The Conformal Method Gives a Poor Parameterization of Initial Data

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Outline

Einstein Constraint Equations

- Given a (vacuum, 3+1) spacetime, can take a space-like slice. It must satisfy

$$R + (\mathrm{tr} K)^2 - |K|^2 = 0$$

$$\mathrm{div} K - \nabla(\mathrm{tr} K) = 0.$$

- If these are satisfied, can evolve a spacetime. (Choquet-Bruhat '52)
- Can we parameterize all initial data?

The Conformal Method

Main idea: split determined data from freely specifiable data.
Given $(M^3, g, \sigma, \tau, N)$, solve

$$8\Delta\phi = R\phi + \frac{2}{3}\tau^2\phi^5 - \left|\sigma + \frac{LW}{2N}\right|^2 \phi^{-7}$$

$$\mathrm{div}\frac{LW}{2N} = \frac{2}{3}\phi^6 d\tau$$

for (ϕ, W). Then

$$\gamma = \phi^4 g$$

$$K = \phi^{-2}(\sigma + LW/2N) + \frac{\tau}{3}\phi^4 g$$

solve the Einstein constraint equations.

τ **is the mean curvature, and controls the coupling.**

Compact Solvability Results

Table: Constant mean curvature (CMC) solvability (Isenberg '95)

	$\tau = 0, \sigma \equiv 0$	$\tau = 0, \sigma \not\equiv 0$	$\tau \neq 0, \sigma \equiv 0$	$\tau \neq 0, \sigma \not\equiv 0$
$Y(g) > 0$	None	Unique	None	Unique
$Y(g) = 0$	"Constants"	None	None	Unique
$Y(g) < 0$	None	None	Unique	Unique

- Thus, straightforward to parameterize CMC initial data.
- Conjecture was that solvability is the same, even if not CMC.
- Small caveat: must be able to solve a prescribed scalar curvature problem if $Y(g) < 0$.

Signs of Weakness

"Near-CMC" if $d\tau$ is small compared to $\tau > 0$.

Table: Near-CMC Solvability, Conjectured Solvability

	$\tau \not\equiv 0, \sigma \equiv 0$	$\tau \not\equiv 0, \sigma \not\equiv 0$
$Y(g) > 0$	None	Unique
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- Maxwell '11: T^3 symmetric data, $Y(g) = 0, \sigma \not\equiv 0$ for a τ that changes sign: Non-existence and Non-uniqueness .
- The-Cang '15: $Y(g) > 0$, σ with limited support, $\tau > 0$, $L(d\tau/\tau) \leq (d\tau/\tau)^2$. Non-existence and Non-uniqueness. $\sigma \equiv 0$ solutions.

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Introduction to Results

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Table: Observed far-from-CMC Solvability, $\tau > 0$

	$\tau \not\equiv 0, \sigma \equiv 0$	$\tau \not\equiv 0, \sigma \not\equiv 0$
$Y(g) > 0$	Existence Non-uniqueness	Non-existence Non-uniqueness
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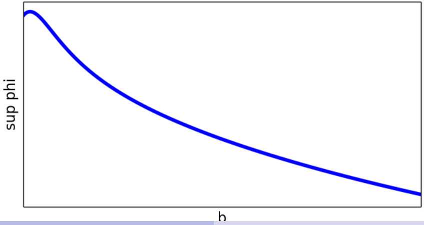
Methodology

- Bifurcation theory attempts to describe the family of solutions as a parameter(s) change.
- AUTO is a program for exploring ODE bifurcations numerically.
- We reduced the conformal method to an ODE by symmetry, e.g. on $S^1 \times S^2$.
- Usually used $\tau = b\xi^a$, with $\xi > 0$, $\sup \xi = 1$.
  - b gives size
  - a gives "distance" from CMC, since $d\tau^a/\tau^a = a\, d\tau/\tau$.

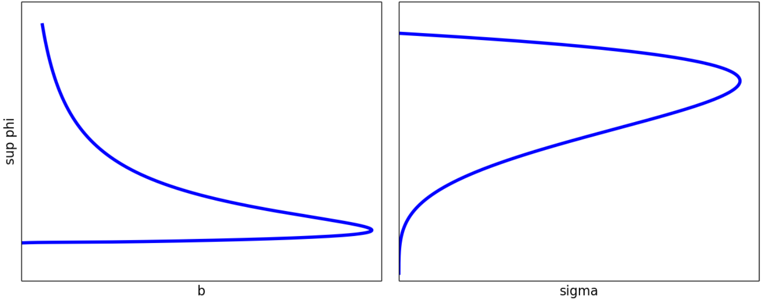


$S^1 \times S^2$, S^1 dependent ($Y(g) > 0$)

- $\tau = b\xi^a$, $\xi > 0$. Unique solutions for all b and a.
- Theorem: There are no S^1 dependent solutions to the "limit equation" (which suggests existence/uniqueness is generic.)
- Also, no τ's that satisfy The-Cang's conditions.

$S^1 \times S^2$, latitude dependent ($Y(g) > 0$)

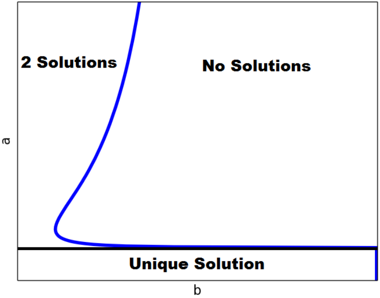
For all $\tau = b\xi^a$ we tested, get same generic picture for "large" a.



- Agrees with The-Cang's results.
- However, none of his conditions seem to be necessary.

$S^1 \times S^2$, latitude dependent ($Y(g) > 0$)

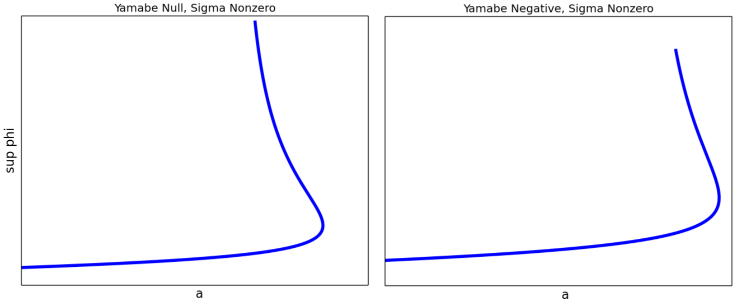
For all $\tau = b\xi^a$ we tested, get same generic existence plot.



- Why does this bend? Why does larger a allow solutions?
- Unique solutions below a horizontal line, which defines "near-CMC". Numerically related to "the" solution of the limit equation. What is this value?

$S^2 \times H^2$, latitude dependent

- As expected, existence and uniqueness for $S^1 \times H^2$, $\tau = b\xi^a$.
- Instead we look at $S^2 \times H^2$, with a compactified H^2. Yamabe class changes as the size of S^2 changes.
- Same picture for $Y(g) > 0$ data.
- For $\sigma \not\equiv 0$, similar results for $Y(g) = 0$, $Y(g) < 0$ data.



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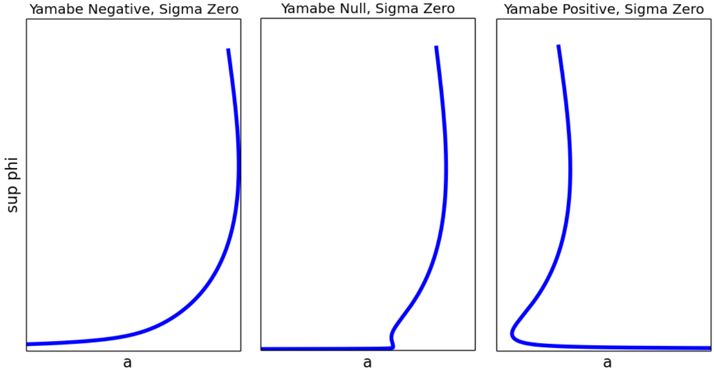

- Can find a fold for all Yamabe classes if $\sigma \equiv 0$.

The Conformal Method is Complicated

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- The conjectured solvability was very wrong for far-from-CMC data.
- If you want a simple 1-1 correspondence between specified data and initial data, the conformal method doesn't work.
- Much of this picture has not been proven analytically.

Why do folds appear?

$$8\Delta\phi = R\phi + \frac{2}{3}\tau^2\phi^5 - \left|\sigma + \frac{LW}{2N}\right|^2 \phi^{-7}$$

$$\operatorname{div}\frac{LW}{2N} = \frac{2}{3}\phi^6 d\tau$$

- ϕ^5 is critical exponent from e.g. the Yamabe problem.
- τ^2 is good sign, but the LW term is a bad sign.
- When far-from-CMC, the bad sign apparently wins out.

Theorem (Premoselli '14)

Consider the case where τ^2 is replaced by $\frac{2}{3}\tau_0^2 - 2\Lambda < 0$, and σ by $a\sigma$. Then there is an $a_* > 0$ such that

- *if $a > a_*$ there are no solutions.*
- *if $a = a_*$ there is exactly one solution.*
- *if $0 < a < a_*$ there are at least two solutions.*

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What else could we try?

- Examine variations of the conformal method.
- Do all (maximal) spacetimes have CMC slices? No.
- Perhaps generic spacetimes have CMC slices. Or perhaps near-CMC slices?
- Maybe there are straightforward conditions required for no CMC slices? (e.g., the initial manifold doesn't allow a Yamabe non-negative metric)

Thank you!