Cohomological Hall algebras and affine quantum groups

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Joint work with Gufang Zhao

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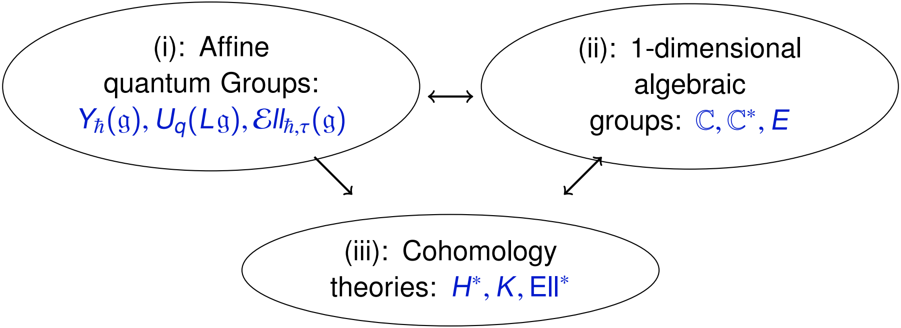
1. Motivations
2. The cohomological Hall algebras
3. Representations
4. Compactibility
5. Shuffle description

Ginzburg-Kapranov-Vasserot correspondence



- $\{Y_\hbar(\mathfrak{g}), U_q(L\mathfrak{g}), \mathcal{E}ll_{\hbar,\tau}(\mathfrak{g})\} \leftrightarrow \{\mathbb{C}, \mathbb{C}^*, E\}$ is by QYBE.
- The correspondence $(ii) \leftrightarrow (iii)$ is well-known.
- The direction $(i) \to (iii)$: [Nakajima, Varagnolo] Let $\mathfrak{M}$ be a Nakajima quiver variety.

$$Y_\hbar(\mathfrak{g}) \curvearrowright H^*_{eq}(\mathfrak{M}), \quad U_q(L\mathfrak{g}) \curvearrowright K_{eq}(\mathfrak{M})$$

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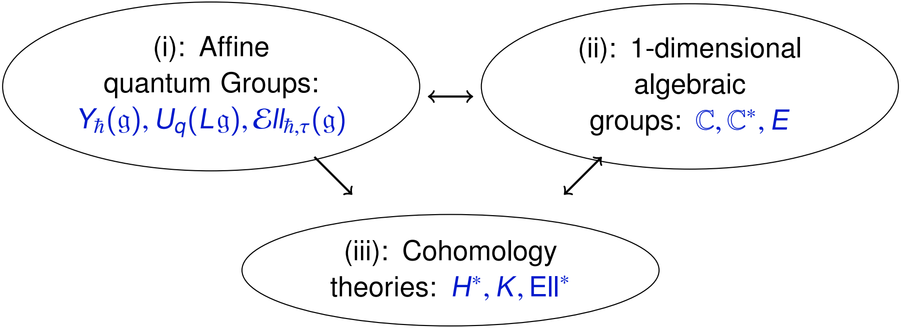

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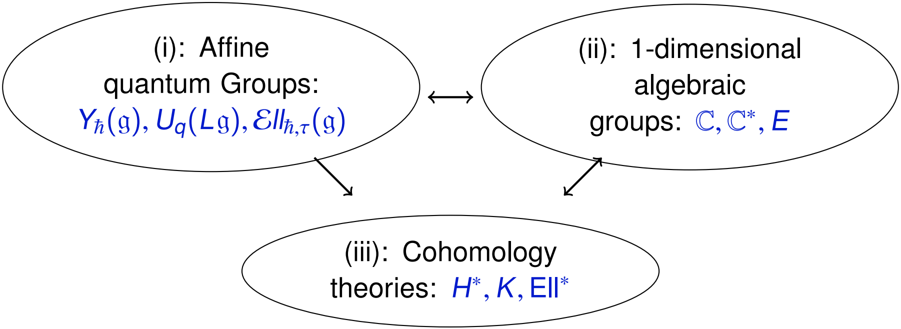

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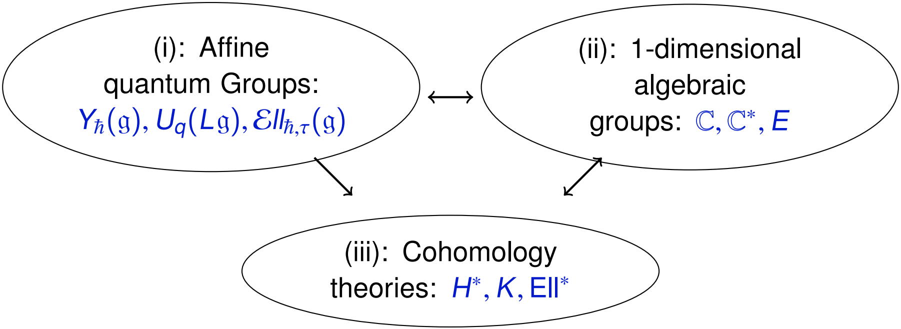

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Main Goals

1. Give a geometric (=cohomological) construction of $Y_\hbar(\mathfrak{g}), U_q(L\mathfrak{g}), \mathcal{E}ll_{\hbar,\tau}(\mathfrak{g})$. (Not just of their representations.)
   - Give a canonical basis.
   - Sheafified version of $\mathcal{E}ll_{\hbar,\tau}(\mathfrak{g})$.
2. Use this to define new affine quantum groups corresponding to arbitrary cohomology theories. (e.g. Cobordism theory.)

Remark

This is an affine analogue of Ringel Hall algebra. For any quiver Q, Ringel defined the Hall algebra

$$\mathcal{H}(Q) = \mathbb{Z}\{[M] : \textit{iso. class of repns. of } Q \textit{ over } \mathbb{F}_q\},$$

By Ringel-Green, there is an isomorphism $\mathcal{H}(Q) \cong U_q(\mathfrak{g})^+$.

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Arbitrary cohomology theory

- Let A be any cohomology theory. A : {top. spaces} $\to$ {*graded rings*}. (E.g.: $A =$ cohomology, K-theory, elliptic cohomology).
- For any $f : X \to Y$, we have smooth pullback

$$f^* : A(Y) \to A(X).$$

- We have proper pushforward

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- Let $Q=(I,H)$ be a quiver. I: vertices, and H: arrows. We fix $\{V_i\}_{i\in I}$ vector spaces of Q, with dim. vector $v=(v_i)_{i\in I}$.
- Rep. Space $\mathrm{Rep}(Q,v)=\bigoplus_{h\in H}\mathrm{Hom}(V_{\mathrm{tail}(h)},V_{\mathrm{head}(h)})$.
- Then: $G_v=\prod \mathrm{GL}_{v_i}\curvearrowright \mathrm{Rep}(Q,v)$.
- Π_Q the preprojective algebra: the path algebra of $Q\cup Q^{op}$, modulo the relations $[x,x^*]=0$.
- $\mathrm{Rep}(\Pi_Q,v)=\{(x,x^*)\mid [x,x^*]=0\}\subset T^*\mathrm{Rep}(Q,v)$
- The moment map $\mu_v: T^*\mathrm{Rep}(Q,v)\to \mathrm{Lie}\, G_v^*$.

$$G_v\times(\mathbb{C}^*)^2\curvearrowright \mu_v^{-1}(0)=\mathrm{Rep}(\Pi_Q,v).$$

Definition (Y-Zhao)

The preprojective CoHA is

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$$\mathrm{Rep}(\Pi_Q, v_1) \times \mathrm{Rep}(\Pi_Q, v_2) \xleftarrow{\phi} Ext_{v_1,v_2} \xrightarrow{\psi} \mathrm{Rep}(\Pi_Q, v_1 + v_2)$$

where

- $Ext_{v_1,v_2} = \{V_1 \to V \to V_2 \mid \dim V_i = v_i, \dim V = v_1 + v_2\}$
- $\phi : V \mapsto (V_1, V_2)$ and $\psi : V \mapsto V$

The Hall multiplication is

$$m_{v_1,v_2} := \psi_* \circ \phi^*.$$

Theorem (Schiffman-Vasserot: Q =Jordan, $A = K$; Y-Zhao: any Q, A)

$\mathcal{P}(A, Q) = \bigoplus_v A_{\mathbb{C}^{*2}}(\mathrm{Rep}(\Pi_Q, v))$ *is an associative algebra.*

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Extended CoHA

Let $\mathcal{P}^0(A) := \mathrm{Sym}(\oplus_{i\in I} A_{G_{e_i}}(\mathrm{pt}))$. E.g. When $A = H^*$, $\mathcal{P}^0 = U(\mathfrak{h}[z])$. When $A = K$, $\mathcal{P}^0 = U(\mathfrak{h}[z^{\pm}])$.

Proposition (Y-Zhao)

There is an action of $\mathcal{P}^0$ on $\mathcal{P}(A, Q)$, compatible with the product structure on $\mathcal{P}$.

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The extended CoHA is

$$\mathcal{P}^{\mathrm{ext}}(A, Q) := \mathcal{P}^0 \ltimes \mathcal{P}(A, Q).$$

The algebra $\mathcal{P}^{\mathrm{ext}} := \mathcal{P}^0 \ltimes \mathcal{P}$ is the (Borel part) of the affine quantum groups for any A.

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Relation with the Yangian: $A = H^*$

Theorem (Y-Zhao)

For any Q without edges loops. We have the algebra isomorphism

$$\mathcal{P}^{\mathrm{ext}}|_{t_1=t_2=\frac{\hbar}{2}} \cong Y_\hbar^{\geq 0}(\mathfrak{g})$$

Remark (in progress)

- *There exists a coproduct ∇ on $\mathcal{P}^{\mathrm{ext}}$.*
- *For $A = K$, $\mathcal{P}^{\mathrm{ext}}$ is expected to be $U_q^{\geq 0}(L\mathfrak{g})$.*
- *For $A = \mathrm{Ell}^*$ of [Lurie, Ando, Chen, Gepner, Goerss-Hopkins, $\cdots$], we get a sheafified elliptic quantum group $\mathcal{E}ll^+_{\tau,\hbar}(\mathfrak{g})$. It is an algebra object in a certain monoidal category of sheaves on the $\{E^{(v)}\}_{v\in N^I}$.*
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Nakajima quiver varieties

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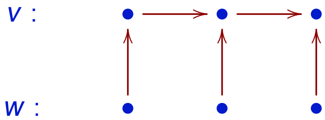

- Let $\mu_{v,w} : T^* \operatorname{Rep}(Q^\heartsuit, v, w) \to \operatorname{Lie} G_v^*$ be the moment map. For stability condition θ, the Nakajima quiver variety is

$$\mathfrak{M}(v, w); = \mu_{v,w}^{-1}(0)//_\theta G_v.$$

Example

- When $Q = \bullet$, then $\mathfrak{M}(r, n) = T^* \operatorname{Gr}(r, n)$ is the cotangent bundle of the Grassmannian.
- When Q is the Jordan quiver, $\mathfrak{M}(n, 1) \cong \operatorname{Hilb}^n(\mathbb{C}^2)$.

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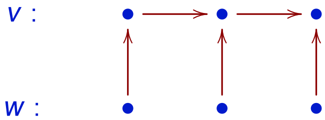

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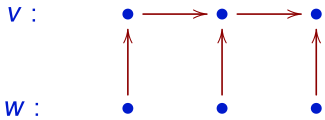

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The action

Theorem (Y-Zhao)

$\forall w \in \mathbb{N}^I$, there is an action

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Action uses the correspondence

$$\mu_{v_1}^{-1}(0) \times \mu_{v_2,w}^{-1}(0)^{ss} \longleftarrow \widetilde{Ext}_{v_1,v_2} \longrightarrow \mu_{v_1+v_2,w}^{-1}(0)^{ss}$$

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Yangians and quiver varieties

1. Let Q be the quiver without edge-loop.

Theorem (Nakajima 1999, Varagnolo 2000)

The Yangian $Y_\hbar(\mathfrak{g})$ acts on $H^*_{G_w\times\mathbb{C}^*}(\mathfrak{M}(w))$, for $\mathfrak{M}(w)=\sqcup_{v\in N^I}\mathfrak{M}(v,w)$.

2. Let Q be any quiver.
   Maulik-Okounkov constructed another Yangian Y_{MO}, based on the RTT-presentation.

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Relation between $Y_\hbar(\mathfrak{g})$ and Y_{MO}

Theorem (McBreen: $Q = ADE$; Y-Zhao: any Q)

- *Q without edge loops. There is an embedding $Y_\hbar(\mathfrak{g}) \hookrightarrow Y_{\mathrm{MO}}$.*
- *This embedding is compatible with the actions on $H^*_{G_w \times \mathbb{C}^*}(\mathfrak{M}(w))$.*

The compatibility

Theorem (Y-Zhao)

Assume $A = H^*$, Q has no edges loops.

1. *There is an embedding $i : \mathcal{P}^{\text{ext}} \hookrightarrow Y^{\geq 0}_{\text{MO}}$, which is compatible with the two actions:*

$$\text{Y-Zhao} : \mathcal{P}^{\text{ext}} \curvearrowright H_{G_w \times \mathbb{C}^*}(\mathfrak{M}(w))$$

$$\text{Maulik-Okounkov} : Y^{\geq 0}_{\text{MO}} \curvearrowright H_{G_w \times \mathbb{C}^*}(\mathfrak{M}(w))$$

2. *i induces an isomorphism $i : \mathcal{P}^{\text{ext}} \cong Y^{\geq 0}_{\hbar}(\mathfrak{g}) \subset Y^{\geq 0}_{\text{MO}}$. This isomorphism is compatible with the two actions:*

$$\text{Y-Zhao} : \mathcal{P}^{\text{ext}} \curvearrowright H_{G_w \times \mathbb{C}^*}(\mathfrak{M}(w))$$

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2. The cohomological Hall algebras
3. Representations
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5. Shuffle description

Cohomology theories and formal groups

For $\pi : L \to X$ a line bundle with zero section s, the first Chern class:

$$c_1(L) := s^* s_*(1).$$

For any two line bundles L, M on X. If $A = H^*$, $c_1(L \otimes M) = c_1(L) + c_1(M)$.

Theorem (Quillen)

There is a unique formal power series $F(u, v) \in A(pt)[\![u, v]\!]$, such that:

$$c_1(L \otimes M) = F(c_1(L), c_1(M)) \in A^*(X).$$

The series $F(u, v)$ is a formal group law:

- $F(u, v) = u + v + \cdots$
- $F(u, v) = F(v, u)$
- $F(F(u, v), w) = F(u, F(v, w))$.

Example

1. Let $A = H^*$, then $F_a(u, v) = u + v$.
2. Let $A = K$, then $F_m(u, v) = u + v - uv$.
3. Let $A =$ cobordism theory, $F(u, v)$ is the universal formal group law.

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Shuffle description of $\mathcal{P}(A, Q)$

Now, let's go back to general A:

- Shuffle algebra $\mathbb{S}(A, Q)$.
- Example: Let Q be the Jordan quiver. Let F be the formal group law.
  - $\mathbb{S} = \bigoplus_{n \in \mathbb{N}} \mathbb{S}_n$ with $\mathbb{S}_n = \mathbb{Q}[[t_1, t_2]][[x_1, \dots, x_n]]^{\mathfrak{S}_n}$.
  - The multiplication $\mathbb{S}_n \otimes_{\mathbb{Q}[[t_1^{\pm}, t_2^{\pm}]]} \mathbb{S}_m \to \mathbb{S}_{n+m}$

$$f(x_1, \dots, x_n) \star g(x_1, \dots, x_m) = \sum_{\sigma \in \mathrm{Sh}(n,m)} \sigma\Big(f(x_1, \dots, x_n) g(x_{n+1}, \dots, x_{n+m}) \cdot h\Big)$$

where
$h = h(Q, F) = \prod_{i \in [1,n], j \in [n+1, n+m]} \frac{(x_i -_F x_j +_F t_1 +_F t_2)(x_j -_F x_i +_F t_1)(x_j -_F x_i +_F t_2)}{x_j -_F x_i}$.

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There is an embedding $\mathcal{P}(A, Q) \hookrightarrow \mathbb{S}(A, Q)$.

Corollary:

- Canonical basis of $\mathcal{P}(A, Q)$.
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Thank You!!!