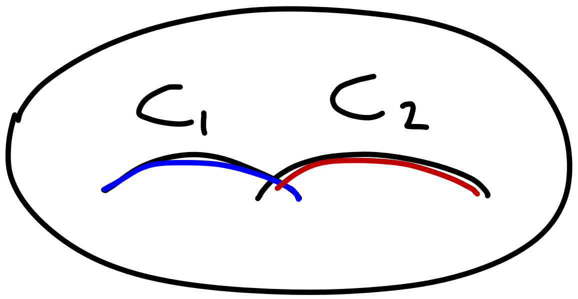
# Twists & braids for gen^l 3-fold flops.

<u>Goal</u>: $X$ a q.proj 3-fold

1. describe: $\mathrm{Aut}\, D(X)$
2. relate to: $\mathrm{Bir}\, X$

<u>ex 1</u>




$C_i \cong \mathbb{P}^1$, rigid, flopping

$\bigcup C_i$ flopping

$$\mathrm{Bir}\, X \cong \left[\begin{tikzcd} & X_1 \arrow[r, leftrightarrow, dashed] & X_{12} \arrow[dr, leftrightarrow, dashed] & \\ X \arrow[ur, leftrightarrow, dashed] \arrow[dr, leftrightarrow, dashed] & & & X_{121} = X_{212} \\ & X_2 \arrow[r, leftrightarrow, dashed] & X_{21} \arrow[ur, leftrightarrow, dashed] & \end{tikzcd}\right]$$

directed graph $\Gamma$

<u>Claim</u> $\mathrm{PBr}_3 \longrightarrow \mathrm{Aut}\, D(X)$

$\boxed{\text{gen}^{\underline{l}}\text{ze}}$

Dynkin diag $\Delta \rightsquigarrow$

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\text{min. res}"] & \mathcal{Y} \\ \mathbb{C}^2 / S_\Delta & \end{tikzcd}$$

versal deform$^{\underline{n}}$

$\#$ params $= \mathrm{rk}(\mathfrak{g}_\Delta)$

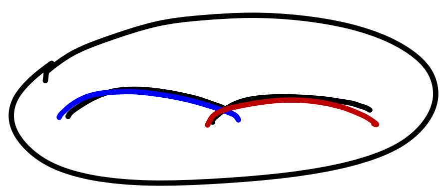
$\sigma \in$ Weyl gp $\rightsquigarrow$ flop $\mathcal{Y}_\sigma$

<u>ex</u> $\circ\!-\!\circ$

$S = \mathbb{Z}/3\mathbb{Z}$

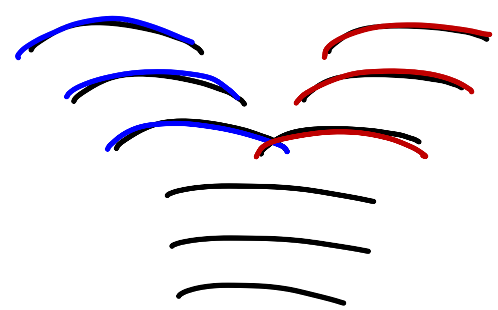
$W = \mathfrak{S}_3$

$Y$



$\mathfrak{g} = \mathfrak{sl}_3$

$\mathcal{Y}$



**Def** $\mathfrak{h}^\circ$ = Cartan $\mathfrak{h}_\Delta$ − root h'planes

$$\mathfrak{h}^\circ \hookleftarrow \mathfrak{h}^\circ_{\mathbb{R}} = \coprod_{G} \text{Weyl chambers } \mathcal{U}_G$$

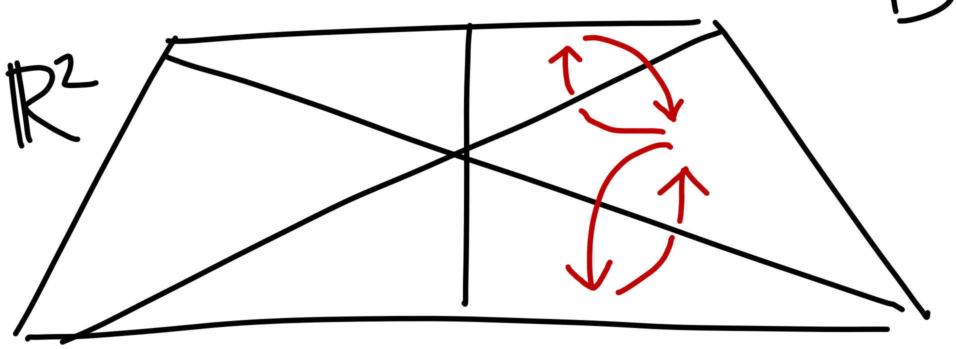
**Thm 0** (type A) w/ E. Segal
$$\pi_1(\mathfrak{h}^\circ, \mathfrak{h}^\circ_{\mathbb{R}}) \longrightarrow \mathrm{Aut}\{D(Y_G)\}$$
fund. groupoid $\qquad$ aut. groupoid

**Pf**

$$p \in \mathcal{W}_G \longmapsto D(Y_G)$$
$$\text{path} \longmapsto \text{flop functor}$$

**ex** $\mathfrak{h}^\circ \subset \mathfrak{h} = \mathbb{C}^2$

$\mathbb{R}^2$



$D(Y_{G_1})$

$D(Y)$

$D(Y_{G_2})$

rel$^{\underline{n}}$s :

- $Y_G$ toric
- window equiv$^{\underline{ce}}$s [HL / BFK]

(now see HL-Sam)

**Thm 0** (type A) $\underset{\text{fund. groupoid}}{\pi_1(\mathfrak{h}^\circ, \mathfrak{h}^\circ_{\mathbb{R}})} \longrightarrow \underset{\text{aut. groupoid}}{\mathrm{Aut}\{D(Y_c)\}}$

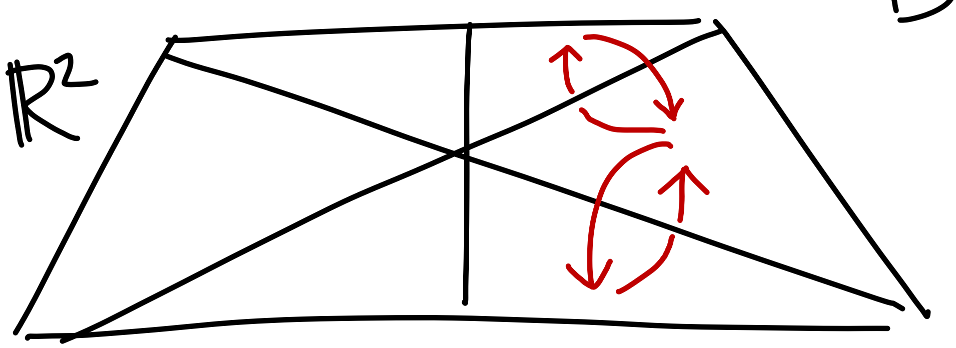
**Cor** $PBr_{n+1} = \pi_1(\mathfrak{h}^\circ, *) \curvearrowright D(Y)$

**Cor** $Br_{n+1} = \pi(\mathfrak{h}^\circ/W, *) \curvearrowright D(Y)$

cf [ST], deduce actions faithful

**ex** $\mathfrak{h}^\circ \subset \mathfrak{h} = \mathbb{C}^2$

$\mathbb{R}^2$



$$\begin{tikzcd} D(Y_{c_1}) \arrow[d, shift left] \\ D(Y) \arrow[u, shift left] \arrow[d, shift left] \\ D(Y_{c_2}) \arrow[u, shift left] \end{tikzcd}$$

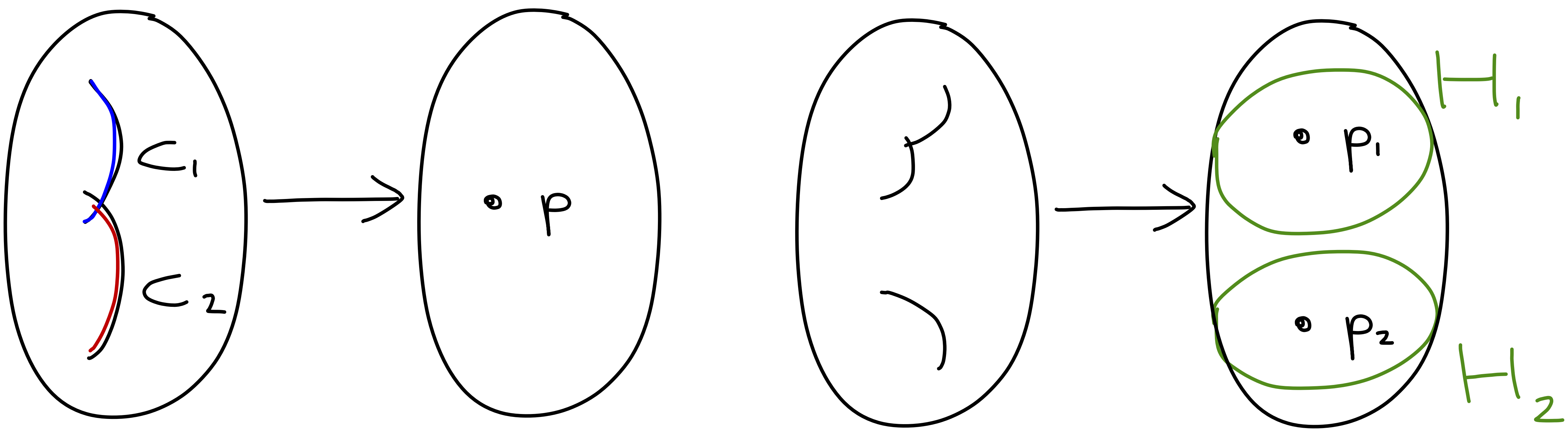
**3-folds** $\quad X \xrightarrow{f} X_{con}$ flopping contract$^{n}$ (normal, q.proj)

$\mathrm{Exc}_f \longmapsto \{p_i\}_1^n$

$\rotatebox{90}{$\supset$}$

Assume $\exists N$ nbhd, Gor term & $\mathbb{R}f_*\mathcal{O} = \mathcal{O}$

**ex 1** $\qquad\qquad\qquad\qquad\qquad\qquad$ **ex 2**




**Lem** [Reid]

$$\begin{array}{ccc} X_{con} & \ni & p_i \\ \cup & & \parallel \\ \underset{gen}{H} & \ni & p_i \longleftarrow SL_2\text{-quot sing} \end{array}$$

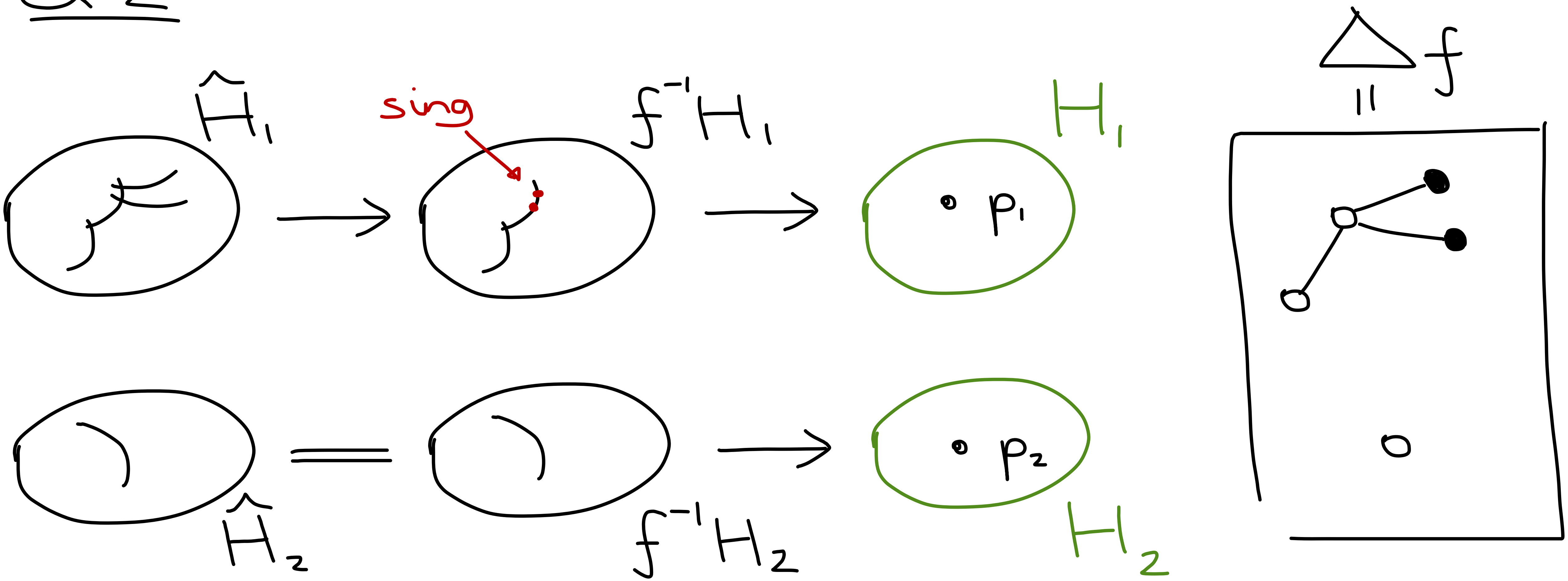
$f^{-1}H \to H$ partial res $\longleftrightarrow$ marked Dynkin diag

**Lem** [Reid]

$$\begin{array}{ccccc} X_{con} & \ni & p_i & & \\ \cup & & \parallel & & \\ H & \ni & p_i & \longleftarrow & SL_2\text{-quot sing} \end{array}$$

$f^{-1}H \to H$ partial res $\longleftrightarrow$ marked Dynkin diag

**ex 2**




min. res

**Def** $V = \langle \text{marked roots} \rangle_{\mathbb{C}} \subset \mathfrak{h}_{\Delta_f}$

$V^\circ = V - \text{root h'planes}$. $\boxed{\dim_{\mathbb{C}} = n}$

**Thm** [80s] $\pi_0(V^\circ_{\mathbb{R}}) = \{\text{min. models } X_\tau \leftarrow\rightarrow X\}$

**Def** $C_J = \bigcup_{j \in J} C_j$ for $J \subseteq \{1 \_\_ n\}$

**Thm 1** w/ Wemyss If $C_J$ flop algebraically $\forall J$ then $\exists$

$$\mathbb{G} := \pi_1(V^\circ, V^\circ_{\mathbb{R}}) \longrightarrow \mathrm{Aut}\{D(X_\tau)\}$$

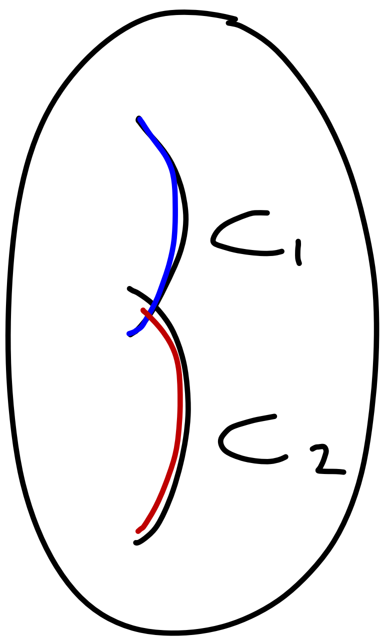
**Conj** faithful

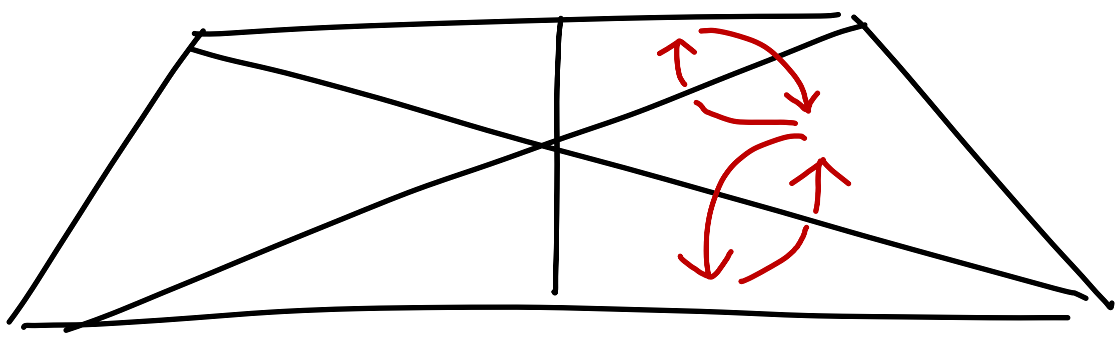
**Thm 1** w/ Wemyss: If $C_J$ flop algebraically $\forall J$ then $\exists$

$$\mathbb{G} := \pi_1(V^\circ, V^\circ_{\mathbb{R}}) \longrightarrow \operatorname{Aut}\{D(X_\tau)\}$$

**ex 1** $V^\circ = \mathbb{C}^2 - 3L$,

$\mathbb{G} = \mathbb{B}r_3$ braid groupoid

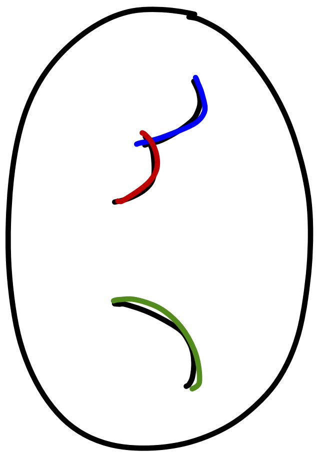


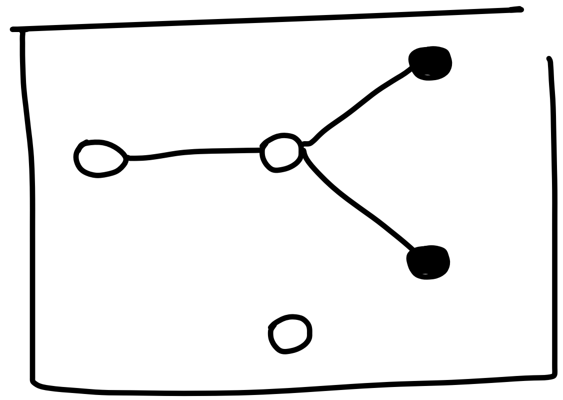
$\mathbb{R}^2 \subset \mathbb{C}^2$

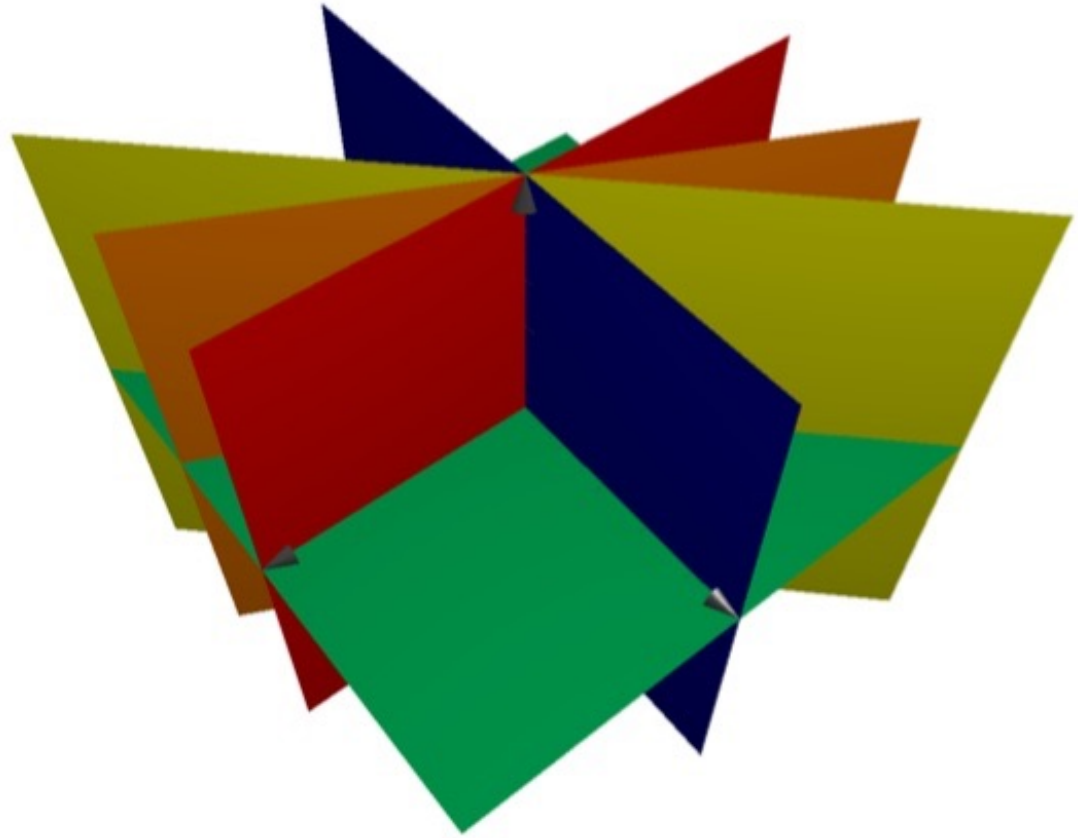
$$\begin{tikzcd} D(X_{\tau_1}) \arrow[d, shift left] \\ D(X) \arrow[u, shift left] \arrow[d, shift left] \\ D(X_{\tau_2}) \arrow[u, shift left] \end{tikzcd}$$

## ex 2 $\quad V^\circ = \mathbb{C}^3 - 5H$

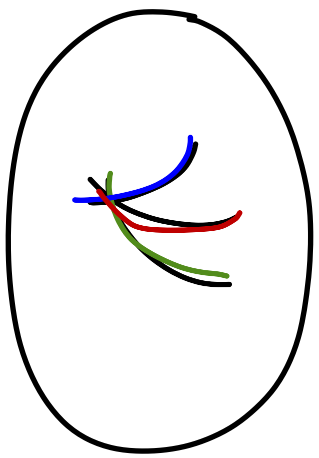


braid rel$^n$ order 4.

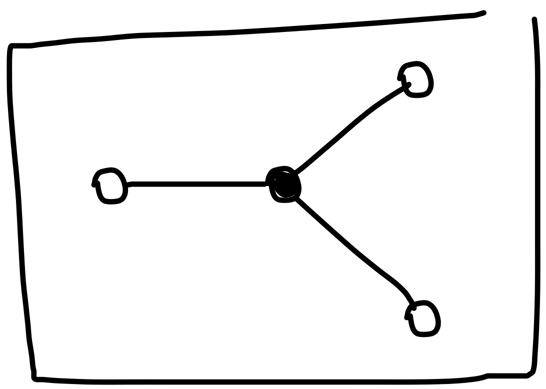
$\Delta =$ 

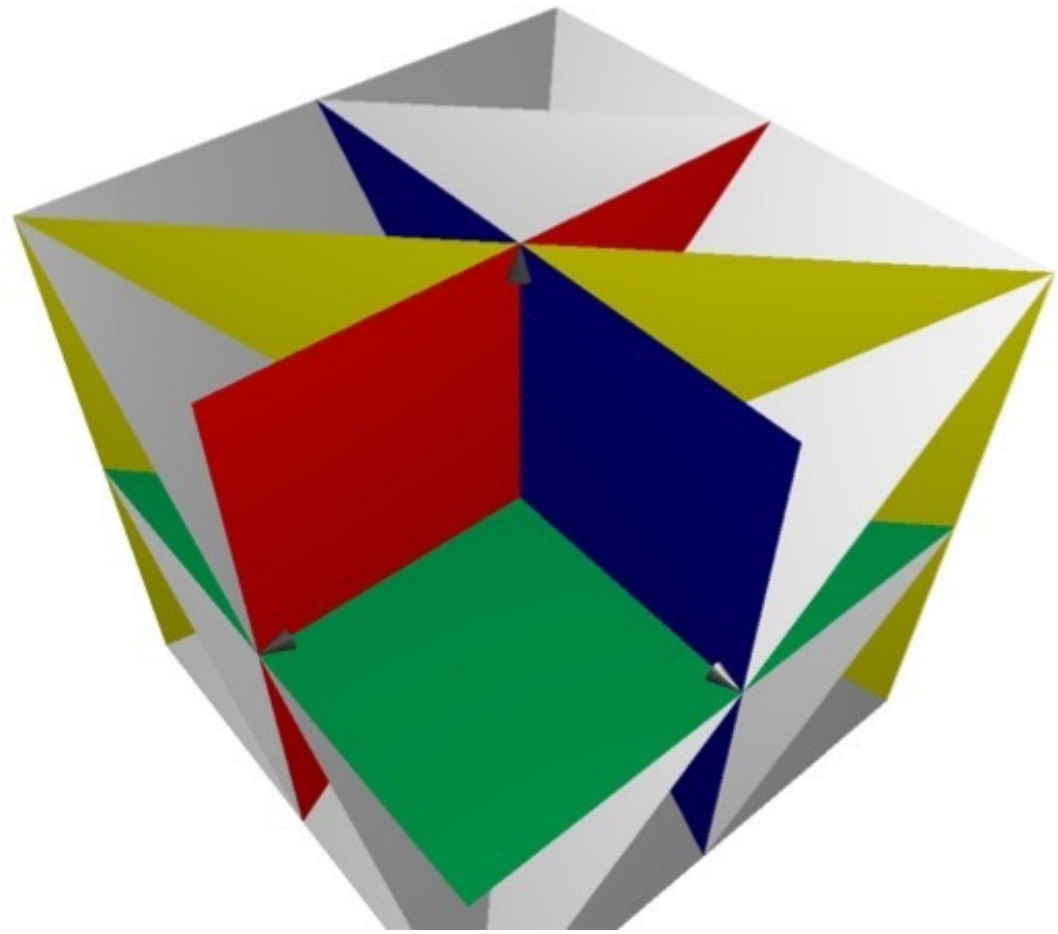


## ex 3 $\quad V^\circ = \mathbb{C}^3 - 7H$



pairwise braid rel$^n$s order 3.

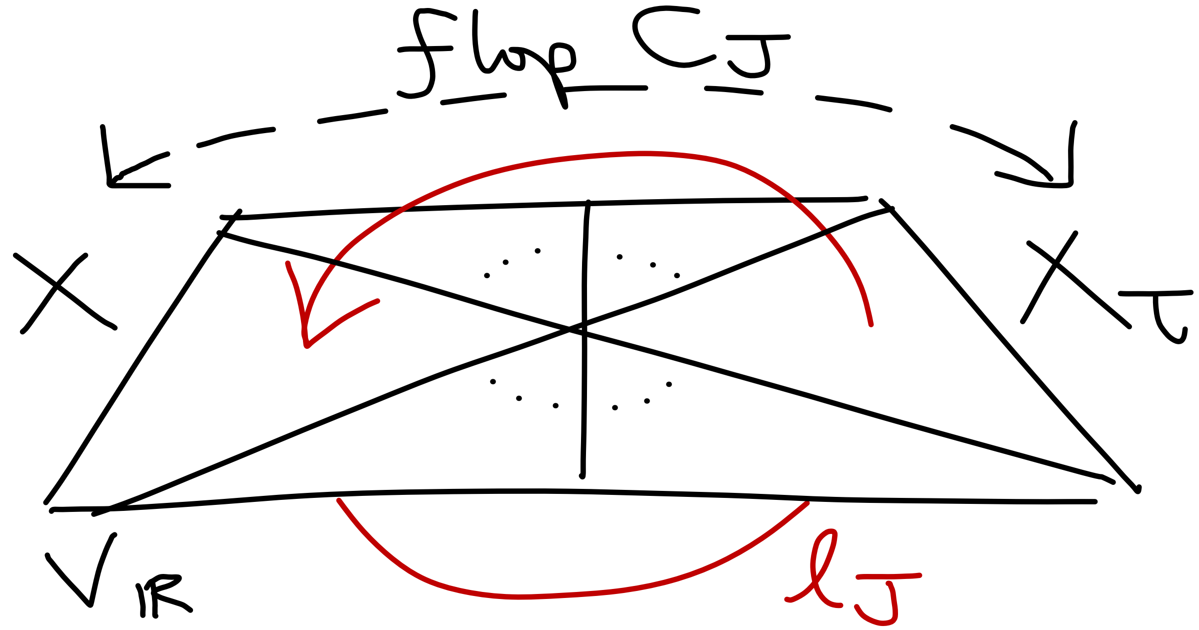
$\Delta =$ 



Rem simplicial h'plane arr. $\Longrightarrow V^\circ$ is $K(\pi,1)$

**Def** loop $l_J$ in $V^\circ$

$G := \langle l_J \rangle \subset \pi_1(V^\circ)$




**Thm 2** $G \curvearrowright D(X)$ **Conj** faithful
w/ Wemyss

**Conj** $G = \pi_1(V^\circ) \iff \Delta$ fully marked

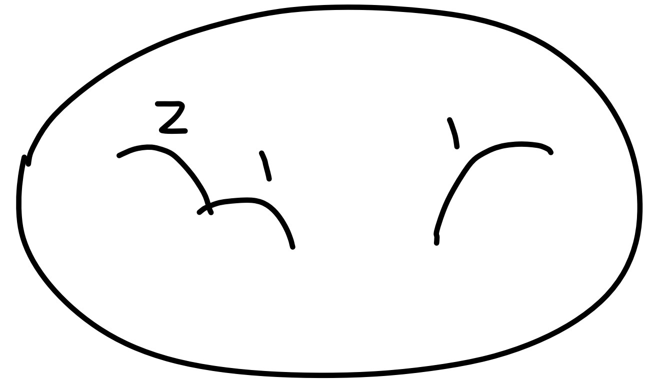
**Pf** $l_J \longmapsto T_{C_J} := \text{twist}(\mathcal{E}_A)$

- $A = \text{nc defs}\left(\bigoplus_{j \in J} \mathcal{O}_{C_j}(-1)\right) \longleftarrow$ aug $\mathbb{C}^{|J|}$-alg
- $\mathcal{E} \in D^b(\mathcal{O}_X \otimes A)$ is univ fam.

$\boxed{\text{gen}^{\text{l}}\text{zes Toda}}$

$f^{-1}p_i$ may be <u>non-red</u>.

<u>ex 2</u>




multiplicities $l_i \in [1 — 6]$

<u>Thm 3</u> w/ Wemyss $\exists\, T_{fib} = T_{\mathcal{O}_{\{f^{-1}p\}}} \in \mathrm{Aut}\, D(X)$

<u>Thm</u> * [Kawamata] $\forall\, \underline{J} = (w_1 — w_n)$, $w_i \in [0 — l_i]$

$\exists\, T_{C_{\underline{J}}} = T_{\bigoplus_i \mathcal{O}_{\varepsilon_i}(-1)} \in \mathrm{Aut}\, D(X)$

$\boxed{w_i^{th} \text{ order thickening}}$

$$\boxed{\text{Q: find } V^\circ \text{ st } \pi_1(V^\circ) \to D(X) \text{ faithfully via } T_{fib},\ T^*_{C_{\underline{J}}}\ \forall\, \underline{J}\ ?}$$

THANK YOU

## * Correction

My description of Prof Kawamata's result was incorrect.

See arXiv:1512.06170v1, Theorem 6.4, which produces many new classes of autoequivalences (Examples 6.5–9) and a new construction of the $T_{C_J}$ in certain smooth cases.

The objects $\mathcal{O}_{E_i}(-1)$ above satisfy some assumptions of this theorem, but it is not claimed they satisfy the rest. It is not clear whether there exist associated $T_{C_{\underline{J}}} \in \operatorname{Aut} D(X)$.