Vertex-primitive graphs having vertices with almost equal neighbourhoods, and vertex-primitive graphs of valency 5

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Easy exercise: If a vertex-primitive digraph has distinct vertices with the same neighbourhood, then it is empty or universal.

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For example, if $f(1,2,3,4) = (2,2,3,2)$ then f has kernel type $(3,1)$.

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They asked about the case when f has kernel type $(3, 2, 1, \ldots, 1)$.

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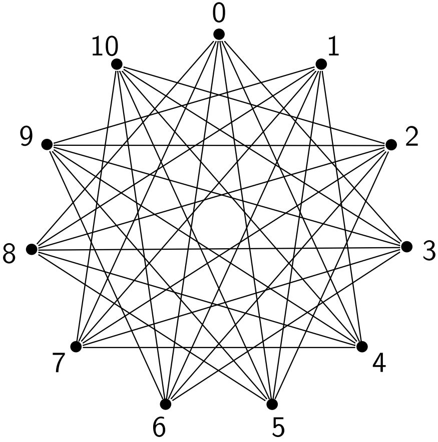
Examples

- K_n.
- C_p when p is prime.

A computer search suggested that, apart from K_n, all examples have prime order.

Typical example:



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For example, Γ_0 is the graph with two vertices adjacent if they have the same neighbourhood.

A consequence of a more general theorem

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Theorem (Spiga, Verret, 2015)

If Γ is a non-trivial vertex-primitive digraph on Ω, then either

1. $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$ *and* $(n-1)(d-\kappa) = d(d-1)$, *or*
2. *there exists* $i \in \{\kappa, \ldots, d-1\}$ *such that* Γ_i *has valency at least* 1 *and at most* $\kappa^2 + \kappa$.

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In particular, for any specific value of κ, this is a "finite" problem.

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Using our theorem, this would require classifying vertex-primitive graphs of valency at most 6.

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Difficult cases: exceptional groups of Lie type, Thompson sporadic group.

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We closed the gap: there are no examples of valency 10, but infinitely many examples of valency 12.

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Given the graphs, one still needs to do some extra work.