The structure of subdegree finite primitive permutation groups

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Permutation groups

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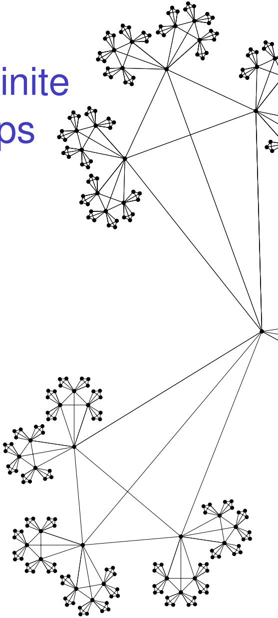

Infinite permutation groups

Throughout: $G \leq \mathrm{Sym}(\Omega)$ is transitive and Ω is countably infinite

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When studying infinite permutation groups, one typically wishes to impose some kind of finiteness condition on G

E.g:

- G has only finitely many orbits on Ω^n, for all $n \in \mathbb{N}$ (Oligomorphic)
- G_α has only finite orbits, for all $\alpha \in \Omega$ (Subdegree finite)

Subdegree finite permutation groups

All automorphism groups of connected, locally finite graphs are subdegree finite

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- (clc) Connected locally compact groups; and
- (tdlc) Totally disconnected locally compact groups

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When studying locally compact groups, one essentially needs to understand:

- (clc) Connected locally compact groups; and
- (tdlc) Totally disconnected locally compact groups

All tdlc groups have a natural permutation representation that is transitive and subdegree finite.

Permutation topology

Write $\Omega = \{\gamma_1, \gamma_2, \ldots\}$.

There is a natural complete metric d on $\mathrm{Sym}(\Omega)$ whereby if permutations g, h agree on $\gamma_1, \ldots, \gamma_n$ but disagree on γ_{n+1}, then set

$$d'(g, h) := 2^{-n},$$

and define

$$d(g, h) := \max\{d'(g, h), d'(g^{-1}, h^{-1}).\}$$

A group $G \leq \mathrm{Sym}(\Omega)$ is closed if it contains all its limit permutations.

E.g. The group $FS(\Omega)$ of permutations with finite support has closure:

$$\overline{FS(\Omega)} = \mathrm{Sym}(\Omega).$$

What is known about infinite primitive permutation groups G?

- Cheryl Praeger & Dugald Macpherson in 1993

  Classified G when G has a closed minimal closed normal subgroup that itself has a closed minimal normal subgroup

- Dugald Macpherson & Anand Pillay in 1993

  Classified G when G has finite Morley rank

- Tsachik Gelander & Yair Glasner in 2008

  Classified G when G is countable non-torsion & linear

- S. in 2014

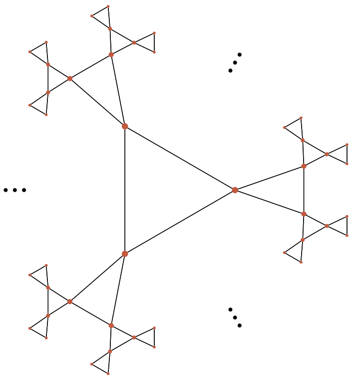
  Classified G when G has finite point stabilisers

The box product: intuition

Suppose $H \leq \mathrm{Sym}(\Delta)$ is transitive and $m \in \mathbb{N}$

Let Λ be a graph whose vertex set is Δ, such that $H \leq \mathrm{Aut}(\Lambda)$

Let $\Gamma(m, \Lambda)$ be the (infinite) graph such that every vertex x lies in m copies of Λ, and these copies only intersect at x



$\Gamma(2, K_3)$

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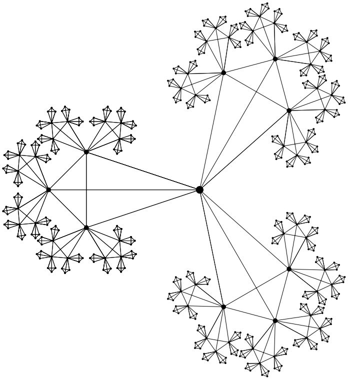
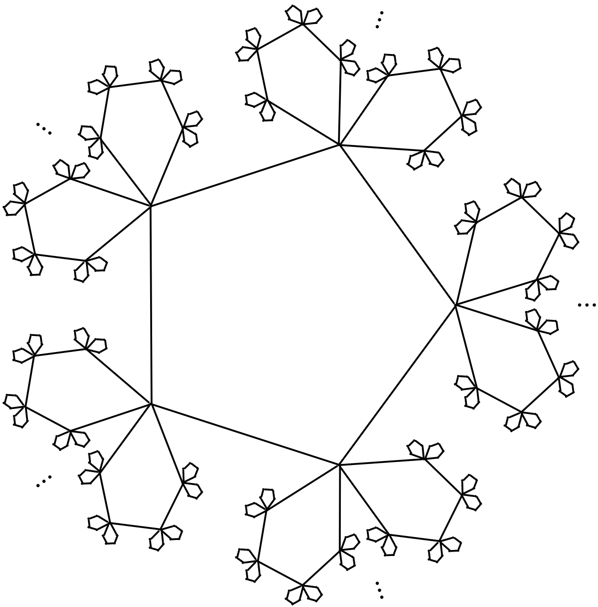


$\Gamma(3, K_4)$

The box product: intuition

The box product $H \boxtimes S_m$ is the largest transitive subgroup of $\mathrm{Aut}\left(\Gamma(m, \Lambda)\right)$ that induces H on each of the lobes

An intuitive description of $D_{10} \boxtimes S_3$



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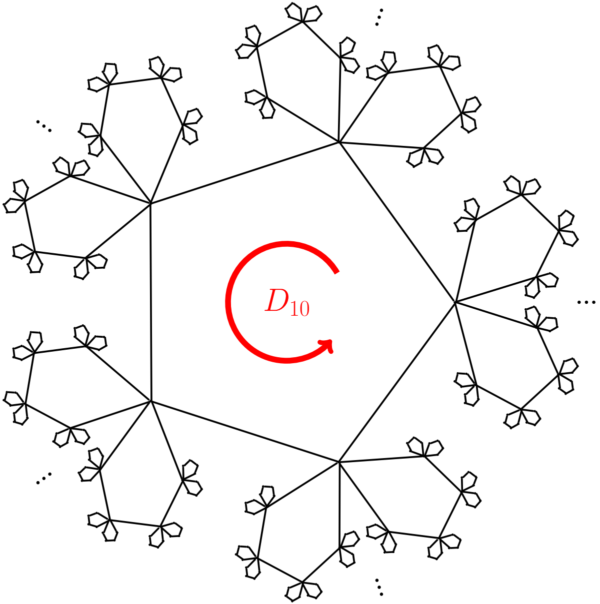

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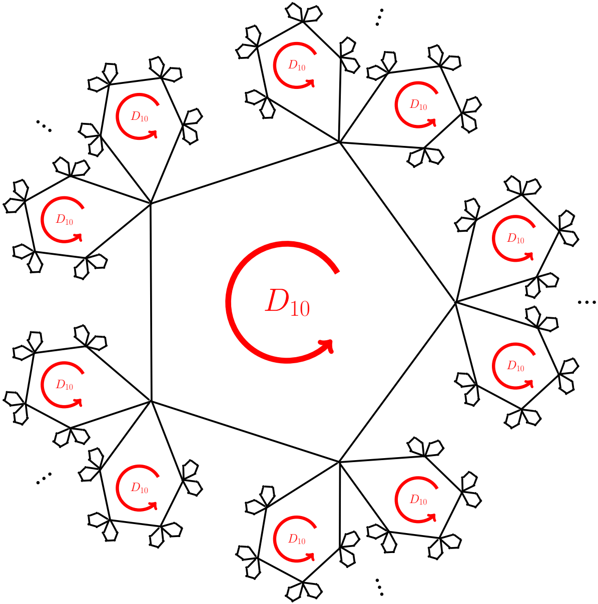

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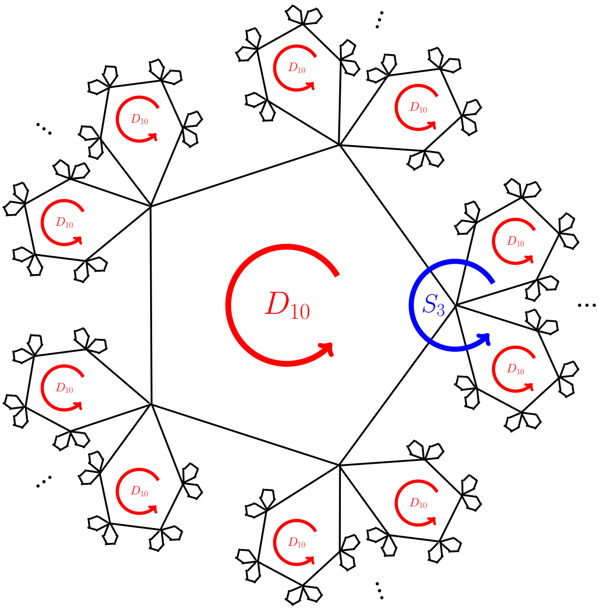

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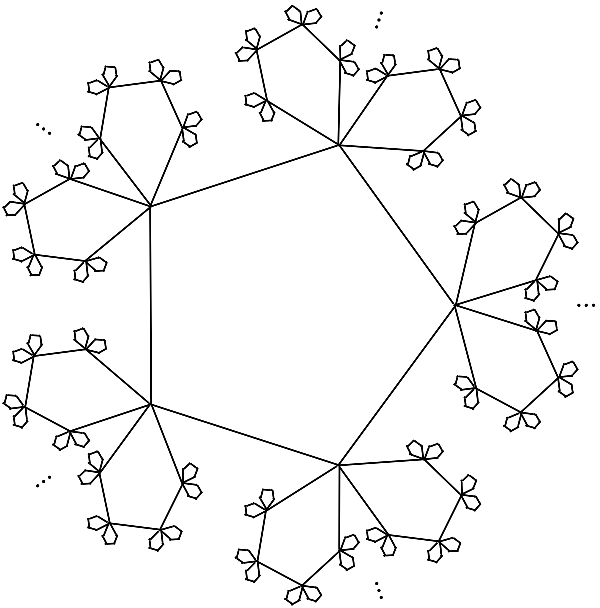

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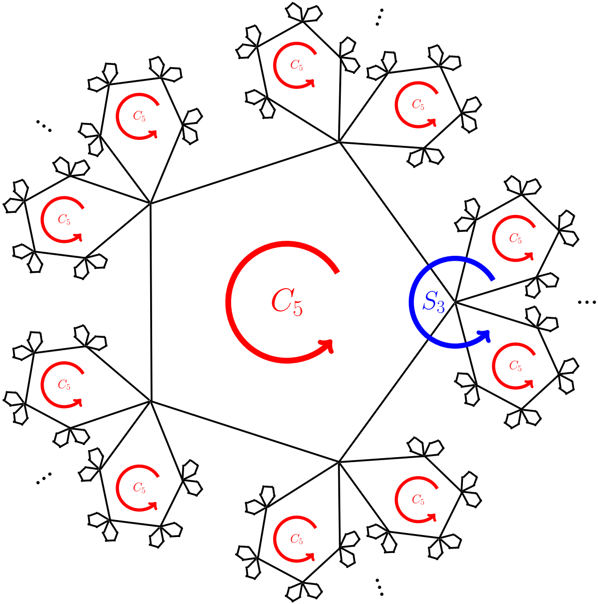

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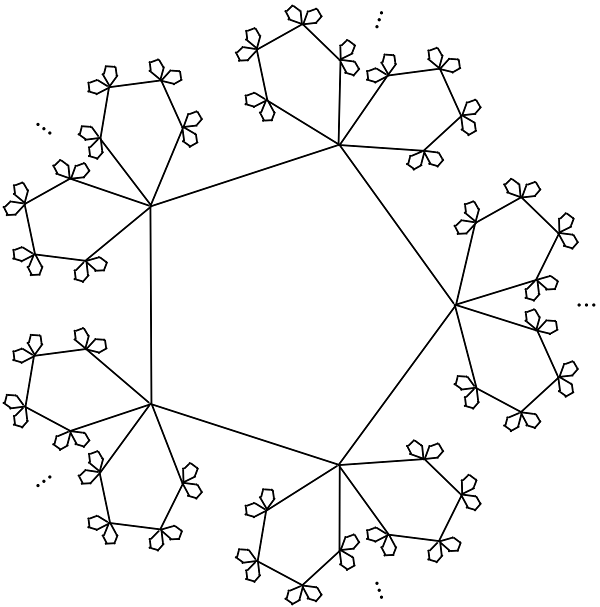

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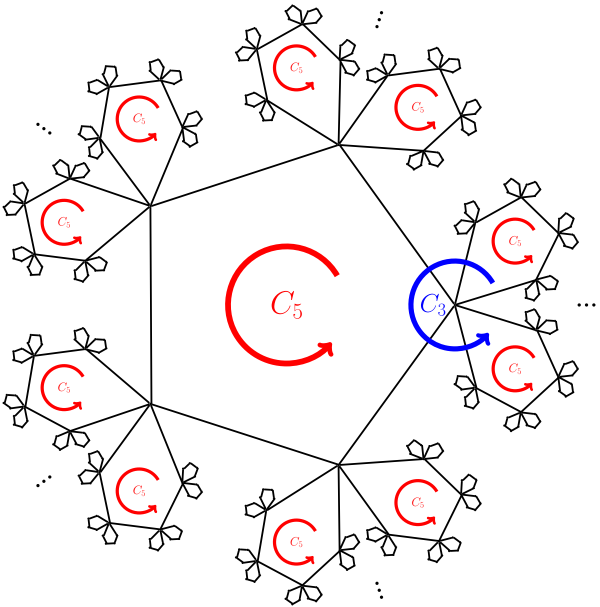

The box product: formal definition

Fix $M \leq \mathrm{Sym}(X)$ and $N \leq \mathrm{Sym}(Y)$ (not necessarily finite).

Form a biregular tree T where:

- vertices in one part V_X of the bipartition have valency $|X|$
- vertices in the other part V_Y have valency $|Y|$

A group $G \leq \mathrm{Aut}\ T$ is locally-(M, N) if G preserves V_X & V_Y and the group induced on the neighbours of v by G_v is:

- M if $v \in V_X$
- N if $v \in V_Y$

Theorem (S. ’15) If M and N are transitive, there exists a universal locally-(M, N) group, $U(M, N)$ which is itself locally-(M, N).

Definition (S. ’15) The box product $M \boxtimes N$ is $U(M, N)\big|_{V_Y}$.

Theorem (poss. attributable to W. Manning, early 20th C)

$M \operatorname{Wr} N$ acting on X^Y with its product action is primitive $\iff$

- M is primitive and not regular and
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Theorem (S., '15)

$M \boxtimes N$ acting on V_Y is primitive $\iff$

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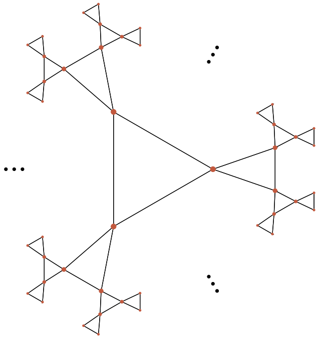
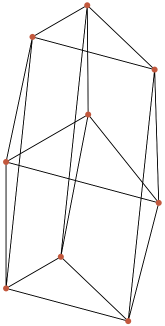
- N is transitive

Geometry

One can see the “shape” of a permutation group $G \leq \mathrm{Sym}(\Omega)$ by looking at an orbital graph Γ.



$\mathrm{Sym}(3) \boxtimes \mathrm{Sym}(2)$



$\mathrm{Sym}(3)\,\mathrm{Wr}\,\mathrm{Sym}(2)$

Structure of subdegree finite primitive groups

Theorem (S. '16) Suppose $G \leq \mathrm{Sym}(\Omega)$ is closed, infinite, subdegree finite and primitive, then G satisfies precisely one of the following:

- **[OAS]** here G is one-ended & almost topologically simple
- **[PA]** here G is a transitive subgroup of $H \operatorname{Wr} S_m$ acting with its product action for some finite $m \geq 2$, where H is the group induced on a fibre by its stabiliser in G, and H is primitive and not regular, subdegree-finite and infinite of type OAS or BP;
- **[BP]** here G is a transitive subgroup of $H \boxtimes S_n$ for some finite $n \geq 2$, where H is the group induced on a lobe by its stabiliser in G, and H is primitive and not regular, subdegree-finite and either finite of degree at least three or infinite of type OAS or PA.

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Preprint coming soon

(For the box product see: arXiv:1407.5697)

Thank you