Highest weights for certain algebras constructed from Yangians

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February 11, 2016

Introduction

Outline:

1. Truncated shifted Yangians $Y_\mu^\lambda(\mathbf{R})$
2. Motivation for these algebras
3. Highest weight theory

Joint with J. Kamnitzer, P. Tingley, B. Webster, and O. Yacobi

Notation

- G a simple algebraic group over $\mathbb{C}$
- $\mathfrak{g} = \mathrm{Lie}(G)$
- For an affine algebraic variety X over $\mathbb{C}$, denote the coordinate ring by $\mathbb{C}[X]$

Quantum duality principle

- Suppose K is a Poisson-Lie group, with $\mathrm{Lie}(K) = \mathfrak{k}$.

$$U_h(\mathfrak{k}) \xrightarrow{\text{restricted dual}} U_h(\mathfrak{k})^* \cong \mathbb{C}_h[K]$$

- Quantum duality principle (Drinfeld, Gavarini):

$$U_h(\mathfrak{k}) \xrightarrow{\text{QDP}} U_h(\mathfrak{k})' \cong \mathbb{C}_h[K^*]$$

where K^* is a Poisson-Lie group with $\mathrm{Lie}(K^*) = \mathfrak{k}^*$.

- Yangian case:

$$\mathfrak{k} = \mathfrak{g}[t], \quad \mathfrak{k}^* = t^{-1}\mathfrak{g}[[t^{-1}]], \quad K^* = G_1[[t^{-1}]]$$

Yangians

- The *Yangian* $Y = Y(\mathfrak{g})$ is the associative $\mathbb{C}$-algebra with generators

$$E_i^{(r)}, H_i^{(r)}, F_i^{(r)} \quad \text{for } i \in I, \; r \geq 1$$

and relations $[E_i^{(r)}, F_j^{(s)}] = \delta_{i,j} H_i^{(r+s-1)}$, etc.

- Y is filtered by $\deg_{NC} X^{(r)} = r - 1$, and

$$\mathrm{gr}_{NC} Y \cong U(\mathfrak{g}[t])$$

where $X^{(r)}$ corresponds to Xt^{r-1}.

Yangians

- Consider a different filtration on Y, where $\deg X^{(r)} = r$ $\implies \operatorname{gr} Y$ is commutative!
- Let $G_1[[t^{-1}]] := \operatorname{Ker}\left(G(\mathbb{C}[[t^{-1}]]) \xrightarrow{t\to\infty} G\right)$

Theorem (Kamitzer-Webster-W-Yacobi)

1. *$G_1[[t^{-1}]]$ is a Poisson-Lie group (via Yang's Manin triple)*
2. *There is an (explicit!) isomorphism of graded Poisson algebras*

$$\operatorname{gr} Y \cong \mathbb{C}\left[G_1[[t^{-1}]]\right]$$

Example: $Y(\mathfrak{sl}_2)$

- For $G = SL_2$,

$$G_1[[t^{-1}]] = \left\{ M(t) \in M_2(\mathbb{C}[[t^{-1}]]) \ : \ M(\infty) = I, \ \det M(t) = 1 \right\}$$

- Consider $H(u) = 1 + \sum_{r>0} H^{(r)} u^{-r}$.There exist unique $A^{(s)} \in Y$ such that

$$H(u) = \frac{1}{A(u)A(u-1)}, \quad A(u) = 1 + \sum_{r>0} A^{(r)} u^{-r}$$

- Write $M(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. Lifts in $Y(\mathfrak{sl}_2)$ are

$$A(u), \quad B(u) := A(u)E(u), \quad C(u) := F(u)A(u),$$

$$A(u)D(u-1) - B(u)C(u-1) = 1$$

Shifted Yangians (case $\mathfrak{g} = \mathfrak{sl}_2$)

- Fix a dominant coweight μ (i.e. a non-negative integer)
- The *shifted Yangian* $Y_\mu \subset Y$ is the subalgebra generated by

$$E^{(r)}, H^{(r)} \text{ for } r > 0, \qquad F^{(s)}, \text{ for } s > \mu$$

- Y_μ quantizes a certain homogeneous space for $G_1[[t^{-1}]]$
- Y_μ is a left coideal subalgebra

Truncated shifted Yangians (case $\mathfrak{g} = \mathfrak{sl}_2$)

- Fix a dominant coweight λ, with $\lambda - \mu = 2m \geq 0$
- Fix a monic polynomial $R(u) \in \mathbb{C}[u]$ of degree λ
- There are unique elements $A^{(s)} \in Y_\mu$ such that

$$H(u) = \frac{R(u)}{u^\lambda (1 - u^{-1})^m} \frac{1}{A(u)A(u-1)}$$

Definition (KWWY)

The *truncated shifted Yangian* is the quotient

$$Y_\mu^\lambda(R) := Y_\mu / \langle A^{(s)} : s > m \rangle$$

Motivation: The affine Grassmannian

- $\mathrm{Gr}_G := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$
- Consider $\mathrm{Gr}_\mu^\lambda := \overline{G[[t]]t^\lambda} \cap G_1[t^{-1}]t^\mu$
- Gr_μ^λ is a finite-dim affine Poisson variety

Theorem (KWWY)

There is a map of graded Poisson algebras

$$\mathrm{gr}\, Y_\mu^\lambda(\mathbf{R}) \longrightarrow \mathbb{C}[\mathrm{Gr}_\mu^\lambda]$$

which is an isomorphism modulo the nilradical of the LHS.

Conjecture

The map is an isomorphism, and $Y_\mu^\lambda(\mathbf{R})$ provides the universal deformation quantization of Gr_μ^λ.

Motivation: Type A

- Case $G = SL_2$,

$$\mathrm{Gr}_0^{2m} = \{ M(t) \in G_1[[t^{-1}]] : \text{poles of order } \leq m \text{ at } t = 0 \}$$

- Isomorphic to Slodowy slice intesect nilpotent orbit closure (Mirkovíc-Vybornov)

$$\mathrm{Gr}_\mu^\lambda \cong \overline{\mathbb{O}_\lambda} \cap \mathcal{S}_\mu \quad \subset \mathfrak{gl}_N$$

  On the quantum level:

  1. $Y_\mu^{N\varpi_1^\vee}$ is a finite W-algebra of type A (Brundan-Kleshchev)
  2. Y_μ^λ is a "parabolic" W-algebra of type A (Webster-W-Yacobi)

Motivation

1. BLPW: Can do "Lie theory" for general Poisson varieties

   $\Longrightarrow$ rep theory of $Y_\mu^\lambda(\mathbf{R})$ should reflect geometry of Gr_μ^λ

2. Geometric Satake: $IH_*(\mathrm{Gr}_\mu^\lambda) \cong V(\lambda)_\mu$, both $G^\vee$ weight spaces

   $\Longrightarrow$ rep theory of $Y_\mu^\lambda(\mathbf{R})$ should be related to $V(\lambda)_\mu$

3. Symplectic duality: Gr_μ^λ should be "symplectic dual" to a Nakajima quiver variety $\mathcal{M}(\lambda, \mu)$

   $\Longrightarrow$ rep theory of $Y_\mu^\lambda(\mathbf{R})$ should be related to geometry of $\mathcal{M}(\lambda, \mu)$

Highest weights: case $\mathfrak{g} = \mathfrak{sl}_2$

Theorem (BK, KTWWY)

There is a bijection

$$\left\{ \begin{array}{c} \textit{highest weights} \\ \textit{for } Y_\mu^\lambda(R) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{monic } S(u) \in \mathbb{C}[u], \\ \deg S(u) = m, \\ S(u) \textit{ divides } R(u) \end{array} \right\}$$

- Write $R(u) = (u - r_1)^{\ell_1} \cdots (u - r_n)^{\ell_n}$
- Both sets above in bijection with basis for $\mathfrak{sl}_2$ weight space

$$\Big(V(\ell_1) \otimes \cdots V(\ell_n) \Big)_\mu$$

Digression: B-algebras

- Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded $\mathbb{C}$-algebra
- The *B-algebra* is $B(A) := A_0 / \sum_{n>0} A_{-n}A_n$
- $B(A)$ controls highest weights, i.e. generalized eigenspaces for A_0 where $A_{>0}$ acts by zero:

$$M \in A-\mathrm{Mod} \quad \Longrightarrow \quad M^{\mathrm{high}} \in B(A)-\mathrm{Mod}$$

$$N \in B(A)-\mathrm{Mod} \quad \Longrightarrow \quad A \otimes_{A_{\geq 0}} N \in A-\mathrm{Mod}$$

Highest weights: general case

Theorem (KTWWY)

Suppose $\mathfrak{g} = \mathfrak{sl}_n$ (only a conjecture, otherwise).
For each (integral) $\mathbf{R}$, *there exists a* $\mathbb{C}^\times$ *action on* $\mathcal{M}(\lambda, \mu)$, *and*

$$\left\{ \begin{matrix} \text{highest weights} \\ \text{for } Y_\mu^\lambda(\mathbf{R}) \end{matrix} \right\} \longleftrightarrow \pi_0\Big(\mathcal{M}(\lambda, \mu)^{\mathbb{C}^\times} \Big)$$

- Nakajima described components combinatorially via the "monomial crystal"

Conjecture

With data as above, there is an isomorphism

$$B(Y_\mu^\lambda(\mathbf{R})) \cong H^* \left(\mathcal{M}(\lambda, \mu)^{\mathbb{C}^\times} \right)$$

Highest weight theory: Expectations

There is a notion of category $\mathcal{O}$ for $Y_\mu^\lambda(\mathbf{R})$

Expectations/Goals:

1. $\mathfrak{g}^\vee$-crystal structure on

$$\mathcal{B}(\lambda, \mathbf{R}) := \bigcup_\mu \left\{ \text{highest weights for } Y_\mu^\lambda(\mathbf{R}) \right\}$$

2. $\mathfrak{g}^\vee$-action on

$$V(\lambda, \mathbf{R}) := \bigoplus_\mu K_0\left(\mathcal{O}(Y_\mu^\lambda(\mathbf{R})) \right)$$

3. Categorical $\mathfrak{g}^\vee$-action on $\bigoplus_\mu \mathcal{O}(Y_\mu^\lambda(\mathbf{R}))$

Thank you for listening!

I refuse to answer that question on the grounds that I don't know the answer.

- Douglas Adams