Constructions of outermost apparent horizons with non-trivial topology

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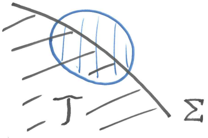
Outermost apparent horizons

- (M^n, g) asymptotically Euclidean Riemannian manifold.
  - A bounding hypersurface is *outer trapped* if mean curvature $H < 0$ in direction of the asymptotically euclidean end.
  - A domain is a *trapped set* if its boundary is outer trapped.
  - The *trapped region* $\mathcal{T}$ is the union of all trapped sets.
  - The *outermost apparent horizon* is the boundary of the trapped region.
- **Theorem.** (Eichmair and others) Assume $n \leq 7$. If $\mathcal{T} \neq \emptyset$ then (M^n, g) has an outermost apparent horizon Σ which is
  - a smooth stable minimal hypersurface,
  - outer area minimizing.



Positive scalar curvature

- **Theorem.** (Hawking, Galloway-Schoen, Galloway) An outermost apparent horizon has a metric of positive scalar curvature.
- Is existence of a PSC metric only obstruction for a bounding manifold to be an outermost apparent horizon?
- Examples:
  - Emparan-Reall Black rings, Black Saturn, etc...,
  - Schwartz $S^p \times S^q$.
- Construction of PSC metrics on compact manifolds:
  - Schoen-Yau, Gromov-Lawson: codim ≥ 3 surgery,
  - Carr: "tubes" around codim ≥ 3 embedded cell complexes.

Constructing examples

- $S \subset \mathbb{R}^n$ smooth submanifold, $\dim(S) = m$, $\epsilon > 0$,

$$U_\epsilon(x) := 1 + \epsilon^{n-m-2} \int_S |x - y|^{-(n-2)} \, dy.$$

- Riemannian metric on $\mathbb{R}^n \setminus S$

$$g_\epsilon := U_\epsilon^{4/(n-2)} \delta,$$

  where δ is the Euclidean metric.

- $\Delta U_\epsilon = 0 \Rightarrow (\mathbb{R}^n \setminus S, g_\epsilon)$ is scalar flat.
- $U \to 1$ at infinity $\Rightarrow (\mathbb{R}^n \setminus S, g_\epsilon)$ is asymptotically Euclidean.

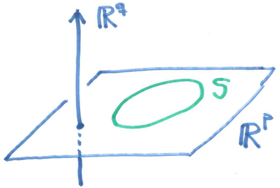
Theorem

Theorem (D.-Larsson)

Let $p \geq 1$ and $q \geq 2$. Suppose that $n = p + q \leq 7$. Let

$$S \subset \mathbb{R}^p \times \{0\} \subset \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n.$$

be a smooth, embedded, compact submanifold of dimension $m < p$. For small $\epsilon > 0$ it holds that $(\mathbb{R}^n \setminus S, g_\epsilon)$ has an outermost apparent horizon Σ_ϵ, which is diffeomorphic to a tube around S.

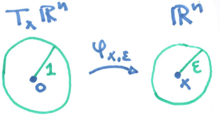


Works also if S has components of different dimensions.

Proof: rescaling and localization

For $(x, \epsilon) \in \mathbb{R}^n \times \mathbb{R}^+$ define $\varphi_{x,\epsilon} \colon T_x\mathbb{R}^n \to \mathbb{R}^n$

$$\varphi_{x,\epsilon}(\zeta) := \exp^\delta_x(\epsilon\zeta) = \text{``}x + \epsilon\zeta\text{''}$$



If $x \in S$ and $\epsilon \to 0$ then

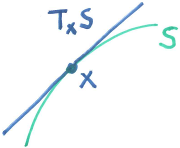
$$\epsilon^{\gamma - m} \int_S |\varphi_{x,\epsilon}(\zeta) - y|^{-\gamma}\, dy \to \int_{T_xS} |\zeta - \eta|^{-\gamma}\, d\eta$$

and

$$\epsilon^{-2}(\varphi_{x,\epsilon})^*(g_\epsilon) \to U_\infty^{4/(n-2)}\delta$$

where

$$U_\infty(\zeta) := 1 + \int_{T_xS} |\zeta - \eta|^{-(n-2)}\, d\eta.$$



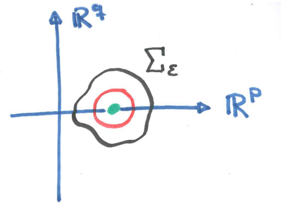
Proof: inner bound

Proposition. $\exists C_{\text{inner}}$ so that the tubular hypersurface $\mathrm{Tub}(S, C_{\text{inner}}\epsilon)$ is outer trapped in $(\mathbb{R}^n \setminus S, g_\epsilon)$.

$$H^{\delta}_{\mathrm{Tub}(S,C\epsilon)} = (n-m-1)(C\epsilon)^{-1} + O(1)$$

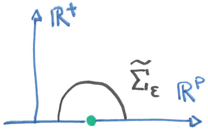
$$H^{g_\epsilon}_{\mathrm{Tub}(S,C\epsilon)} = U_\epsilon^{-2/(n-2)} \left(H^{\delta}_{\mathrm{Tub}(S,C\epsilon)} + 2\frac{n-1}{n-2} d(\ln U_\epsilon)(\nu) \right)$$

$\Rightarrow$ horizon Σ_ϵ in $(\mathbb{R}^n \setminus S, g_\epsilon)$ outside $\mathrm{Tub}(S, C_{\text{inner}}\epsilon)$.



Proof: the horizon after symmetry

SO(q)-symmetry of $(\mathbb{R}^p \times \mathbb{R}^q, g_\epsilon) \Rightarrow \Sigma_\epsilon$ has quotient $\tilde{\Sigma}_\epsilon$ in $\mathbb{R}^p \times \mathbb{R}^+$.



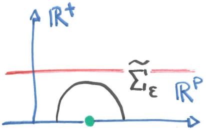
Height function z = projection on $\mathbb{R}^+$.

$$H_\Sigma^{g_\epsilon} = U_\epsilon^{-2/(n-2)} \left((q-1)\frac{dz(\nu)}{z} + H_{\tilde{\Sigma}}^\delta + 2\frac{n-1}{n-2} d(\ln U_\epsilon)(\nu) \right)$$

$\Rightarrow$ equation for $\tilde{\Sigma}$:

$$(q-1)\frac{dz(\tilde{\nu})}{z} + H_{\tilde{\Sigma}}^\delta + 2\frac{n-1}{n-2} d(\ln U_\epsilon)(\tilde{\nu}) = 0.$$

Proposition. $\exists C_{\mathrm{upper}}$ so that $z(x) < C_{\mathrm{upper}}\epsilon$ for $x \in \Sigma_\epsilon$.



Proof: evaluate

$$(q-1)\frac{dz(\tilde{\nu})}{z} + H^{\delta}_{\tilde{\Sigma}} + 2\frac{n-1}{n-2}d(\ln U_\epsilon)(\tilde{\nu}) = 0.$$

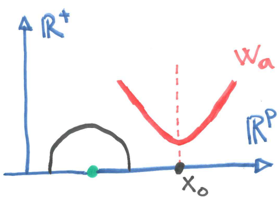
at maximum of z.

Proof: sideways bound

- Construct surfaces graph(W_a) rotationally symmetric in $\mathbb{R}^p$ around $x_0 \in \mathbb{R}^p$ solving
$$\beta \frac{dz(\nu)}{z} + H^{\delta}_{\mathrm{graph}(W_a)} = 0,$$
with minimum at height a above x_0.
- $W_a(x) := w_a(|x - x_0|)$ for w_a satisfying
$$\begin{cases} \dfrac{\ddot{w}_a(t)}{1 + (\dot{w}_a(t))^2} = \dfrac{\beta}{w_a(t)} - \dfrac{\dot{w}_a(t)}{t}, \\ w_{a,\delta}(0) = a, \quad \dot{w}_{a,\delta}(0) = 0. \end{cases}$$
- Regularize ODE $\Rightarrow$ existence and properties of solutions w_a.

Proof: sideways bound (continued)

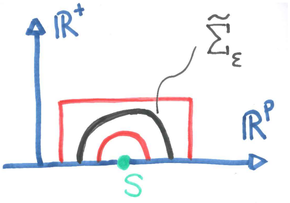
- Contradiction if graph(W_a) tangent to Σ_ϵ far from S.



- **Proposition.** $\exists C_{\text{side}}$ so that the the horizontal distance from Σ_ϵ to S is at most $C_{\text{side}}\epsilon$.

Proof: convergence

- Bounds scaling $\sim \epsilon$.



- Study convergence of rescaled Σ_ϵ. For $\epsilon_k \to 0$ and $x \in S$ set

$$h_k := \epsilon_k^{-2} \left(\varphi_{x,\epsilon_k}\right)^* \left(g_{\epsilon_k}\right) \to h_\infty := U_\infty^{4/(n-2)} \delta$$

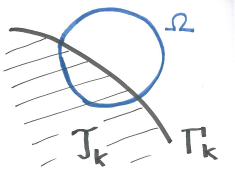
and

$$\Gamma_k := \left(\varphi_{x,\epsilon_k}\right)^{-1} \left(\Sigma_{\epsilon_k}\right).$$

Proof: convergence (continued)

- **Theorem.** (Schoen-Simon, Wickramasekera) $n \leq 7$, Γ_k smooth stable minimal surface for metric h_k,
    - uniform area bound,
    - all intersect a compact set,

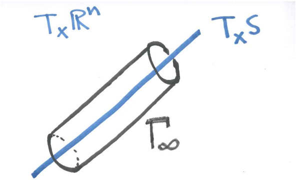
  $\Rightarrow$ subsequence converges smoothly to Γ_∞ stable minimal for h_∞.
- Outward area minimizing property $\Rightarrow$ uniform area bound.



$$\mathrm{Vol}(\Gamma_k \cap \Omega) \leq \mathrm{Vol}(\partial\Omega)$$

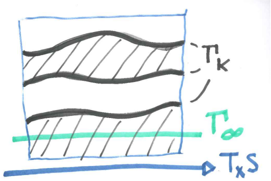
Proof: identifying the limit

- In the metric h_∞ we have foliation of $T_x\mathbb{R}^n \setminus T_xS$ by CMC cylinders around T_xS.
- Maximum principle argument $\Rightarrow \Gamma_\infty$ is the unique zero mean curvature cylinder.



Proof: identifying the limit (continued)

- $\Gamma_k \to \Gamma_\infty$ smooth convergence with multiplicities $\Rightarrow$ Γ_k finite number of graphs over the limit.



- Outward area minimizing $\Rightarrow$ only one graph over Γ_∞ = cylinder around $T_x S$.
- Patching $\Rightarrow$ Σ_{ϵ_k} diffeomorphic to tube around S.