The Number of Automorphisms of Random Trees

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Automorphisms of graphs

Definition

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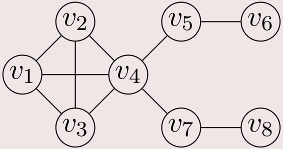
Example

The bijection α defined by

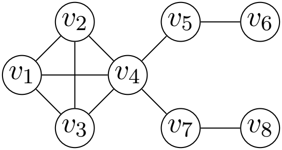
$$\alpha(v_1) = v_2,\ \alpha(v_2) = v_3,\ \alpha(v_3) = v_1,\ \alpha(v_4) = v_4,$$
$$\alpha(v_5) = v_7,\ \alpha(v_6) = v_8,\ \alpha(v_7) = v_5,\ \alpha(v_8) = v_6,$$

is an automorphism of the graph



The automorphism group

The automorphisms of a graph G form a group $\mathrm{Aut}(G)$ with respect to composition. In our example, this automorphism group is isomorphic to $S_2 \otimes S_3$, which has twelve elements.



Automorphism group and enumeration

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Theorem

A graph G with n vertices can be labelled with labels $1, 2, \ldots, n$ in

$$\frac{n!}{|\operatorname{Aut}(G)|}$$

different ways.

Big and small automorphism groups

The size of the automorphism group can vary greatly between trees of the same size: a tree with n vertices can have only one automorphism (the identity), but also as many as $(n-1)!$ automorphisms.

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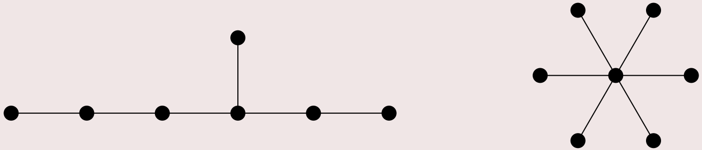
Two trees with seven vertices whose automorphism groups have order 1 and 720 respectively:



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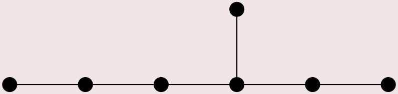

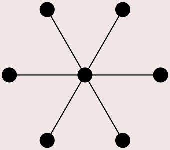

This poses the natural question for the *typical* order of the automorphism group of a tree (given the number of vertices).

Prior results

- McKeon (1991,1996): asymptotic behaviour of the mean number of automorphisms for certain tree classes.

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- Yu (2012): asymptotic behaviour of mean and variance of $\log|\mathrm{Aut}(T)|$ for random labelled trees as $|T| \to \infty$; concentration property. Lognormal limit law is conjectured.

The main theorem

Theorem

Let T_n be a labelled tree of order n chosen uniformly at random. There exist positive constants $\mu \approx 0.052290$ and $\sigma^2 \approx 0.039498$ such that mean and variance of $\log|\operatorname{Aut} T_n|$ are $\mu_n = \mu n + O(1)$ and $\sigma_n^2 = \sigma^2 n + O(1)$ respectively, and the renormalised random variable

$$\frac{\log|\operatorname{Aut} T_n| - \mu_n}{\sigma_n}$$

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Analogous statements hold for unlabelled trees and other families of trees (e.g. plane trees, d-ary trees) as well.

Rooted trees: recursive calculation

The structure of the automorphism group of trees is particularly well understood: it is always obtained from symmetric groups by iterated direct products and wreath products (Jordan 1869). One can derive a simple recursive formula for the order of the automorphism group of a rooted tree in terms of its branches from this fact:

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Suppose that the root branches are rooted trees $T_1, T_2, \ldots, T_k$ with multiplicities $r_1, r_2, \ldots, r_k$. Then we have

$$|\operatorname{Aut} T| = \prod_{j=1}^{k} r_j! |\operatorname{Aut} T_j|^{r_j}.$$

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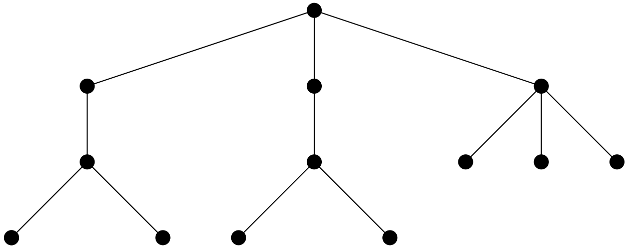
Simply put, an automorphism of T acts as an automorphism within branches and also possibly permutes branches that are isomorphic.

Rooted trees: an example



In this example, $|\operatorname{Aut}(T)| = (2! \cdot 2!^2) \cdot 3! = 48.$

Cayley trees and Pólya trees

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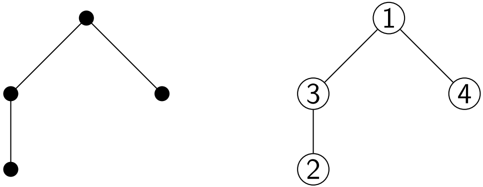
The figure shows a Pólya tree and a possible labelling that represents a Cayley tree.



Generating functions

We consider the two bivariate generating functions associated with Pólya and Cayley trees respectively:

$$Y_{\mathcal{P}}(x,t) = \sum_{T \in \mathcal{P}} x^{|T|} |\operatorname{Aut} T|^t$$

and

$$Y_{\mathcal{C}}(x,t) = \sum_{T \in \mathcal{C}} \frac{x^{|T|}}{|T|!} |\operatorname{Aut} T|^t.$$

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Since every Pólya tree T can be labelled in $|T|!/|\operatorname{Aut}(T)|$ ways to yield a Cayley tree, we have the elementary relation

$$Y_{\mathcal{C}}(x,t) = Y_{\mathcal{P}}(x,t-1).$$

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which translates to

$$Y_{\mathcal{P}}(x,t) = x \prod_{T \in \mathcal{P}} \Big(\sum_{n \geq 0} n!^t x^{n|T|} |\operatorname{Aut} T|^{nt} \Big),$$

making use of the recursive formula for the size of the automorphism group.

A functional equation

Next we manipulate the equation

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$$\begin{aligned}
Y_{\mathcal{P}}(x,t) &= x \exp\Big(\sum_{T \in \mathcal{P}} \log \sum_{n \geq 0} n!^t x^{n|T|} |\operatorname{Aut} T|^{nt} \Big) \\
&= x \exp\Big(\sum_{T \in \mathcal{P}} \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \Big(\sum_{n \geq 1} n!^t x^{n|T|} |\operatorname{Aut} T|^{nt} \Big)^k \Big) \\
&= x \exp\Big(\sum_{T \in \mathcal{P}} \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\lambda_1 + \lambda_2 + \cdots = k} \binom{k}{\lambda_1, \lambda_2, \dots} \prod_{m \geq 1} \big(m!^t x^{m|T|} |\operatorname{Aut} T|^{mt} \big)^{\lambda_m} \Big) \\
&= x \exp\Big(\sum_{k \geq 1} \sum_{j \geq 1} (-1)^{k-1} (k-1)! \sum_{\substack{\lambda_1 + \lambda_2 + \cdots = k \\ \lambda_1 + 2\lambda_2 + \cdots = j}} \prod_{m \geq 1} \Big(\frac{m!^{\lambda_m t}}{\lambda_m !} \Big) \sum_{T \in \mathcal{P}} x^{j|T|} |\operatorname{Aut} T|^{jt} \Big).
\end{aligned}$$

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Thus for certain coefficients $a(j,t)$, we have

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and consequently, by the relationship between Pólya trees and Cayley trees,

$$Y_{\mathcal{C}}(x,t) = x \exp\Big(\sum_{j\geq 1} a(j,t-1) Y_{\mathcal{C}}(x^j, jt-j+1) \Big).$$

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We have $a(1,t) = 1$ for all t and $|a(j,t)| \leq 2^{j-1}$ for $\mathrm{Re}(t) \leq 0$. It follows that

$$Y_{\mathcal{C}}(x,t) = x \exp\Big(Y_{\mathcal{C}}(x,t) + R(x,t)\Big),$$

where $R(x,t)$ is an analytic function of x and t if $|x| < \frac{1}{2}$ and $\mathrm{Re}(t) \leq \frac{1}{2}$.

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Now let W denote Lambert's W-function, defined implicitly by $x = W(x)e^{W(x)}$. We can write

$$Y_{\mathcal{C}}(x,t) = -W(-x\exp(R(x,t))).$$

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It is well known that $W(x)$ can be analytically continued to the entire complex plane, except for a branch cut at $x = -e^{-1}$, where we have the asymptotic expansion

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It follows that

$$xR(x,t) = \frac{1}{e} - \frac{1}{e}\Big(1+\rho(t)\frac{\partial}{\partial x}R(x,t)\Big|_{x=\rho(t)}\Big)(1-x/\rho(t)) + O(|1-x/\rho(t)|^2).$$

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Hence $Y_{\mathcal{C}}(x,t) = -W(-x\exp(R(x,t)))$ has a square root singularity with asymptotic expansion

$$Y_{\mathcal{C}}(x,t) = 1 - \left(2\Big(1+\rho(t)\frac{\partial}{\partial x}R(x,t)\Big|_{x=\rho(t)}\Big)\right)^{1/2}\sqrt{1-x/\rho(t)} + O(|1-x/\rho(t)|).$$

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Now we can apply the *Flajolet-Odlyzko singularity analysis*, which yields

$$[x^n]Y_{\mathcal{C}}(x,t) \sim \left(\frac{1}{2\pi}\Big(1+\rho(t)\frac{\partial}{\partial x}R(x,t)\Big|_{x=\rho(t)}\Big)\right)^{1/2} n^{-3/2}\rho(t)^{-n}.$$

uniformly in t on compact subsets of the half-plane $\{t \in \mathbb{C} : \mathrm{Re}(t) \leq \frac{1}{2}\}$.

Deducing the limiting distribution

$$[x^n]Y_{\mathcal{C}}(x,t) = \frac{1}{n!} \sum_{\substack{T \in \mathcal{C} \\ |T|=n}} |\operatorname{Aut} T|^t \sim \left(\frac{1}{2\pi} \left(1 + \rho(t) \frac{\partial}{\partial x} R(x,t) \Big|_{x=\rho(t)} \right) \right)^{1/2} n^{-3/2} \rho(t)^{-n}.$$

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Since there are n^{n-1} Cayley trees with n vertices, the moment generating function of $\log|\operatorname{Aut} T|$ for random Cayley trees with n vertices is

$$\frac{1}{n^{n-1}}\sum_{\substack{T\in\mathcal{C}\\|T|=n}} e^{t\log|\operatorname{Aut} T|} = \frac{1}{n^{n-1}}\sum_{\substack{T\in\mathcal{C}\\|T|=n}} |\operatorname{Aut} T|^t,$$

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In fact, one can use a general result, known as *Hwang's quasi-power theorem*, to obtain the desired result.

The trouble with Pólya trees

Our approach fails to apply to Pólya trees, because the generating function

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Thus we need a somewhat different approach, where extremely large contributions to the automorphism group are neglected.

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Hence, if a class $\mathcal{T}$ of rooted trees has outdegrees bounded by Δ, then

$$Y_{\mathcal{T}}(x,t) = \sum_{T \in \mathcal{T}} x^{|T|} |\operatorname{Aut} T|^t$$

does in fact have nonzero radius of convergence for every t, and the techniques that we used for Cayley trees can still be applied.

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where $T_1, T_2, \ldots$ are the branches of T with multiplicities $r_1, r_2, \ldots$, we replace the factor $\prod_{j=1}^{k} r_j!$ by $\prod_{j=1}^{k} \min(M, r_j!)$ to obtain the recursive definition of a quantity $A_M(T)$:

$$A_M(T) = \prod_{j=1}^{k} \min(M, r_j!) A_M(T_j)^{r_j}.$$

Now we first prove a central limit theorem for A_M and let M go to infinity.

From rooted to unrooted trees

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As it turns out, artificially fixing one vertex (the root) cannot influence the size of the automorphism group too much:

Lemma

Let T_r be a rooted version of some tree T (rooted at a vertex r). The sizes of the automorphism groups of T and T_r satisfy the inequalities

$$|\operatorname{Aut} T_r| \leq |\operatorname{Aut} T| \leq |T| |\operatorname{Aut} T_r|.$$

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The first inequality is trivial, for the second we note that the number of different rooted labellings of T is $|T| \cdot \frac{|T|!}{|\operatorname{Aut} T|}$, while the number of labellings of T with r as the root is $\frac{|T|!}{|\operatorname{Aut} T_r|}$.

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imply that

$$\log|\operatorname{Aut} T_r| = \log|\operatorname{Aut} T| + O(\log|T|),$$

so the central limit theorem carries over (the error is of lower order than the standard deviation).

Conclusion

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- For many different classes of random trees, the number of automorphisms asymptotically follows a log-normal limit law.
- Similar results can also be proven for trees that are not uniformly chosen at random, but rather follow a growth process (Ralaivaosaona + SW, 2016+)
- We expect similar results to hold for classes of graphs that are "tree-like", in particular so-called subcritical graph classes (which include e.g. cacti, outerplanar graphs and series-parallel graphs).