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Wave-Number Explicit Convergence Analysis for Galerkin-type discretizations of the Helmholtz equation

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joint work with J.M. Melenk (TU Wien) and I.G. Graham (U Bath)

Setting, Main Assumptions I

Let $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain and $\omega \geq \omega_0 > 0$. Let $c \in L^\infty(\Omega)$ and $\mathbf{A} \in \mathbf{W}^{1,\infty}\left(\Omega^*, \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)$ be given for some domain $\Omega^* \supset \overline{\Omega}$ with

$$0 < \alpha := \inf_{\mathbf{x}\in\Omega^*} \lambda_{\min}(\mathbf{A}(\mathbf{x})) \leq \sup_{\mathbf{x}\in\Omega^*} \lambda_{\max}(\mathbf{A}(\mathbf{x})) =: \beta < \infty,$$
$$0 < c_{\min} := \inf_{\mathbf{x}\in\Omega} c(x) \leq \sup_{\mathbf{x}\in\Omega} c(x) =: c_{\max} < \infty.$$

Let $\mathcal{H} \subset H^1(\Omega)$ be a closed subspace with norm*

$$\|u\|_{\mathcal{H},\Omega} := \left(\|\nabla u\|^2 + \left\|\frac{\omega}{c} u\right\|^2\right)^{1/2}.$$

* $(\cdot,\cdot)$: $L^2(\Omega)$ scalar product, $\|\cdot\|$: $L^2(\Omega)$ norm.

Variational Formulation of Helmholtz equation, Main Assumptions II

Sesquilinear form $A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ can be split into $A = a - b$ with

definition of a	$a\,(u, v) = (\mathbf{A}\nabla u, \nabla v) - \left(\frac{\omega}{c}u, \frac{\omega}{c}v\right)$,
continuity of b	$\forall u, v \in \mathcal{H} \quad |b\,(u, v)| \leq C_b \, \|u\|_{\mathcal{H}} \, \|v\|_{\mathcal{H}}$
definiteness of b	$(u \in \mathcal{H} \wedge \mathsf{Im}\, b\,(u, u) = 0) \implies u|_{\partial\Omega} = 0$,
	$\forall u \in \mathcal{H} \quad \mathsf{Re}\, b\,(u, u) \leq 0$,
b : boundary operator	$(uv)|_{\partial\Omega} = 0 \implies b\,(u, v) = 0$,

Helmholtz Problem. For $F \in \mathcal{H}'$ seek $u \in \mathcal{H}$ such that

$$A(u, v) = F(v) \quad \forall v \in \mathcal{H}.$$

Theorem 1. Let all assumptions be satisfied. The Helmholtz problem and its adjoint equation have a unique solution.

The proof (e.g., Leis) uses Fredholm theory, elliptic regularity, unique continuation principle.

Stability results

Let $F(v) = \int_\Omega f\overline{v}$ for some $f \in L^2(\Omega)$. Let $u \in \mathcal{H}$ denote the solution of the Helmholtz problem. Theorem 1 implies that there is a constant $C_{\mathsf{stab}} > 0$ independent of f such that

$$\|u\|_{\mathcal{H}} \leq C_{\mathsf{stab}} \|f\| .$$

Question: How does C_{stab} depend on data?

Proposition (Melenk, 1995). Let Ω be a bounded star-shaped domain with smooth boundary or a bounded convex domain. Let $\mathbf{A} = \mathbf{I}$ and $c = 1$. Let $b(u, v) = \pm \mathrm{i}\,\omega \int_{\partial\Omega} u\bar{v}$. Then, there is $C_{\mathsf{stab}} > 0$ depending only on Ω such that for any $f \in L^2(\Omega)$ the solution u of the Helmholtz problem satisfies

$$\|u\|_{\mathcal{H}} \leq C_{\mathsf{stab}} \|f\|,$$
$$|u|_{H^2(\Omega)} \leq C_{\mathsf{stab}} (1 + \omega) \|f\|.$$

The proof employs elliptic regularity and the "Rellich trick", i.e., using the test function $v = \langle \mathbf{x}, \nabla u \rangle$ and integration by parts.

Theorem (Melenk, 2012, Melenk/Esterhazy 2012). Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain. Let $\mathbf{A} = \mathbf{I}$ and $c = 1$. Let $b\left(u, v\right) = \pm \mathrm{i}\,\omega \int_{\partial\Omega} u\bar{v}$. Then, there is $C > 0$ depending only on Ω such that for any $f \in L^2\left(\Omega\right)$ the solution u of the Helmholtz problem satisfies

$$\|u\|_{\mathcal{H}} \leq C\omega^{5/2} \|f\| .$$

The proof uses a) Fourier analysis of the full space problem, b) "Helmholtz" extension operators, c) layer potentials.

Theorem (Betcke et al. 2010).

Geometric assumptions: Let $\mathcal{E} := \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 < 1 \right\}$ for some $a_1 > a_2 > 0$ and let $\Omega := \mathbb{R}^2 \backslash \overline{\mathcal{E}}$.

Functional assumption: $\mathbf{A} = \mathbf{I}$ and $c = 1$. $\mathcal{H} := H^1_{w,0}(\Omega)$: weighted Sobolev space for the exterior Helmholtz equation and $b(u,v) = 0$.

Then there exists a sequence $0 < \omega_0 < \omega_1 < \ldots \to \infty$ such that the inverse of the $\eta-$combined acoustic double layer operator $A^{-1}_{\omega_m,\eta}$ satisfies

$$\left\| A^{-1}_{\omega_m,\eta} \right\| \geq C\, \mathrm{e}^{\gamma \omega_m} \left(1 + \frac{|\eta|}{\omega_m}\right)^{-1} \quad \text{for some } \gamma > 0,\ C > 0.$$

Galerkin Discretization

Let $S \subset \mathcal{H}$ a finite dimensional subspace. For given $f \in L^2(\Omega)$ find $u_S \in S$ such that

$$A(u_S, v) = (f, v) \quad \forall v \in \mathcal{H}.$$

Discrete Stability

Dual problem:
For given $f \in L^2(\Omega)$ find $z \in \mathcal{H}$ such that $A(v, z) = (v, f) \quad \forall v \in \mathcal{H}$.

Dual solution operator: $Q^* : L^2(\Omega) \to \mathcal{H}$ with $Q^* f := z$.

Definition (Adjoint approximability) (see Schatz 74, S. 06, Banjai, S. 07, Dörfler, S. 13) For a finite dimensional subspace $S \subset \mathcal{H}$, the *adjoint approximability* for the Helmholtz problem is

$$\sigma(S) := \sup_{g \in L^2(\Omega) \setminus \{0\}} \frac{\inf_{v \in S} \left\| Q^* \left(\left(\frac{\omega}{c} \right)^2 g \right) - v \right\|_{\mathcal{H}}}{\left\| \frac{\omega}{c} g \right\|_{L^2(\Omega)}}.$$

Theorem (stability) (S. 06)**.** Let the main assumptions be satisfied and

$$\sigma(S) < \frac{\min\{\alpha, 1\}}{\max\{\beta, 1\} + C_b}.$$

Then, the Galerkin discretization has a unique solution.

Theorem (S. 06)**.** Let the main assumptions be satisfied and

$$\sigma(S) < \frac{\min\{\alpha, 1\}}{4(\max\{\beta, 1\} + C_b)}.$$

Then, the Galerkin solution exists, is unique, and the error $e = u - u_S$ satisfies

$$\|e\|_{\mathcal{H}} \leq 2\frac{\max\{\beta, 1\} + C_b}{\min\{\alpha, 1\}} \inf_{v \in S} \|u - v\|_{\mathcal{H}},$$

$$\left\|\frac{\omega}{c} e\right\|_{L^2(\Omega)} \leq (\max\{\beta, 1\} + C_b)\, \sigma(S)\, \|e\|_{\mathcal{H}}.$$

Estimate of adjoint approximability $\sigma(S)$

The key rôle for the estimate of the adjoint approximability $\sigma(S)$ is played by the *splitting lemma*; see:

Melenk & S. 2010, 2011,

Melenk 2012,

Parsania, Melenk & S. 2013,

Esterhazy & Melenk 2012.

Splitting Lemma. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain; either with analytic boundary or Ω is a bounded polygonal domain. Let $\mathbf{A} = \mathbf{I}$, $c = 1$, and $b(u, v) = \pm \mathrm{i}\,\omega \int_{\partial\Omega} u\bar{v}$. Then there exist constants $C, \gamma > 0$ independent of $\omega \geq \omega_0$ such that for every $f \in L^2(\Omega)$ the solution of the Helmholtz equation can be written as $u = u_{H^2} + u_{\mathcal{A}}$ with

$$\omega \left\| u_{H^2} \right\|_{\mathcal{H}} + \left\| u_{H^2} \right\|_{H^2(\Omega)} \leq C \, \|f\|_{L^2(\Omega)},$$

$$\left\| \Phi_{n,k} \nabla^{n+2} u_{\mathcal{A}} \right\|_{L^2(\Omega)} \leq C^{\frac{C_{\mathsf{stab}}}{\omega}} \max\{n, \omega\}^{n+2} \gamma^n \, \|f\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0,$$

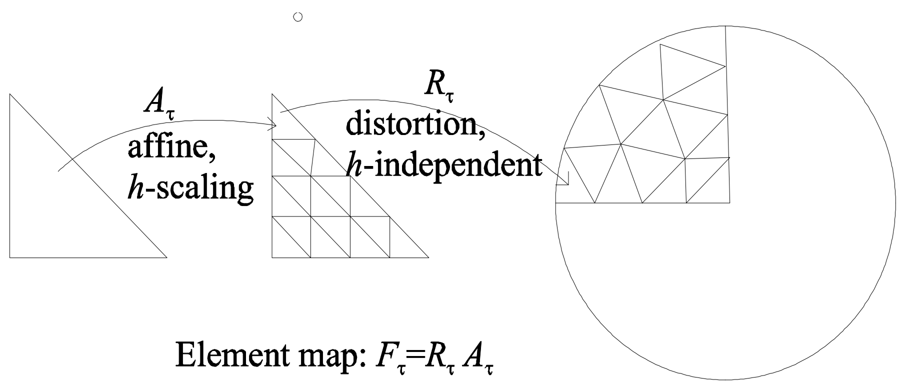
where $\Phi_{n,k}$ denotes the weight function lifting the corner singularities. It holds $\Phi_{n,k} = 1$ for analytic domains.

hp-Finite Elements

$\Omega \subset \mathbb{R}^d$ is a bounded domain with analytic boundary.

$\mathcal{T} = \{\tau_i : 1 \leq i \leq q\}$ is a conforming finite element triangulation

$h_\tau := \mathrm{diam}\,\tau, \qquad h_{\mathcal{T}} := \max\{h_\tau, \tau \in \mathcal{T}\}.$



Element map: $F_\tau = R_\tau A_\tau$

Remark 1. For polygonal domains we assume that $\mathcal{T}$ is constructed from a quasi-uniform, shape-regular triangulation by refining those elements geometrically which touch the vertices by using $L \approx p$ refinement layers.

hp-Finite Element Space

$$S_{\mathcal{T}}^{p} := \left\{ u \in C^{0}\left(\overline{\Omega}\right) \mid \forall \tau \in \mathcal{T} : u|_{\tau} \circ F_{\tau} \in \mathbb{P}_{p} \right\}.$$

Theorem (Melenk, S. 2010,11)**.** Let the assumptions of the splitting lemma be satisfied. There exist constants c_1, c_2 independent of ω, h, and p such that

$$\omega h/p \leq c_1 \qquad \text{together with} \qquad p \geq c_2 \log \omega$$

implies the existence and uniqueness of the finite element solution.

The minimal dimension of the corresponding $h - \log \omega$ finite element space satisfies

$$\dim S = O\left(\omega^d\right).$$

Theorem (Melenk, S. 2010, 11)**.** Let the assumptions of the previous theorem be satisfied. In the case of a polygonal domain we assume that Ω is convex. Then,

$$\|u - u_S\|_{\mathcal{H}} \lesssim C_{f,g}\frac{h}{p}$$

Variable Wave Speed (joint work with I.G. Graham)

Sesquilinear form $A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is given by

$$A\left(u,v\right) = \left(\nabla u, \nabla v\right) - \left(\frac{\omega}{c}u, \frac{\omega}{c}u\right) - \mathrm{i}\left(\frac{\omega}{c}u, v\right)_{L^2(\partial\Omega)}.$$

Question: How does the stability constant C_{stab} behave if ω is replaced by $\frac{\omega}{c}$? Which assumptions imply $C_{\mathrm{stab}} \lesssim \left(\frac{\omega}{c_{\min}}\right)^{\alpha}$?

The analysis of the problem becomes particularly hard if c' is large, e.g.,

$$c\left(\mathbf{x}\right) = 2 + \prod_{i=1}^{d} \sin\left(m x_i\right) \text{ and } m \sim \omega.$$

Techniques of proofs:

a) Fourier Analysis

For constant wave speed we have $\hat{u}(\xi) = \hat{G}(\xi)\,\hat{f}(\xi)$ with

$$\hat{G}(\xi) = \begin{cases} \frac{\mathrm{i}}{\sqrt{2\pi}\omega}\int_0^\infty \mathrm{e}^{\mathrm{i}\,\omega r}\,\mu(r)\cos(\|\xi\|\,r)\,dr & d=1, \\ \frac{\mathrm{i}}{4}\int_0^\infty H_0^{(1)}(\omega r)\,\mu(r)\,r J_0(r\,\|\xi\|)\,dr & d=2, \\ (2\pi)^{-3/2}\int_0^\infty \mathrm{e}^{\mathrm{i}\,\omega r}\,\mu(r)\,r\frac{\sin(r\,\|\xi\|)}{r\,\|\xi\|}dr & d=3. \end{cases}$$

and the estimate for C_{stab} follows from estimates of $\hat{G}$.

This technique does **not** apply to the case of non-constant wave speed. We tried to employ modulated Fourier expansions but **failed**.

b) The “Rellich trick”

Employ the function $v = \langle \mathbf{x}, \nabla u \rangle$ as a test function in the variational formulation and integrate by parts. This technique does **not** work if ∇c is “large” and “oscillating”.

c) Analyse the Sturm-Liouville problem in one-dimension.

The system matrix for the Sturm-Liouville problem is tri-diagonal but highly indefinite. For the analysis of the inverse one needs lower estimates for its determinant.

$$det=q(1)q(8)+q(2)s(1)q(8)+q(1)q(2)q(3)s(2)q(8)+q(3)s(1)s(2)q(8)+q(1)q(3)q(4)s(3)q(8)+q(2)q(3)q(4)s(1)s(3)q(8)$$

$$+q(1)q(2)q(4)s(2)s(3)q(8)+q(4)s(1)s(2)s(3)q(8)+q(1)q(4)q(5)s(4)q(8)+q(2)q(4)q(5)s(1)s(4)q(8)+q(1)q(2)q(3)q(4)q(5)s(2$$

$$+q(3)q(4)q(5)s(1)s(2)s(4)q(8)+q(1)q(3)q(5)s(3)s(4)q(8)+q(2)q(3)q(5)s(1)s(3)s(4)q(8)+q(1)q(2)q(5)s(2)s(3)s(4)q(8)$$

$$+q(5)s(1)s(2)s(3)s(4)q(8)+q(1)q(5)q(6)s(5)q(8)+q(2)q(5)q(6)s(1)s(5)q(8)+q(1)q(2)q(3)q(5)q(6)s(2)s(5)q(8)$$

$$+q(3)q(5)q(6)s(1)s(2)s(5)q(8)+q(1)q(3)q(4)q(5)q(6)s(3)s(5)q(8)+q(2)q(3)q(4)q(5)q(6)s(1)s(3)s(5)q(8)$$

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$+q(2)q(4)q(5)q(7)s(1)s(4)s(7)+q(1)q(2)q(3)q(4)q(5)q(7)s(2)s(4)s(7)+q(3)q(4)q(5)q(7)s(1)s(2)s(4)s(7)+q(1)q(3)q(5)q(7)s$

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Clearly, the estimate of such determinants is too complicated!

One-dimensional case, piecewise constant wave speed

Let $\Omega = (-1, 1)$ and introduce the points

$$-1 = x_0 < x_1 < \ldots x_n = 1$$

and define the intervals $\tau_i = (x_{i-1}, x_i)$, $1 \leq i \leq n$. Consider piecewise constant wave speed which is given by

$$c\,(x) := c_i \text{ for } x \in \tau_i := (x_{i-1}, x_i)\,, \quad 1 \leq i \leq n$$

where c_i are positive constants.

For a positive wave number $\omega \in \mathbb{R}_{\geq \omega_0}$ we consider the homogenous Helmholtz equation in the strong form

$$-u'' - \left(\frac{\omega}{c}\right)^2 u = 0 \quad \text{in } \Omega = (-1, 1)\,,$$

$$\begin{aligned} -u' - \mathrm{i}\,\frac{\omega}{c_1} u &= g_1 \quad \text{at } x = -1, \\ u' - \mathrm{i}\,\frac{\omega}{c_n} u &= g_2 \quad \text{at } x = 1. \end{aligned}$$

Example (Graham, S. 2016). Let the number n of subintervals be even. Choose $\omega = n$ and

$$x_\ell := \begin{cases} -1 + 2\frac{\ell-1}{n} & \ell \text{ odd}, \\ -1 + \frac{2\ell}{n} & \ell \text{ even}, \end{cases}$$

$$c_\ell := \frac{1}{\pi} \times \begin{cases} 1 & \ell \text{ odd}, \\ 3 & \ell \text{ even}. \end{cases}$$

Then, for $k = 0, 1, 2$, it holds

$$\left\| u^{(k)} \right\|_{L^2(\Omega)} \leq 7 n \omega^{k-1} \max\left\{ |g_1|, |g_2| \right\}.$$

Theorem (Graham, S. 2016)**.** For $k = 2, 4, 6 \ldots$ and $n = 2k + 1$, let the piecewise constant wave speed be defined as before. There exists a sequence of interval partitionings $\left(x_i^{(k)}\right)_{i=0}^{n}$ and wave numbers $\omega_k = n$ such that

$$\|u'\|_{L^2(\Omega)} \geq c_1 \, \mathrm{e}^{\gamma\omega_k} \min \{|g_1|, |g_2|\} .$$

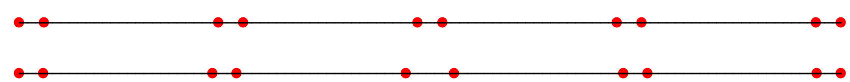
for some c_1, $\gamma > 0$.

Numerical Experiment

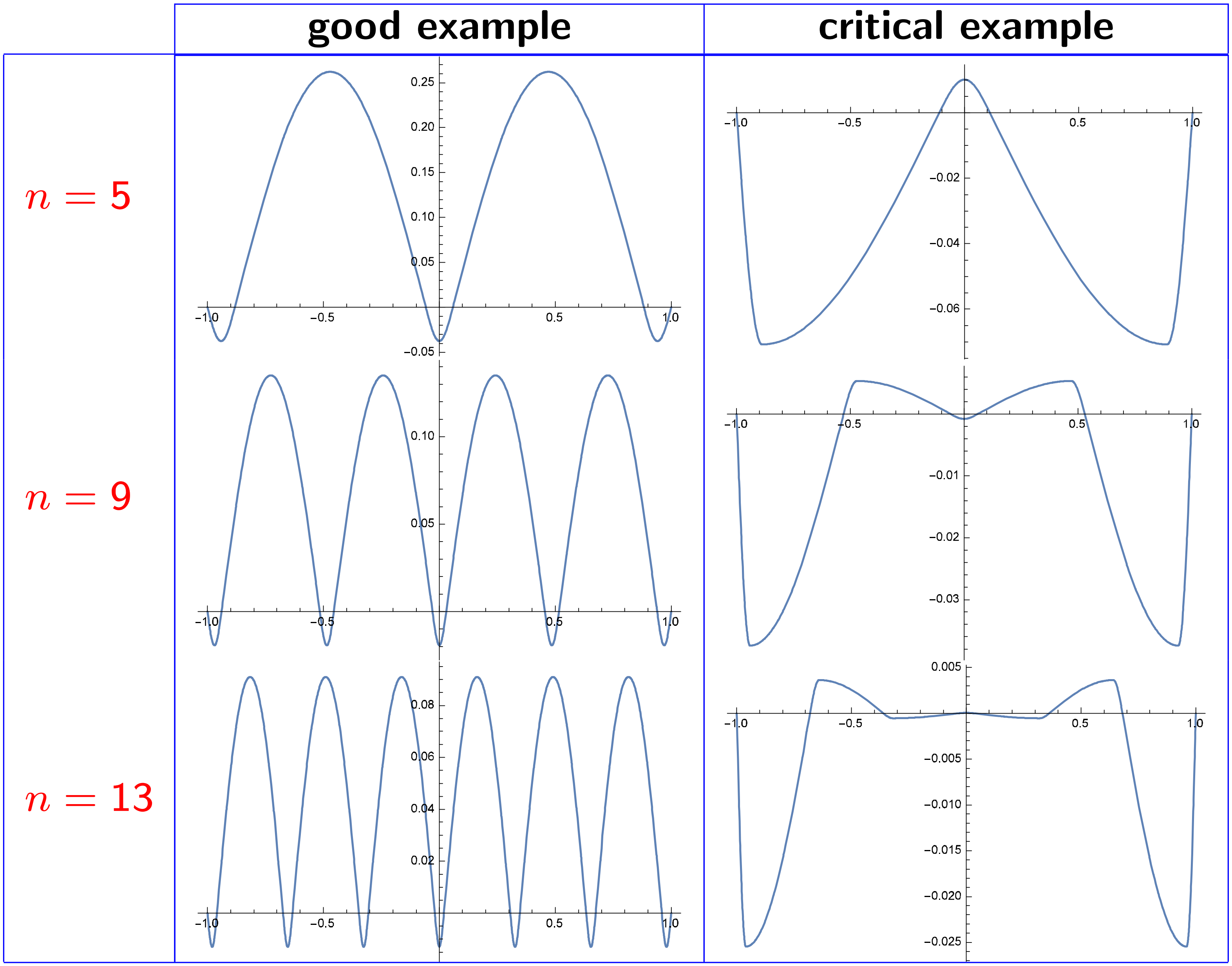
For some $r \in [0, 1[$ and any $n = 4k + 1$ let

$$\omega := \frac{\pi}{2} \times \begin{cases} (n - r) & \text{“good example”}, \\ \left(\frac{n+1}{2} - r\right) & \text{“critical example”}, \end{cases} \quad \text{and} \quad c_\ell := \begin{cases} 1 - r & \ell \text{ odd}, \\ 1 + r & \ell \text{ even}, \end{cases}$$

mesh for good example
mesh for critical example



Robin boundary data: $g_1 = g_2 = 1$.



good example
critical example
$n = 5$
$n = 9$
$n = 13$
−1.0
−0.5
0.5
1.0
0.25
0.20
0.15
0.10
0.05
−0.05
−0.02
−0.04
−0.06
−0.01
−0.03
0.08
0.06
0.04
0.02
0.005
−0.005
−0.010
−0.015
−0.020
−0.025