Partial linear spaces with a primitive rank 3 automorphism group of affine type

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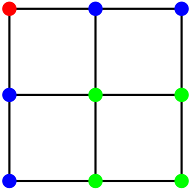

Joint work with John Bamberg, Alice Devillers, and Cheryl Praeger

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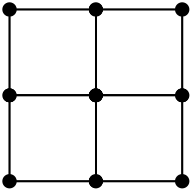

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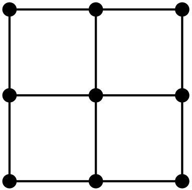

A partial linear space is proper if it is not a linear space or a graph.

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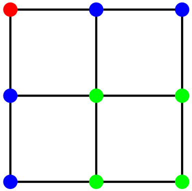

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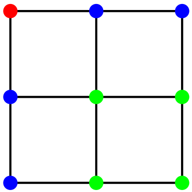

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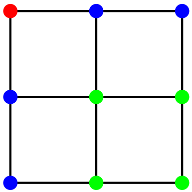

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- These have been classified by Devillers (2005,2008) in the (primitive) almost simple and grid cases.

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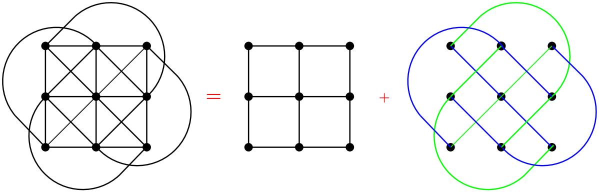

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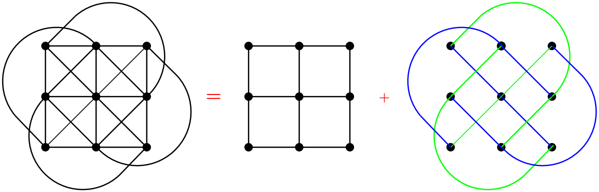

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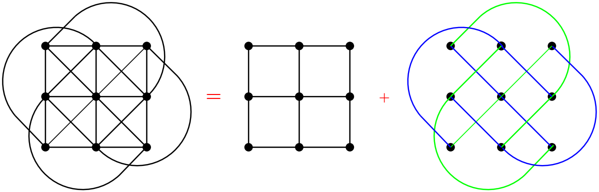

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- Biliotti-Montinaro-Francot (2015): 2-$(v, k, 1)$ designs with a primitive rank 3 affine group on points and 2 line orbits.

Affine groups: $G = V : G_0$ for $G_0 \leqslant \mathrm{GL}_d(p)$ acting on $V = V_d(p)$ where p is prime and V is irreducible $\mathbb{F}_p G_0$-module.

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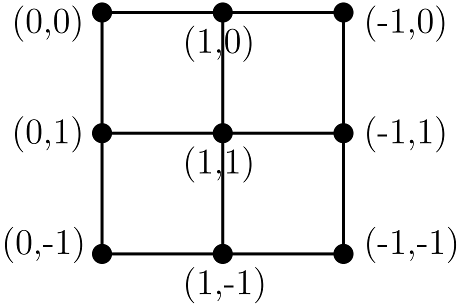
Examples of proper partial linear spaces with rank 3 affine group:

1. $p^n \times p^n$ grid with $V = V_n(p) \oplus V_n(p)$ and $G_0 = \mathrm{GL}_n(p) \wr C_2$.



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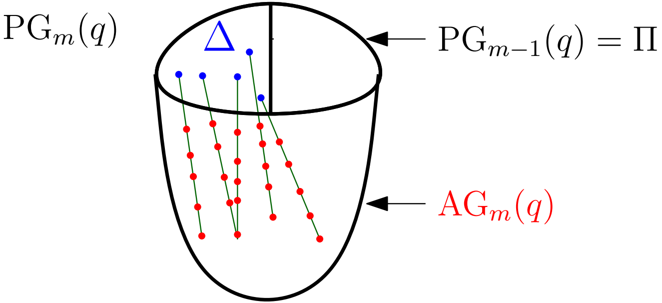

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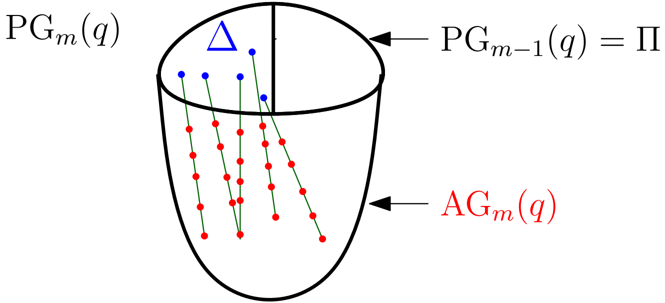

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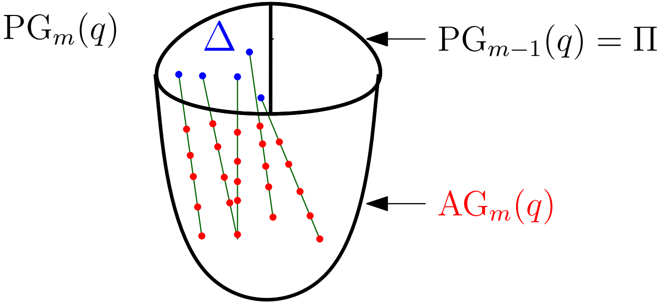

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i.e. $\mathscr{P} = V_m(q)$ and $\mathscr{L} = \{\langle v \rangle + w : v, w \in V_m(q), \langle v \rangle \in \Delta\}$.

3. $V = V_2(q) \otimes V_n(q)$ and $G_0 = \mathrm{GL}_2(q) \otimes \mathrm{GL}_n(q) : \mathrm{Aut}(\mathbb{F}_q)$ where $n \geqslant 2$. Let $\mathscr{L}$ be the set of translates of

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5. Let $V = \mathbb{F}_9^n$ where $n \geqslant 2$, $\mathbb{F}_9^* = \langle \zeta \rangle$ and $\mathrm{Aut}(\mathbb{F}_9) = \langle \sigma \rangle$. Let $G_0 = \mathrm{GL}_n(3)\langle \zeta^2, \zeta\sigma \rangle$ and $\mathscr{L} = \ell^{V:G_0}$ where

$$\ell = \langle e_1 + \zeta e_2, -\zeta e_1 + e_2 \rangle_{\mathbb{F}_3}$$

Here the lines have size 9.

Theorem (Conjecture really)

Let S be a proper partial linear space and $G \leqslant \mathrm{Aut}(S)$ a rank 3 primitive permutation group with socle $V = V_d(p)$. Then

(i) *S lies in one of the 5 infinite families just discussed, or*

(ii) *S is one of finitely many exceptions, or*

(iii) *one of the following holds:*

   (a) $G_0 \leqslant \Gamma\mathrm{L}_1(p^d)$, *or*

   (b) $V = V_n(p) \oplus V_n(p)$ *and* $G_0 \leqslant \Gamma\mathrm{L}_1(p^n) \wr C_2$ *where* $d = 2n$, *or*

   (c) $V = V_2(t^3)$ *and* $\mathrm{SL}_2(t) \trianglelefteq G_0$ *where* $p^d = t^6$.

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- Repeat this process, ruling out examples you know about, until you find $u \neq v \in L \setminus \{0\}$ such that $u - v \notin x^{G_0}$ a contradiction.

Theorem (Liebeck (1987))

Let G be a primitive group of rank 3 with socle $V = V_d(p)$. Then G_0 belongs to one of the following classes.

(A) *Infinite classes. (There are 11 classes.)*

(B) *Extraspecial classes. (Only finitely many.)*

(C) *Exceptional classes. (Only finitely many.)*

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The theorem is true, except possibly for (A9) and (A10).

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Assuming that (A1) and (A5) do not hold, and also that $G_0 \not\leqslant \Gamma L_1(p^n) \wr C_2$ when (A2) holds, we can prove the following.

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Assuming that (A1) and (A5) do not hold, and also that $G_0 \not\leqslant \Gamma L_1(p^n) \wr C_2$ when (A2) holds, we can prove the following.

Theorem

If (A2) does not hold, then only Examples (2), (3), (4) or (5) arise. If (A2) holds, then Examples (1) and (2) arise, as well as finitely many other examples.

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Let $V = V_d(3)$. Suppose that either

1. $d = 5$ and $G_0 = M_{11}$ with subdegrees 132 and 110, or
2. $d = 4$ and $G_0 = M_{10} \simeq A_6.2 \simeq \Omega_4^-(3).2$.

Then there is a partial linear space S with $\mathrm{Aut}(S) = V : G_0$.

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Then there is a partial linear space S with $\mathrm{Aut}(S) = V : G_0$.

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- The former has $243 = |V|$ lines and line size 12.
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Is there a geometric description for these examples? Ideas are welcome!