a partition of $V(T)$ such that, for any two clusters A and B of the partition, the graph distance between A and B is larger than the minimum of the total weights of A and B. The resulting partition does not depend on the grouping order.

Show that this process has a phase transition: there exists $p_c \in (0, 1)$ such that for $p < p_c$ the partition has no infinite cluster and for $p > p_c$ the partition has an infinite cluster.

Ross Kang, Radboud University Nijmegen, The Netherlands
The chromatic number of high dimensional random geometric graphs

Consider n random points i.i.d. on the unit sphere in R^d. Devroye, György, Lugosi and Udina (2011) showed that, holding n fixed (but large) and letting d grow very rapidly like $d \gg 2^{n^2}$, then a suitable random geometric graph on these points is close in total variation distance to a binomial random graph. They also showed that for d rather smaller, say, $d \gg \log^3 n$, the clique number of the random geometric graph is close to the clique number of the corresponding binomial random graph.

Question: what about the chromatic number? Can a polylogarithmic lower bound on d still be sufficient for the chromatic number of the random geometric graph to be close to that of the corresponding binomial random graph? Note that the chromatic number of a random geometric graph is typically not too far from the clique number, while the same is not true for a binomial random graph, and so there should be a ”jump” in behaviour.

Jeannette Janssen, Dalhousie University, Canada
Nested geometric graphs

The familiar geometric graph $RG(n, d)$ can be described as follows: each vertex has a “sphere of influence” centered at the vertex with radius d. Vertices u and v are connected if u falls inside the sphere of influence of v, or vice versa. We are interested in the sparse case, where $d = \Theta(1/n)$. A natural generalization is to consider spheres of different size. The power law geometric graph $PRG(n, A_1, A_2)$ is the following. Vertices $v_1, \ldots, v_n$ are chosen u.a.r. from the unit square (seen as a torus). Vertex v_i has sphere of influence with area $\left(\frac{A_2}{n}\right)\left(\frac{n}{i}\right)^{A_1}$. ($A_2 > 0$, $0 \le A_1 < 1$.) Note that, if $A_1 = 0$, this reverts to the sparse geometric graph. By coupling with this graph, we see that, by choosing A_2 large enough, we can guarantee that this graph has a giant component.

First question: what is the diameter of the giant component? What is the threshold for its appearance?

Second question: Let $v_1, v_2, \ldots v_n$ be chosen u.a.r from the unit square. Let $V_t = \{v_1, \ldots, v_t\}$. Let $\{R_t\}$ be a sequence of graphs, where R_t has vertex set V_t and edges formed according to

$PRG(t, A_1, A_2)$. Now define the union of these graphs: $G = \bigcup_{t=1}^{n} R_t$. What is the diameter of the giant component of G?

Some partial results on the second question for the special case where $A_1 = 0$, can be found in: `https://arxiv.org/abs/1608.01697`
See also:
C. Cooper, A. Frieze, and P. Pralat, Some typical properties of the Spatial Preferred Attachment model, Internet Mathematics 10 (2014), 27-47.