Nonparametric Modal Regression in the Presence of Measurement Error

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August 15, 2016
(Joint work with Zhou, Haiming)

Problem of interest

- Modal model: For $x \in \mathscr{X}$,

$$\begin{aligned}\text{Mode}(Y|X=x) &= M(x) \\ &= \{y \in \mathscr{Y} : p_y(y|x) = 0,\ p_{yy}(y|x) < 0\}.\end{aligned}$$

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- Measurement error model:

$$W_j = X_j + U_j,\ \ j = 1, \ldots, n,$$

— $W_j \sim f_W(w)$: the observed covariate,
— $X_j \sim f_X(x)$: the underlying true covariate,
— $U_j \sim f_U(u)$: the measurement error (m.e.),
$U_j \perp (X_j, Y_j)$.

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- Goal: Estimate $M(x)$ based on $\{(Y_j, W_j),\ j = 1, \ldots, n\}$.

Road map of the talk

- Motivation and relevant literature
- Methodology
- Asymptotic properties
- Implementation
- To be resolved

Motivation: Speed-flow data

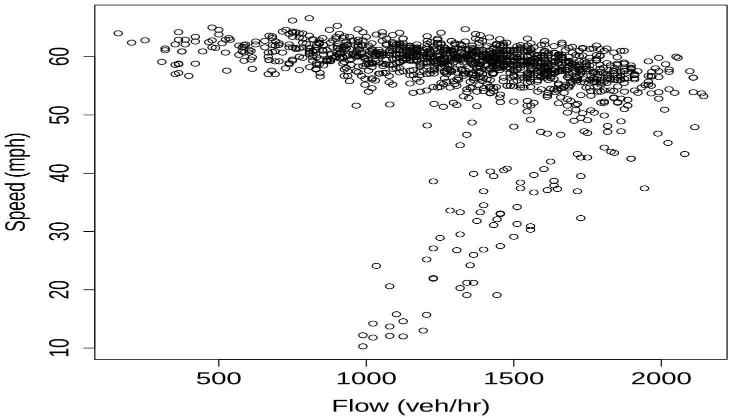

Fig. 1: Speed-flow diagram

Motivation: Temperature data

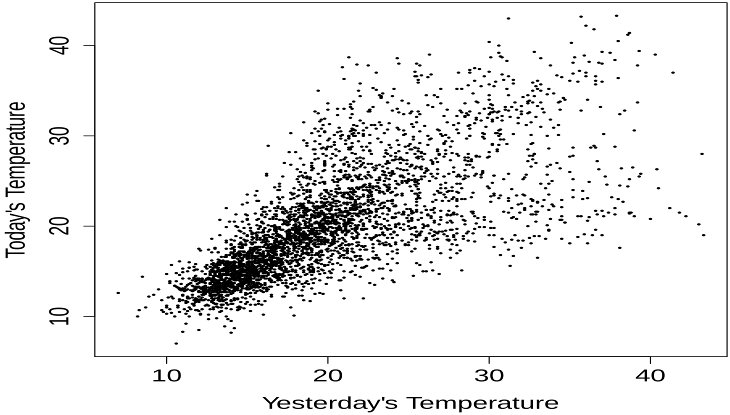

Fig. 2: Each day's temperature vs. the previous day's.

Relevant literature

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- In the presence of m.e.: ???

Nonparametric modal regression accounting for m.e.

(I) Local constant estimator of $M(x)$:

$\hookrightarrow$ Construct the local constant **k**ernel **d**ensity **e**stimator (KDE) of $p(x, y)$;

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  - $E\left\{ K_{U,0}\left(\frac{W - x}{h_1}\right) \middle| X \right\} = K_1\left(\frac{X - x}{h_1}\right)$ (Stefanski & Carroll, 1990).

- Choosing $K_2(\cdot)$ in certain ways (e.g., normal kernel) yields

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$$y^{(k+1)} = \frac{\sum_{j=1}^{n} K_{U,0}\left(\frac{W_j - x}{h_1}\right) K_2\left(\frac{Y_j - y^{(k)}}{h_2}\right) Y_j}{\sum_{j=1}^{n} K_{U,0}\left(\frac{W_j - x}{h_1}\right) K_2\left(\frac{Y_j - y^{(k)}}{h_2}\right)}.$$

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In the absence of m.e. (Fan, Yao & Tong, 1996):

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$$\hat{T}_{n,\ell}'(x, y) = \frac{1}{n h_1 h_2^2} \sum_{j=1}^{n} K_{U,\ell}\left(\frac{W_j - x}{h_1}\right) K_2\left(\frac{Y_j - y}{h_2}\right)\left(\frac{Y_j - y}{h_2}\right).$$

- Setting $\hat{p}_y(y|x) = 0$, a mode estimator $\hat{y}_{M1}(x)$ solves

$$\sum_{j=1}^{n} \left\{ K_{U,0}\left(\frac{W_j - x}{h_1}\right) \hat{S}_{n,2}(x) - K_{U,1}\left(\frac{W_j - x}{h_1}\right) \hat{S}_{n,1}(x) \right\} \times$$
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$$y^{(k+1)} = \frac{\sum_{j=1}^{n} \omega_j^{(k)} Y_j}{\sum_{j=1}^{n} \omega_j^{(k)}},$$

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$$\omega_j^{(k)} = \left\{ K_{U,0}\left(\frac{W_j - x}{h_1}\right) \hat{S}_{n,2}(x) - K_{U,1}\left(\frac{W_j - x}{h_1}\right) \hat{S}_{n,1}(x) \right\} \times K_2\left(\frac{Y_j - y^{(k)}}{h_2}\right).$$

Asymptotic properties of $\hat{y}_M(x)$

Consider

- Pointwise error rate:

$$\Delta_n(x) = |\hat{y}_M(x) - y_M(x)|, \text{ for } x \in \mathscr{X}.$$

- Uniform error rate:

$$\Delta_n = \sup_{x \in \mathscr{X}} \Delta_n(x).$$

- Mean integrated squared error:

$$\text{MISE} = E\left\{\int_{\mathscr{X}} \Delta_n^2(x)dx\right\}.$$

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$$\begin{aligned} \hat{p}_y(x, y_M) &= \hat{p}_y(x, y_M) - 0 \\ &= \hat{p}_y(x, y_M) - \hat{p}_y(x, \hat{y}_M) \\ &= (y_M - \hat{y}_M)\hat{p}_{yy}(x, y^*), \end{aligned}$$

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where y^* lies between y_M and $\hat{y}_M$.

Hence,

$$\hat{y}_M(x) - y_M(x) \approx -p_{yy}^{-1}(x, y_M)\hat{p}_y(x, y_M).$$

Asymptotic error rates

$$\Delta_n(x) = O(h_1^2 + h_2^2) + \begin{cases} O_P\left(\sqrt{\dfrac{1}{nh_1^{1+2b}h_2^3}}\right) & \text{for ordinary smooth } U, \\ O_P\left\{\dfrac{\exp(h_1^{-b}/d_2)}{\sqrt{nh_1^{1-2b_2}h_2^3}}\right\} & \text{for super smooth } U, \end{cases}$$

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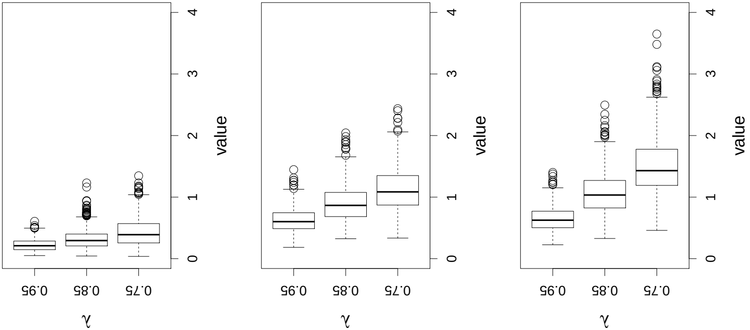
- ordinary smooth U: $|\phi_U(t)| = O(t^{-b})$, as $t \to \infty$, for some $b > 0$,
- super smooth U: $|\phi_U(t)| = O\{t^{b_0}\exp(-t^b/d_2)\}$, as $t \to \infty$, for some $b_0, b, d_2 > 0$.

Simulation study—Case I

- $[Y|X = x] \sim$
  $0.5N\left(m(x) - 2\sigma(x), 2.5^2\sigma^2(x)\right) + 0.5N\left(m(x), 0.5^2\sigma^2(x)\right)$,
- $m(x) = x + x^2$,
- $\sigma(x) = 0.5 + e^{-x^2}$,
- $X \sim \text{Uniform}(-2, 2)$, and
- $U \sim \text{Laplace}(0, \sigma_u/\sqrt{2})$,
- σ_u takes values that lead to $\lambda = 0.75, 0.85, 0.95$, where

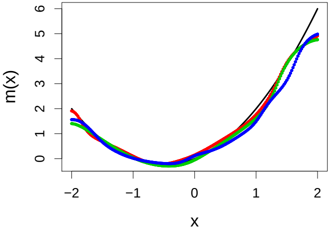
$$\lambda = \frac{\text{Var}(X)}{\text{Var}(X) + \sigma_u^2} = \text{the reliability ratio.}$$

Here, $M(x) = \{y_M(x)\}$ and $y_M(x) \approx m(x)$.



(a) $\hat{y}_{M1}(x)$ (b) $\hat{y}_{M0}(x)$ (c) Naive estimate

Fig. 3: Boxplots of ISEs versus λ.



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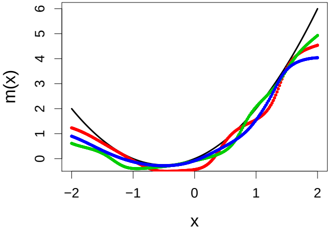

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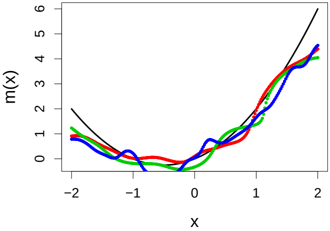


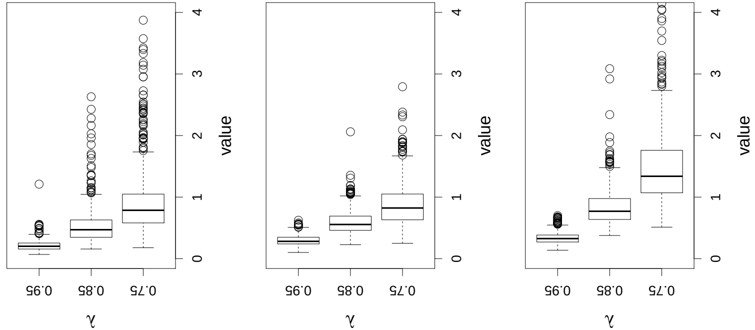
(c) Naive estimate

Fig. 4: Quantile curves. Q_1: red; Q_2: green; Q_3: blue; truth: black.

Simulation study—Case II

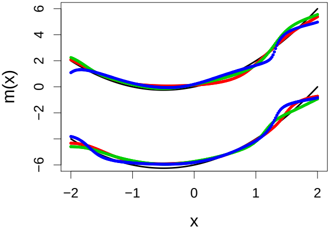
- $[Y|X = x] \sim 0.5N\left(m_1(x), 0.5^2\right) + 0.5N\left(m_2(x), 0.5^2\right)$,
- $m_1(x) = x + x^2$, $m_2(x) = m_1(x) - 6$,
- $X \sim \text{Uniform}(-2, 2)$, and
- $U \sim \text{Laplace}(0, \sigma_u/\sqrt{2})$.

Here, $M(x) = \{y_M^{(1)}(x),\ y_M^{(2)}(x)\}$, where $y_M^{(k)}(x) \approx m_k(x)$, for $k = 1, 2$.

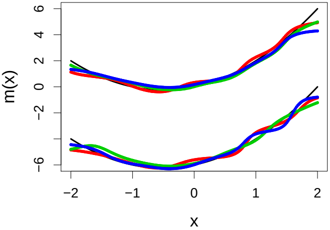


(a) $\hat{y}_{M1}(x)$ (b) $\hat{y}_{M0}(x)$ (c) Naive estimate

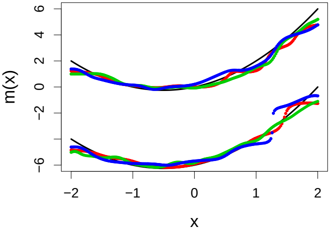
Fig. 5: Boxplots of ISEs versus λ.



(a) $\hat{y}_{M1}(x)$



(b) $\hat{y}_{M0}(x)$



(c) Naive estimate

Fig. 6: Quantile curves. Q_1: red; Q_2: green; Q_3: blue; truth: black.

Bandwidth selection (ideal/unrealistic)

We used the theoretical optimal bandwidths (h_1, h_2) that minimize

$$\text{ISE} = \int_{x \in \mathscr{X}} \left\{ \text{Haus}(\hat{M}(x),\ M(x)) \right\}^2 dx,$$

where

- $\text{Haus}(A, B) = \inf\{r : A \subset B \oplus r, B \subset A \oplus r\}$ is the Hausdorff distance between two sets, A and B,

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- $\text{Haus}(A, B) = \inf\{r : A \subset B \oplus r, B \subset A \oplus r\}$ is the Hausdorff distance between two sets, A and B,
- $A \oplus r = \{x : \inf_{y \in A} |x - y| \le r\}$.

Bandwidth selection (practical)

In the absence of m.e.: Choose (h_1, h_2) to minimize ISE (Fan and Yim, 2004).

- $$\begin{aligned} \text{ISE} &= \int \{\tilde{p}(y|x) - p(y|x)\}^2 f_X(x)dxdy \\ &= \int_{\mathscr{Y}} \int_{\mathscr{X}} \tilde{p}(y|x)^2 f_X(x)dxdy \\ &\quad -2\int_{\mathscr{Y}} \int_{\mathscr{X}} \tilde{p}(y|x)p(y|x)f_X(x)dxdy + ... \end{aligned}$$

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- $$\begin{aligned} & \text{CV}_d(h_1, h_2) \\ = \ & \frac{1}{n} \sum_{j=1}^{n} \int_{\mathscr{Y}} \tilde{p}_{-j}(y|X_j)^2 dy - \frac{2}{n} \sum_{j=1}^{n} \tilde{p}_{-j}(Y_j|X_j). \end{aligned}$$

We propose...

$$\mathrm{CV}_m(h_1, h_2) = \frac{1}{n}\sum_{j=1}^{n} d^2\left(Y_j,\, \hat{M}_{-j}(X_j)\right)\tilde{N}^2_{-j}(X_j)\omega(X_j),$$

- $d(x, S) = \inf_{y\in S}|x-y|$ for a set S,
- $\hat{N}(x) =$ the number of distinct elements in $\hat{M}(x)$.

Case	Ref	Reg	Boot	CV_d	CV_m	Ideal
I	0.59	**0.50**	2.19	6.19	0.60	0.34
	(0.08)	(0.12)	(0.88)	(1.56)	(0.05)	(0.05)
II	3.46	1.22	0.70	2.16	**0.41**	0.36
	(1.84)	(0.66)	(0.45)	(0.73)	(0.10)	(0.05)

Table 1: Average of EISE across 500 MC replicates. Numbers in parentheses are $10\times$ Std. Err.

where the empirical integrated square error (EISE) is

$$\mathrm{EISE} = \sum_{k=0}^{\mathcal{M}} \left\{ \mathrm{Haus}(\hat{M}(x_k), M(x_k)) \right\}^2 \Delta.$$

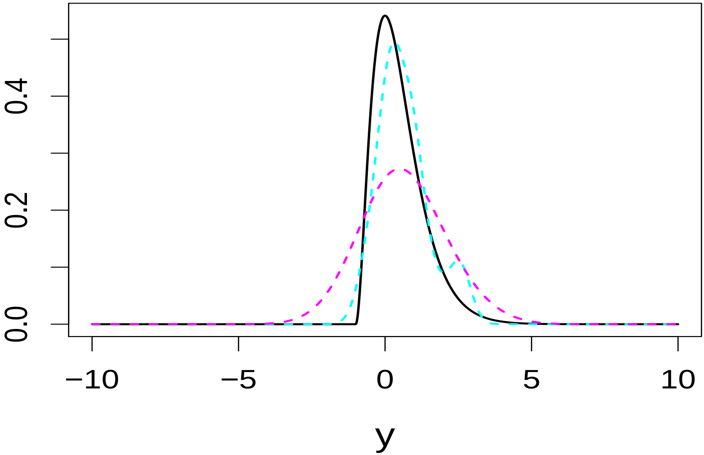

Fig. 7: Black line: the true $p(y|x)$; blue line: the estimate using CV_d; red line: the estimate using our CV_m.

To be resolved...

- Convergence of the mean-shift algorithm
- Bandwidth selection
- Conditional density estimation in the presence of m.e. in both Y and X