Maximal subgroups of groups of intermediate growth

Alejandra Garrido
joint work with Dominik Francoeur

University of Duesseldorf

Permutation Groups, BIRS
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Primitive groups and Maximal subgroups

Primitive permutation actions = "atoms" of permutation actions
$G \curvearrowright X$ is primitive $\Leftrightarrow$ point stabilizers are maximal.

General question

Given a group (not as permutation group), what are its primitive permutation representations? i.e. What are its maximal subgroups?

If G is finitely generated, every proper subgroup is contained in a maximal one.

First basic question

Does a given finitely generated group contain maximal subgroups of infinite index?

The class $\mathcal{IP}$

Let $\mathcal{IP}$ denote the class of f.g. groups with some maximal subgroup of infinite index.
Some known results:

$\notin \mathcal{IP}$

- nilpotent groups
- virtually soluble linear groups [Margulis+Soifer, '81]

$\in \mathcal{IP}$

- free groups
- not v.s. linear groups [Margulis+Soifer, '81]
- mapping class groups, hyperbolic groups, other "geometric" groups (with appropriate caveats) [Gelander+Glasner, '07]

Big and small groups: word growth

Definition

The growth function $\gamma_G(n)$ of G w.r.t finite generating set S gives the number of elements of G of S-length $\leq n$.

Up to equivalence relation, $\gamma_G(n)$ does not depend on S.

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Types of growth (up to equivalence):

- $\gamma_G(n) \approx n^a, a \in \mathbb{N}$ virtually nilpotent [Wolf,Bass,Guivarch; Gromov]
- $\gamma_G(n) \approx \exp(n)$ e.g. free groups, not v.s. linear groups [Tits alternative, '72]
- $\gamma_G(n)$ is super-polynomial and sub-exponential: intermediate growth [first examples by Grigorchuk, '85]

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Question (Cornulier, '06)

Are there groups of intermediate growth in $\mathcal{IP}$?

Let T =rooted, infinite binary tree, Aut T=its group of automorphisms.

Two subgroups of Aut T

$$G_1 = \langle a, \beta, \gamma, \delta \rangle \quad G_2 = \langle a, b, c, d \rangle$$

$$a = \text{"swap" on level 1}$$

$$\beta = (a, \gamma) \qquad b = (a, b)$$

$$\gamma = (a, \delta) \qquad c = (a, d)$$

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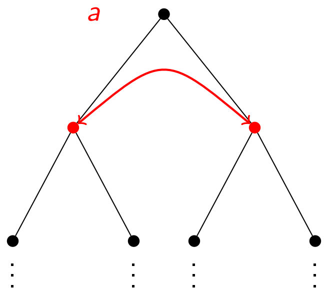

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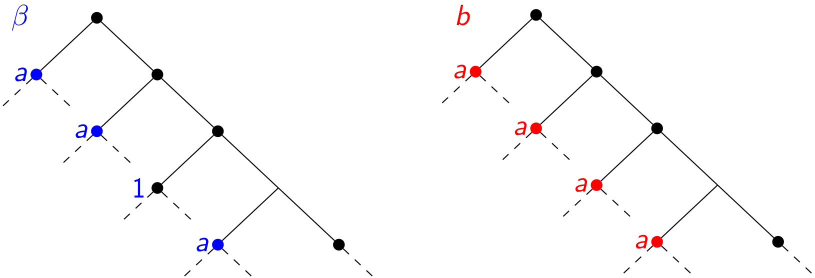

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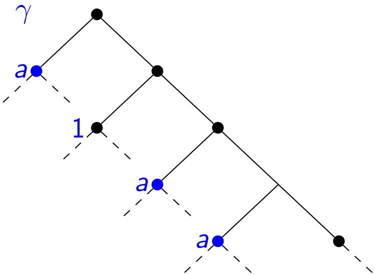

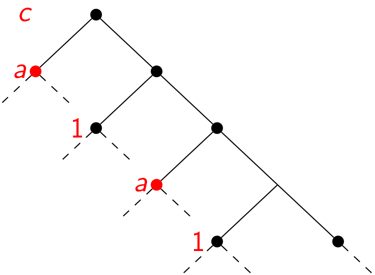

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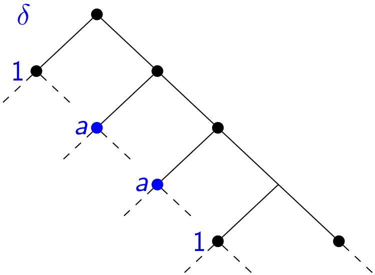

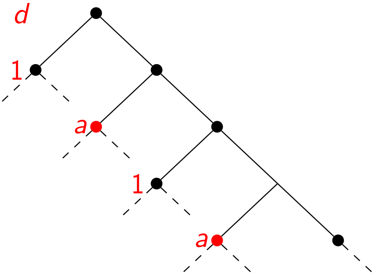

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

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Actually, we prove this for a larger family of "siblings of Grigorchuk's group" defined by Šunić. They are groups of automorphisms of the p-regular tree for p any prime. The ones on the binary tree are all of intermediate growth. We show that the non-torsion ones (which all contain D_∞) are in $\mathcal{IP}$, by finding $\aleph_0$ finitely generated maximal subgroups of infinite index.

Main results [Francoeur + G, '16]

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Additional fact: Each $H(q)$ is conjugate to G_2 in Aut T.

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Corollary

- G_2 *is a primitive permutation group (acts faithfully on cosets of infinite index maximal subgroup).*
- G_2 *has trivial Frattini subgroup (Cfr.* G_1 *has Frattini subgroup of finite index).*

How to find maximal subgroups of infinite index?

Classical idea: find dense subgroups in profinite topology.

Definition

The profinite topology of a group G has $\{N \lhd G \mid |G : N| < \infty\}$ as base of neighbourhood of the identity.

$H \leq G$ is dense if $HN = G$ for every $N \lhd G$ with $|G : N| < \infty$.

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Fact

G has a maximal subgroup of infinite index if and only if it has a proper subgroup which is dense in the profinite topology.

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So suffices to find dense subgroup for Aut T topology (=level stabilizers form base of neighbourhoods of identity).

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Want: $H < G_2$ such that $H\,\mathrm{St}_{G_2}(n) = G_2$ for each $n \in \mathbb{N}$.

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Let T be the rooted, infinite d-regular tree and $G \leq \mathrm{Aut}\ T$ be generated by $g_1, g_2, \ldots$. Then, $\langle g_1^{n_1}, g_2^{n_2}, \ldots \rangle$ is dense in G for the Aut T topology for any $n_1, n_2, \cdots \in \mathbb{N}$ coprime with $d!$.

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Corollary

Let q be an odd integer, then $H(q) = \langle (ab)^q, b, c, d \rangle$ is a dense subgroup of G_2 for the profinite topology.

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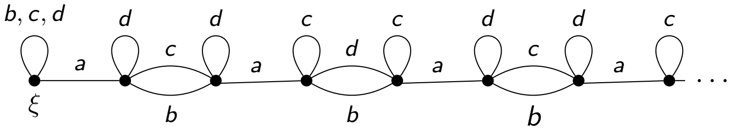
Look at actions of $H(q)$ and G_2 on boundary of tree T. Suffices to consider orbit of ξ =rightmost ray. Thanks to copy of dihedral group $\langle a, b \rangle$, the orbit of ξ under G_2 is isomorphic to $\mathbb{Z}$. But the orbit under $H(q)$ is strictly smaller (corresponds to $q\mathbb{Z}$):



H_q are maximal for q odd prime, not much more

Some technical work, using techniques similar to those of Pervova to show:

Theorem

Let q be an odd prime, then $H(q)$ is maximal and of infinite index in G_2.

Theorem

There are at most $\aleph_0$ maximal subgroups of infinite index in G_2.

They all map onto some $H(q)$.

Thank you!