Representation Stability & the S_n-module Structure in the Partition Lattice

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- joint work with Vic Reiner

(based on paper to appear in IMRN, "Rep'n stability for cohomology of configuration spaces in $\mathbb{R}^d$")

Representation Theoretic Stability

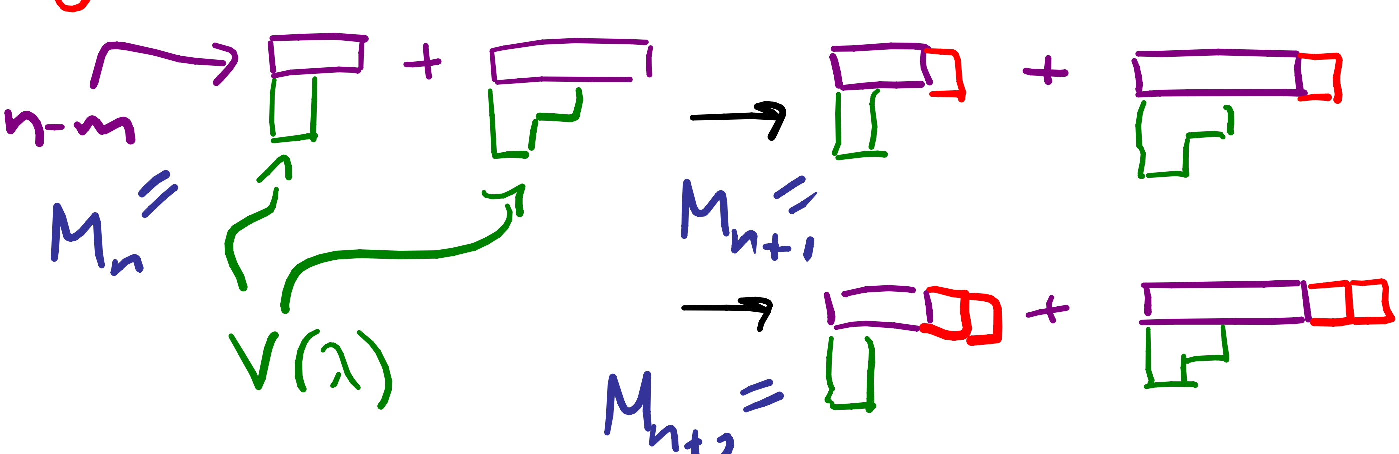
Defn (Church, Farb): A series of S_n-modules $M_1, M_2, \ldots$ stabilizes at $B>0$ if for each $n>B$, we have

$$M_n = \sum_{\lambda \vdash m \le B} c_\lambda V(\lambda)$$

where $V(\lambda) \cong S^{(n-m,\lambda)}$

and where c_λ does not depend on n

e.g.



Our Focus: S_n-rep'ns from partition lattice

Our Starting Point:

Thm (Church-Farb): $H^i(M_n)$ stablilizes for $n \geq 4i$ where M_n is configuration space of n distinct points in plane & i is held fixed.

Thm (Church-Farb): More generally, letting M_n^d be the configuration space of n distinct points on a connected orientable d-manifold, $H^i(M_n^d)$ stabilizes for

$$\begin{cases} n \geq 4i & \text{if } d=2 \\ n \geq 2i & \text{if } d>2 \end{cases}$$

Our First Objective: Sharpen these bounds for $PConf_n(\mathbb{R}^d)$

Church-Farb Method for Orientable Manifolds

- Use Totaro's E_2-page of Leray spectral sequence showing cohom. of manifold M + $H^i(M_n(\mathbb{R}^d))$ determines cohomology of config. space of n distinct pts on M as follows:

$$E_2^{p,(d-1)q} = \bigoplus_{\substack{S \text{ with} \\ |S| = n-q}} H^{q(d-1)}(\underbrace{C_S(\mathbb{R}^d)}_{\text{product of subspace arrangement complements}}) \otimes H^p(M^S)$$

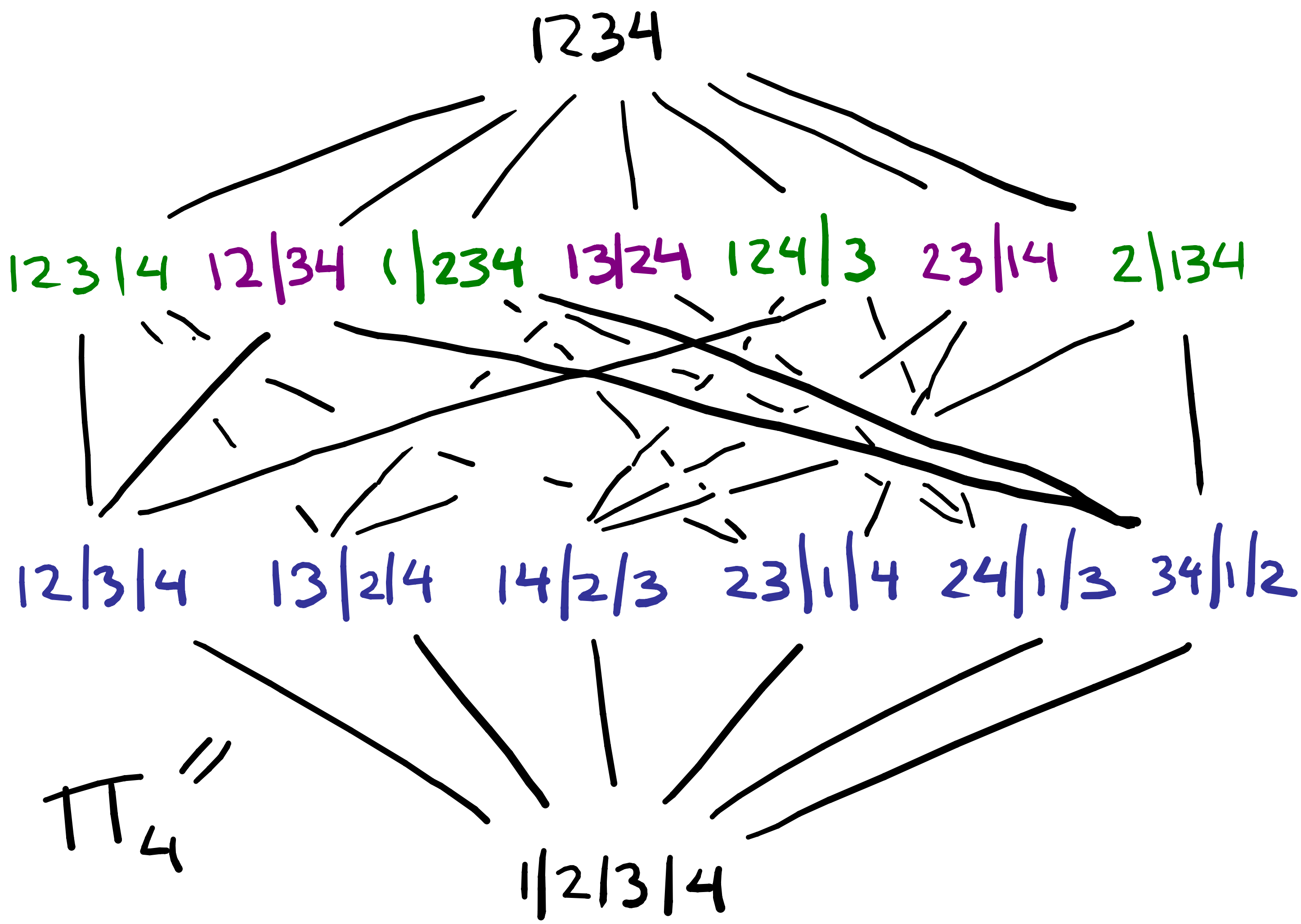
for set partition S with $|S|$ parts &

e.g. for $S = \{1,3\}\{2,4,5\}$

- $C_S(M) := \{\underline{x} \in M^5 \mid x_1 \neq x_3 ; x_2 \neq x_4 \neq x_5\}$
$= C_{\{1,3\}}(M) \times C_{\{2,4,5\}}(M)$
- $M^S := \{\underline{x} \in M^5 \mid x_1 = x_3 ; x_2 = x_4 = x_5\}$

& $E_2^{p,q} = 0$ for $d-1 \nmid q$

Partition Lattice Π_n & its S_n-representions



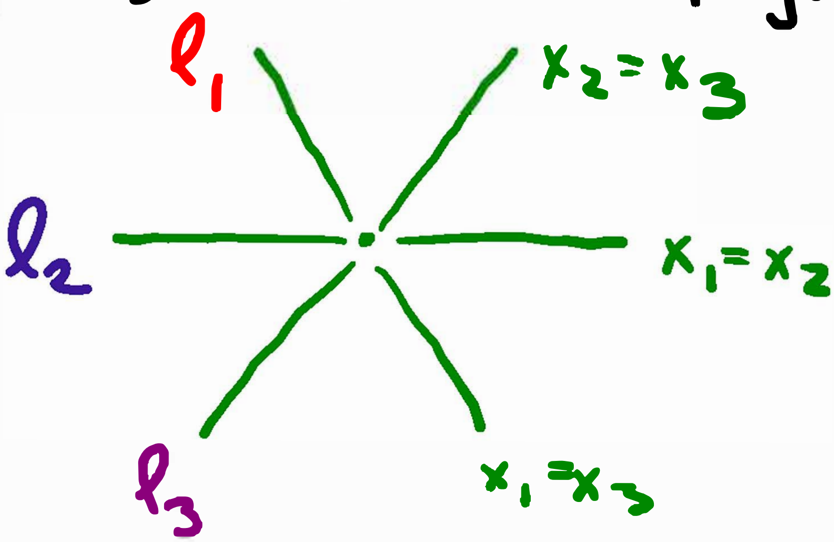
$\Pi_4 =$

- S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

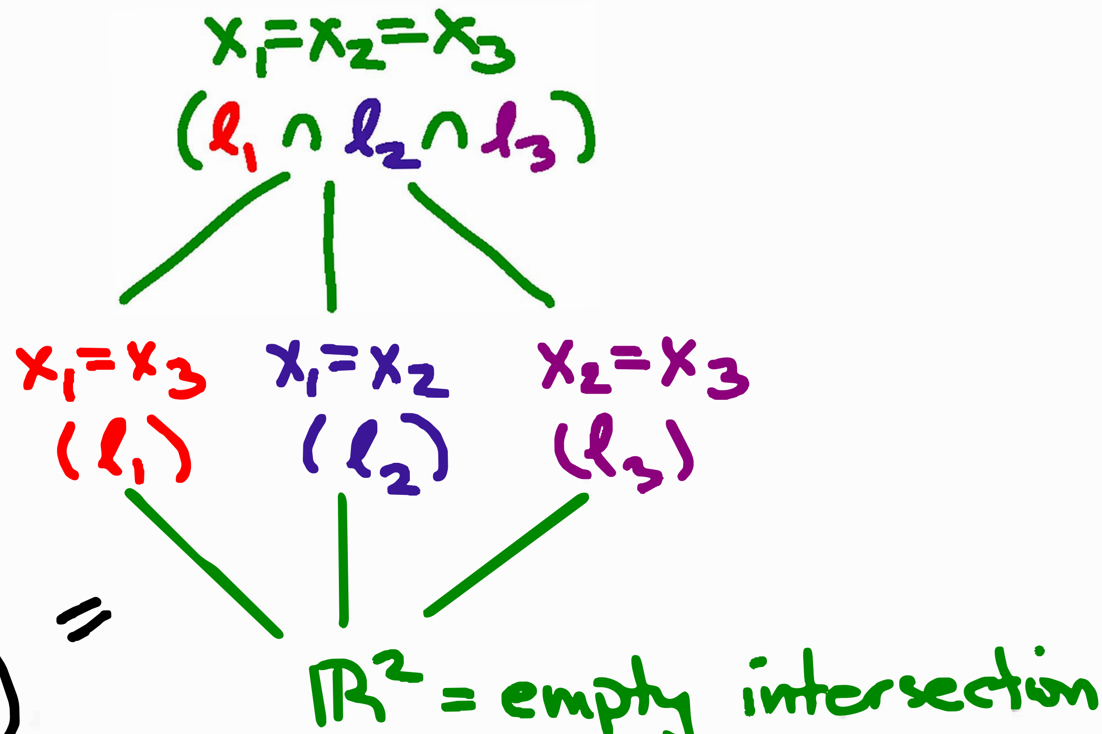
Reinterpreting via Subspace Arrangement Complements

- M_n = complement of type A (complex) braid arr't $\{x_i = x_j \mid 1 \le i < j \le n\}$



- Π_n = intersection poset $\mathcal{L}(A_{n-1})$
- S_n-module structure for $H^i(M_n)$ will translate (via Goresky-MacPherson formula) to "Whitney homology" in Π_n.

$\mathcal{L}(A_2) =$

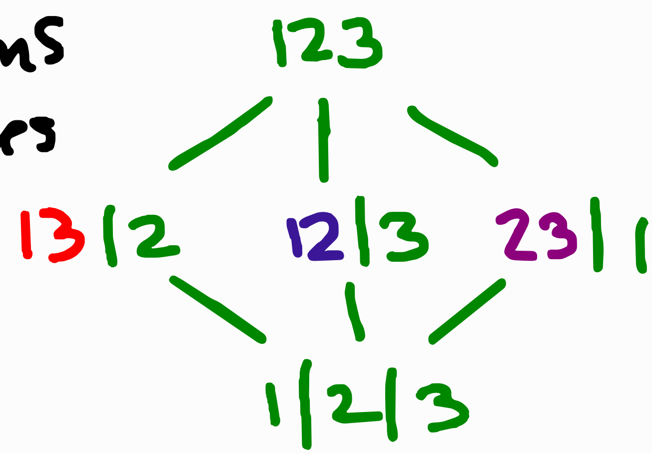


$\parallel$

poset of intersections of subspaces

$\cong$

$\Pi_3 =$



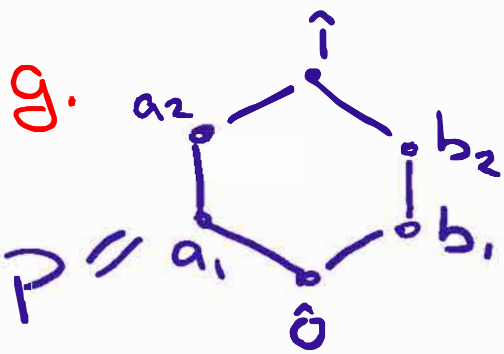
$\parallel$

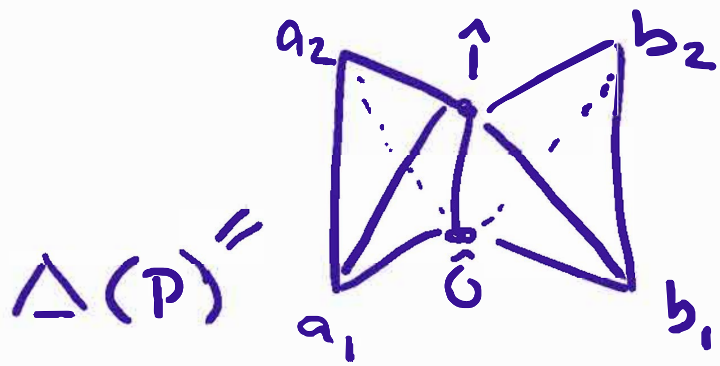
lattice of set partition

$x_i = x_j \iff i,j$ in same block

Def'n: The order complex of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains in P.

e.g.





- Let $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$ e.g. for Π_n

Convention: When we speak of topological properties (homology, etc.) of poset P, we mean $\Delta(P)$ or $\Delta(\overline{P})$.

Poset rank := # steps from bottom

S_n-Representations on Chains (i.e on Faces) & on Homology

- S_n action on set partitions is order-preserving & rank-preserving
- Thus, it induces S_n-action α_S on chains $u_1 < u_2 < \dots < u_j$ with u_r of rank i_r for $1 \le r \le j$ and $S = \{i_1, \dots, i_j\}$, in other words on faces of $\Delta(\overline{\Pi}_n)$ with vertices colored S, where vertices in $\Delta(\overline{\Pi}_n)$ are colored by poset rank.

- S_n-action on chains commutes with simplicial boundary map

$$d(u_0 < \dots < u_r) = \sum_{0 \le i \le r} (-1)^i (u_0 < \dots < \hat{u}_i < \dots < u_r)$$

- Thus, S_n-action on i-faces induces S_n-rep'n on ith homology

- But homology of Π_n is concentrated in top degree due to its "EL-shellability":

Thm (Stanley + Björner): Π_n is supersolvable, hence is EL-shellable.

- Likewise, homology of $\Pi_n^S = \{u \in \Pi_n \mid rk(u) \in S\}$ also concentrated in top degree by:

  Thm (Björner): P graded & EL-shellable $\Rightarrow$ P^S also EL-shellable. In particular, P^S is Cohen-Macaulay.

- The virtual rep'n $\beta_S := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$ is actual S_n-rep'n on top homology of $\Pi_n^S := \{u \in \Pi_n \mid rk(u) \in S\}$

Thm (H-Reiner): $\beta_S(\Pi_n)$ stabilizes at $n \geq 4\max(S)$ for any fixed S.

Upcoming Case for Config. Spaces on Manifolds: $S = \{1, 2, \ldots, i\}$, where we will do better...

Whitney Homology

$WH_i(P) :=$ "i-th Whitney homology of P"

$$= \bigoplus_{rk(u)=i} \tilde{H}_{i-2}(\hat{0}, u) = \bigoplus_{\lambda \text{ has } n-i \text{ blocks}} WH_\lambda(P)$$

$$WH_\lambda(P) := \bigoplus_{\substack{u \in P \\ type(u)=\lambda}} \tilde{H}_{top}(\hat{0}, u)$$

Thm (Sundaram): $WH_i(P) \cong \beta_{1\ldots i}(P) + \beta_{1\ldots i-1}(P)$

Observations:

1. $WH_i(\Pi_n)$ stabilizes at same bound as $\beta_{1\ldots i}(\Pi_n)$.

2. $\alpha_S = \sum_{T \subseteq S} \beta_T$ & $\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$

allowing transfer of stability bounds

Goresky-MacPherson formula

(for cohomology of subspace arr't)

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}}^{>0}} \tilde{H}_{\mathrm{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection lattice

Plan: Apply to braid arrangement using S_n-equivariant version due to Sundaram-Welker yielding Whitney homology.

G-Equivariant Enrichment of Goresky-MacPherson Formula

Thm (Sundaram-Welker): Let A be a G-arrangement of $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$ for G a finite subgroup of $GL_n(\mathbb{C})$. Then

$$\tilde{H}^i(M_A) \cong_G \bigoplus_{x \in (L_A^{>0})/G} \mathrm{Ind}_{\mathrm{Stab}(x)}^{G} \tilde{H}_{\mathrm{codim}(x)-i-2}(\hat{0}, x)$$

(in our case)

$$= WH_i(L_{A_{n-1}}) = WH_i(\Pi_n)$$

Note: Config. space of n distinct points in $\mathbb{R}^{2d}$ gives generating subspaces of real codim $2d$ & $\tilde{H}^i(M_A) = 0$ unless $i = (2d-1)\mathrm{rk}(x)$ for some x, i.e. unless $2d-1$ divides i.

Upshot for Stability:

- $\beta_{1,j,i}(\Pi_n)$ stabilizes at $B>0$

  $\iff WH_i(\Pi_n)$ stabilizes at $B>0$

  $\iff H^i(M_n)$ stabilizes at $B>0$

Thm (H-Reiner): $H^i(M_n)$ stabilizes sharply at $3i+1$. More generally, $H^i(M_n^{2d})$ for M_n^{2d} = config. space of n distinct pts in $\mathbb{R}^{2d}$ stabilizes sharply for $n \geq 3\frac{i}{2d-1}+1$.

Thm (H-Reiner): $\langle H^i(M_n^{2d}), S^{(n-|\nu|,\nu)}\rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu|+i & \text{for } d \text{ odd} \\ |\nu|+i+1 & \text{for } d \text{ even} \end{cases}$

Proof Techniques & Results We'll Use

Thm (Hanlon–Stanley): $\Pi_n \cong \mathrm{sgn} \otimes \left(\xi_n \uparrow_{C_n}^{S_n} \right)$

Method: Calculate $\mu_{\Pi_n^g}(\hat{0},\hat{1}) = \chi_{\Pi_n}(g)$

Thm (Joyal): $\mathrm{lie}_n \cong \xi_n \uparrow_{C_n}^{S_n}$

Thm (Barcelo): Explicit S_n-equivt bijection yielding $\Pi_n \cong \mathrm{sgn} \otimes \mathrm{lie}_n$

Thm (Kraskiewicz & Weyman):

$$\mathrm{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \mathrm{maj}(T) \equiv 1 \pmod{n}}} S^{\lambda(T)}$$

* **Key Fact for Stability:** $u \in \Pi_n$ of rank i has at most $2i$ letters in nontrivial blocks

Thm (Sundaram):

S_j-rep'n on top homology of Π_j

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\pi_j] \prod_{j \text{ even}} e_{m_j}[\pi_j]$$

$$= \underbrace{(h_{m_1})}_{j=1 \text{ part}} \underbrace{\left(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\pi_j]\right)\left(\prod_{j \text{ even}} e_{m_j}[\pi_j]\right)}_{\text{"}\widehat{Wh_\lambda}\text{" has degree} \leq 2i \text{ by } \star}$$

where ch = "Frobenius characteristic" isom.

$ch(f) = \sum_\mu f(\mu)\frac{p_\mu}{z_\mu}$ from S_n class functions to ring of symmetric fn's

$$h_n := \sum_{1 \leq i_1 \leq i_2 \leq \ldots} x_{i_1} x_{i_2} \cdots x_{i_n} = ch(\text{trivial rep'n})$$

$$e_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = ch(\text{sgn rep'n})$$

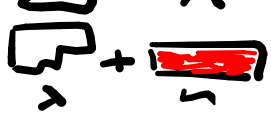
$$M\left(\sum c_\mu x^\mu\right) := \sum c_\mu \left(x^\mu \otimes \mathbb{1}_{n-|\mu|}\uparrow_{S_{|\mu|}\times S_{n-|\mu|}}^{S_n}\right)$$

Thm (H-Reiner): Holding i fixed & letting n grow, $\beta_{1,i}(\Pi_n)$, $WH_i(\Pi_n)$ & $H^i(M_n)$ stabilize sharply as S_n-rep'ns at $n = (2i) + (i+1)$.

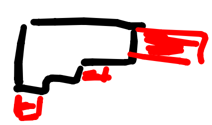
Idea: $\widehat{WH}_i = \bigoplus_{\lambda \vdash n} WH_\lambda(\Pi_n)$ stabilizes at $n = 2i$ since at most $2i$ letters in nontrivial blocks. Obtain upper bound of $i+1$ on length of 1st row in S^λ in $\widehat{WH}_i$. Pieri Rule says multiplying $ch(\widehat{WH}_i) = \bigoplus c_\lambda s_\lambda$ by $h_{\lambda_1 + j}$ is stable.

Pieri Rule:

$$h_n s_\lambda = \sum s_\mu$$



$= \mu$



$= \mu$

Key Properties of Symmetric Functions

- $S^\lambda \leftrightsquigarrow$ schur fn $s_\lambda = \displaystyle\sum_{\substack{T \text{ SSYT} \\ \text{shape } \lambda}} x^T$

$$T = \begin{array}{l} \overbrace{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline \end{array}}^{\lambda_1} \\ \begin{array}{|c|c|}3 & 4 \\ \hline \end{array} \end{array} \rightsquigarrow \underbrace{x_1^2 x_2^2 x_3 x_4}_{x^T \;=}$$

$\Longrightarrow$ $\boxed{\text{$s_\lambda$ must include some monomial divisible by $x_1^{\lambda_1}$.}}$

- Wreath product of rep'ns $\leftrightsquigarrow$ plethysm of symmetric fns

$\Longrightarrow$ $\boxed{\text{$f$ includes x_1^a \& g includes x_1^b then $f[g]$ includes x_1^{ab}.}}$

Using Symmetric Functions to Bound λ_1 and $\ell(\lambda)$

- $\lambda_1(f\cdot g) = \lambda_1(f) + \lambda_1(g)$
- $\ell(f\cdot g) = \ell(f) + \ell(g)$
- $\lambda_1(f[g]) = \lambda_1(f)\cdot\lambda_1(g)$
- $\ell(f[g]) = \ell(f)\cdot\ell(g)$
- $\lambda_1(\Pi_n) \leq n-1$ for $n>2$
- $\lambda_1(e_n[\Pi_2]) = n+1$

Motivations from Number Theory:

- Church-Ellenberg-Farb & Matchett/Wood-Vakil, & others:

$$\langle H^i(PConf_n(\mathbb{C})), V\rangle_{S_n} = \dim_{\mathbb{Q}_\ell} H^i_{et}(Conf_n; V)$$

coefs twisted by V

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" & counting fixed pts of Frobenius map

e.g. # $\square$-free degree n polys $= q^n - q^{n-1}$

Remark: Applications to number theory focus on $M = \mathbb{R}^2$ case

Translating "Polynomial Characters" into Symmetric fns (to get Improved "Power Saving Bounds")

- Any polynomial $P(x_1, x_2, x_3, \ldots)$ gives a class fn for S_n by letting $x_i = \#$ i-cycles in conjugacy class
- The elements $\binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$ where λ has m_i parts of size i form a basis for $\mathbb{Q}[x_1, x_2, x_3, \ldots]$

Prop'n (H-Reiner): $ch(\chi_P) = \begin{cases} \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$

for $P = \binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$

Combining with Earlier Results ...

Prop'n (H-Reiner): $ch(x_P) = \begin{cases} \frac{p_\lambda}{z_\lambda} h_{n-|\lambda|} & \text{for } n \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$

for $P = \binom{X}{\lambda} = \binom{x_1}{m_1}\binom{x_2}{m_2}\cdots$

- guarantees for all $P \in \mathbb{Q}[x_1, x_2, \ldots]$, $x_P = M\left(\sum_\mu c_\mu x^\mu\right)$ s.t. $|\mu| \leq \deg(P)$ $\forall \mu$.
- analyze $\langle x_P, H^i(M_n^{2d}) \rangle$ via:

Thm (H-Reiner): $\langle H^i(M_n^{2d}), S^{(n-|\nu|, \nu)} \rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu| + i & \text{for } d \text{ odd} \\ |\nu| + i + 1 & \text{for } d \text{ even} \end{cases}$

Upshot: $\langle x_P, H^i P(\mathrm{Conf}(\mathbb{C})) \rangle_{S_n}$ is constant for $n \geq \max\{2\deg(P), \deg(P) + i + 1\}$.

Wiltshire-Gordon Conjectures & Related Results

Defn (Wiltshire-Gordon):

$$V_n^k = \bigoplus_{\substack{|\lambda|=n \\ \ell(\lambda)=n-k \\ \lambda \text{ has no parts of size } 1}} WH_\lambda(\Pi_n)$$

Thm (H-Reiner):

$$\mathrm{Ind}\left(\mathrm{Res}\left(V_n^k\right) \oplus V_{n-1}^k\right) \cong \mathrm{Res}\left(V_{n+1}^{k+1}\right)$$

(conjectured by Wiltshire-Gordon)

Idea: Symmetric fns & generating fns

Qn: Pf by clever homology bases?

Refinement for equation (by block structure)?

Thm (H-Reiner):

$$V_n = ch\Big(\bigoplus_k V_n^k\Big) \cong \bigoplus S^{\lambda(T)}$$

T is "Whitney generating" SYT

where T is **Whitney generating** if either

(1) $T = \emptyset$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

or

(2) $T\big|_{\{1,2,3,4\}}$ is one of the four shapes:

$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ $\quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$

$T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ $\qquad$ $T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$

with the following further restrictions:

(a) If T_3, then the first ascent k with $k \geq 4$ is odd

(b) If T_4, then the first ascent k with $k \geq 4$ is even

ascent := i such that $i+1$ in weakly higher row

Idea: Both sides satisfy same recurrence: categorified $d_n = n d_{n-1} + (-1)^n$

Qn: Refined formula for individual k?

Sharp Stability of Rank-Selected Homology in Π_n?

Conjecture (H-Reiner): for fixed $S \subseteq \{1,2,\ldots,n-2\}$ with $i = \max(S)$, the rank-selected homology $\beta_S(\Pi_n)$ stabilizes sharply at $n = 4i - |S| + 1$.

Rk: For $S = \{1,2,\ldots,i\}$, this yields $3i+1$ & for $S = \{i\}$, this yields $4i$.

Comparison: $WH_{S,i}(\Pi_n) := \beta_S + \beta_{S \setminus \{i\}}$ has component (where rank i has all blocks of size 2 & 1) that does not stabilize earlier than this.

Thm (H-Reiner): $\beta_S(\Pi_n)$ for any fixed S stabilizes for $n \geq 4\max S$.

Idea: Show $ch(\alpha_S(\Pi_n))$ has upper bound of $2\max(S)$ on length of 1st row in $\widehat{\alpha_S(\Pi_n)}$

- This gives stability bound $n \geq 4\max S$ for $\alpha_S(n)$ & likewise for $\alpha_T(n)$ for each $T \subseteq S$.

- Deduce same bound for $\beta_S(n)$ using that $\beta_S(n) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(n)$

Thm: $\langle 1, \beta_S(\Pi_n)\rangle$ stabilizes for $n \geq 2\max S - \binom{|S|-1}{2}$

Idea: Use partitioning for $\Delta(\Pi_n)/S_n$ from (H., 2003) and consequent combinatorial interpret. for $\langle 1, \beta_S(\Pi_n)\rangle$.

$$\langle 1, \beta_S(\Pi_n)\rangle = h_S(\Delta(\Pi_n)/S_n)$$

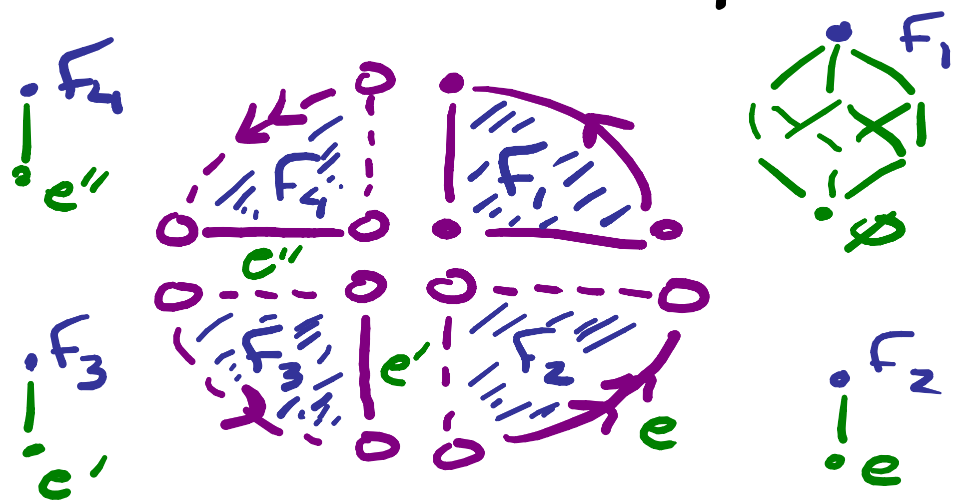
- Injection $\varphi_n : \left\{\begin{array}{c}\text{facets} \\ \text{contrib.} \\ \text{to } b_S(n)\end{array}\right\} \to \left\{\begin{array}{c}\text{facets} \\ \text{contrib} \\ \text{to} \\ b_S(n+1)\end{array}\right\}$ eventually also a surjection.

Rk: This is sharp for $S = \{i\}$ but not for every single choice of S.

e.g. (Hanlon): $\langle 1, \beta_{\{1,\ldots,i\}}(\Pi_n)\rangle = 0$ for $n > 2$.

Partitioning: Δ is partitionable if face poset $F(\Delta)$ decomposes into disjoint union of boolean algebras w/ facets as top elements

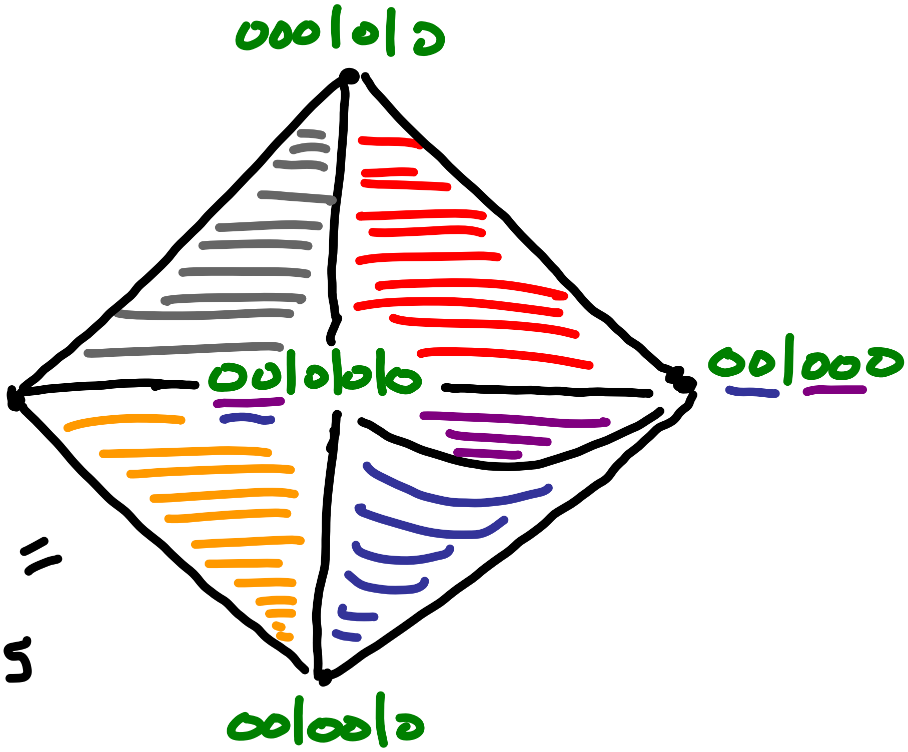
e.g.



- $h_S(\Delta(\Pi_n)/S_n) = \langle 1, \beta_S(\Pi_n) \rangle$ as # saturated chain orbits with "topological descent set" S.
- $\Delta(\Pi_n)/S_n$ "almost" shellable but $lk_\Delta(F) \cong \mathbb{RP}^n$ for some F

Depicting Faces & a Chain Labeling

e.g.



$\Delta(\Pi_5^*)/S_5 =$

- Label saturated chain S_n-orbits in Π_n^* with sequence of separator insertions positions each as far left as possible

e.g. $\lambda(0\overset{}{\underset{3}{|}}00\underset{1}{|}00\underset{2}{|}00) = 351\ldots$

- $0|0|00|000|000|00000|00000$

$\varphi_n = \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \leftarrow n \cdot \max S - 1$

Some Further Questions:

1. (Farb) How fast does the multiplicity of any particular $V(\lambda)$ stabilize within M_n?

2. (H-Reiner) How fast does the multiplicity of $V(\lambda)$ stabilize within $\beta_S(\Pi_n)$ as S is held fixed & n grows?

(Note: Qn 1 is special case of Qn 2 with $S = \{1, 2, \ldots, i\}$)

3. Is there an explicit homology basis for Π_n explaining:

Thm (Kraskiewicz & Weyman):

$$\Pi_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \mathrm{maj}(T) \equiv 1 \pmod{n}}} S^{\lambda(T)^{\text{transpose}}}$$

Note: Solving 3 might help with 2 if homology basis helps us find homology basis for each Π_n^S.

4. (Farb) What rep'ns do we get after stabilization occurs?

5. Decomposition into irreducibles for each V_n^R individually?