Horizontal profiles of forests: scaling limits and time-change equations

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Joint work with overlapping subsets of
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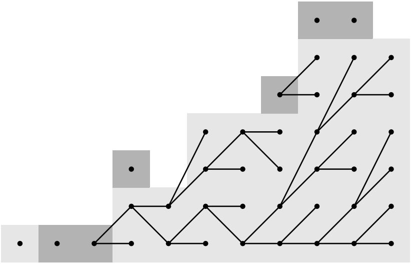
Galton-Watson Processes with Immigration



Typical Construction

- $(\chi_{n,i})_{n,i}$ independent with distribution μ on $\{0, 1, 2, \ldots\}$.
- (η_n) independent (also with χ) with distribution ν.
- Set $Z_0 = k$ and

$$Z_{n+1} = \eta_{n+1} + \sum_{i=1}^{Z_n} \chi_{n,i}.$$

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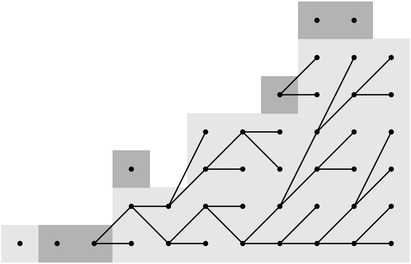

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- Atypical construction ($\nu = 0$): Let $X = (X_n)$ be a random walk such that $\mathbb{P}(X_n - X_{n-1} = k) = \mu_{k+1}$. Define $Z_0 = k$ and

$$Z_{n+1} = k + X_{C_n} \quad \text{where} \quad C_n = Z_0 + \cdots + Z_n.$$

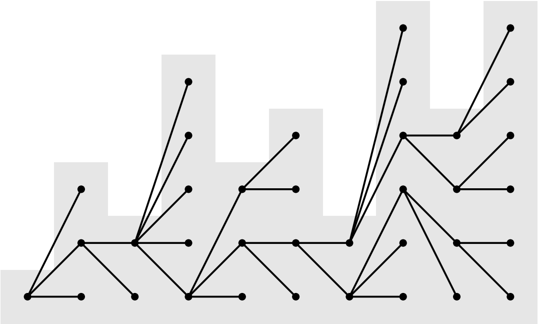
- Branching property: $\mathbb{P}^{\mu,\nu}_{k_1} * \mathbb{P}^{\mu}_{k_2} = \mathbb{P}^{\mu,\nu}_{k_1+k_2}$.
- Implication: $\mathbb{E}^{\mu,\nu}_k\left(s^{Z_n}\right) = \mathbb{E}^{\mu}_k\left(s^{Z_n}\right)^k \mathbb{E}^{\mu,\nu}_0\left(s^{Z_n}\right)$.

Galton-Watson trees and forests

Rooted plane trees

A tree in which one has ordered all siblings.
Formally realized as sets of words $u \in \mathscr{U} = \cup_{n \geq 0} \mathbb{Z}_+^n$ such as $\{\emptyset, 1, 2, 3, 1, 21, 22, 221, 222, 223, 224, 225, , 2211, \ldots\}$, satisfying

1. $\emptyset \in \tau$
2. If $uj \in \tau$ then $u \in \tau$
3. If $u \in \tau$, there exists $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.



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Galton-Watson trees

Built recursively via an iid sequence $(\chi_u, u \in \mathscr{U})$.

- $\emptyset \in \Theta$
- If $u \in \Theta$, add uj for all $j \leq \chi_u$.

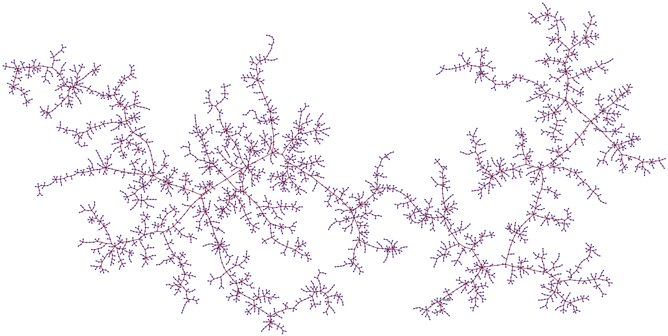
If Z_k is the size of the k-th generation, the process $Z = (Z_k)$ is a Galton-Watson process whose offspring distribution is the same as that of χ.

$$\Theta \text{ is almost surely finite if and only if } \mathbb{E}(\chi_u) \leq 1.$$

Θ^n: Θ conditioned on having size n.

Theorem (Aldous 1993)

If $\mathbb{E}(\chi_i) = 1$, $\mathrm{Var}(\chi_i) = \sigma^2 \in (0, \infty)$ and χ_u is aperiodic then $\left(\Theta^n, \sigma d_n/\sqrt{n}\right)$ converges distribution as $n \to \infty$ to a random metric space (τ, d) called the Continuum Random Tree.

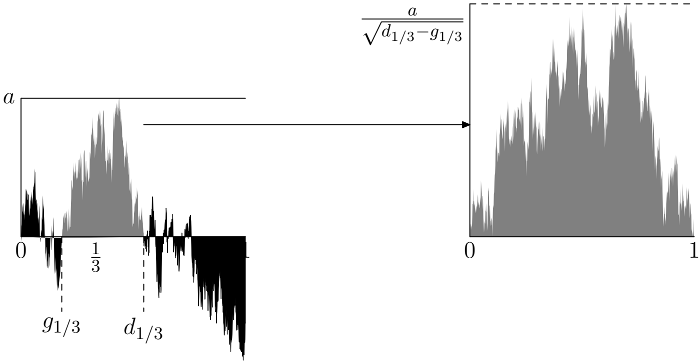


↑, *The continuum random tree. III*, Ann. Probab. **21** (1993), no. 1, 248–289. MR: 1207226

Conjecture (Aldous 1991)

If $\mathbb{E}(\chi_i) = 1$, $\mathrm{Var}(\chi_i) = \sigma^2 \in (0, \infty)$ and χ_u is aperiodic then $(Z^n_{\sqrt{n}k}/\sqrt{n}, k \geq 0)$ converges in distribution $n \to \infty$ to $(\sigma/2 Z_{\sigma t/2}, t \geq 0)$, where

$$\int_0^1 f(e_s)\, ds = \int f(t)\, Z_t\, dt.$$



David Aldous, *The continuum random tree. II. An overview*, Stochastic analysis (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 167, Cambridge Univ. Press, Cambridge, 1991, pp. 23–70. MR: 1166406

Theorem (Drmota and Gittenberger, 1997)

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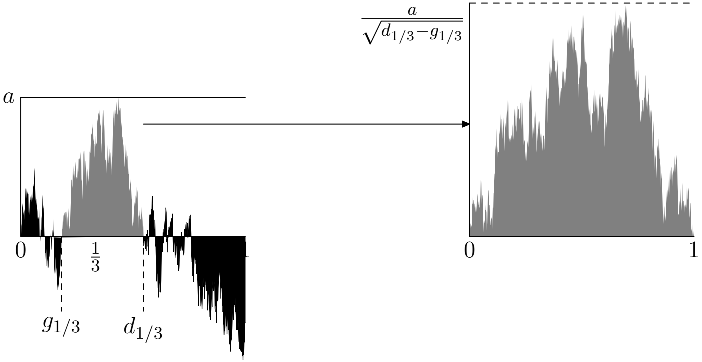
and e is a normalized Brownian excursion.



Michael Drmota and Bernhard Gittenberger, *On the profile of random trees*, Random Structures Algorithms **10** (1997), no. 4, 421–451. MR: 1608230 (99c:05176)

Jeulin's theorem

Theorem

The local time process Z defined by

$$\int_0^1 f(e_s)\, ds = \int f(t)\, Z_t\, dt$$

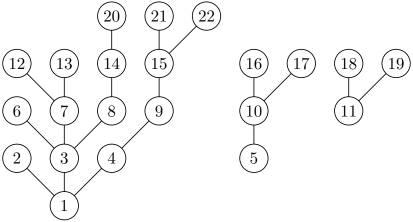
has the same law as the unique solution $\tilde{Z}$ of

$$\tilde{Z}_t = e_{\int_0^t \tilde{Z}_s\, ds}$$

which is non-zero to the right of zero.

Th. Jeulin and M. Yor (eds.), *Grossissements de filtrations: exemples et applications*, Lecture Notes in Mathematics, vol. 1118, Springer-Verlag, Berlin, 1985. MR: 884713

Time-change equations: Coding discrete populations

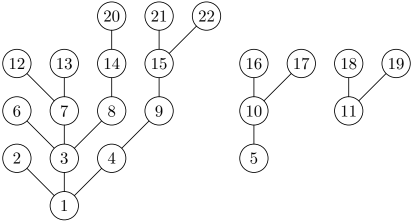


- χ_i: # children of individual i.

- y_n: # immigrants up to generation n.
- c_n: # individuals up to generation n.

$$c_n = c_0 + y_n + \chi_1 + \cdots + \chi_{c_{n-1}}$$

Time-change equations: Coding discrete populations



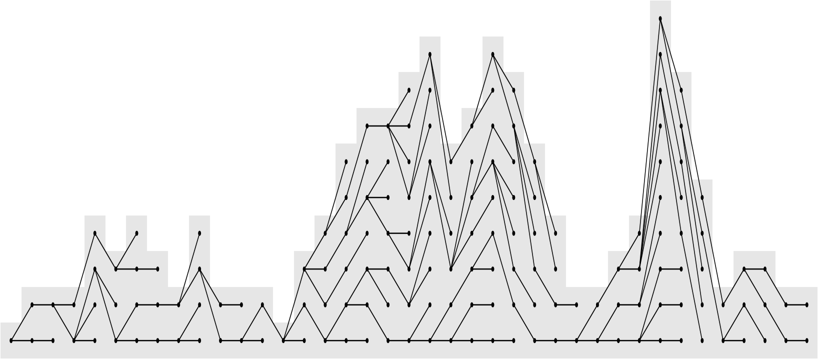
- χ_i: # children of individual i.
- $x_n = \chi_1 + \cdots + \chi_n - n$.
- y_n: # immigrants up to generation n.
- c_n: # individuals up to generation n.
- z_n: # individuals comprising generation n.

$$c_n = c_0 + y_n + \chi_1 + \cdots + \chi_{c_{n-1}}$$
$$= z_0 + \cdots + z_n$$
$$z_n = c_0 + x_{c_{n-1}} + y_n$$

A representation of GWI processes

- μ reproduction law, ν immigration law.
- $\tilde{\mu}(k) = \mu(k+1)$.
- X a random walk with step distribution $\tilde{\mu}$.
- Y an independent random walk with step distribution ν.
- $Z_0 = k$ and for $n \geq 1$:

$$Z_n = k + X_{Z_0 + \cdots + Z_{n-1}} + Y_n.$$



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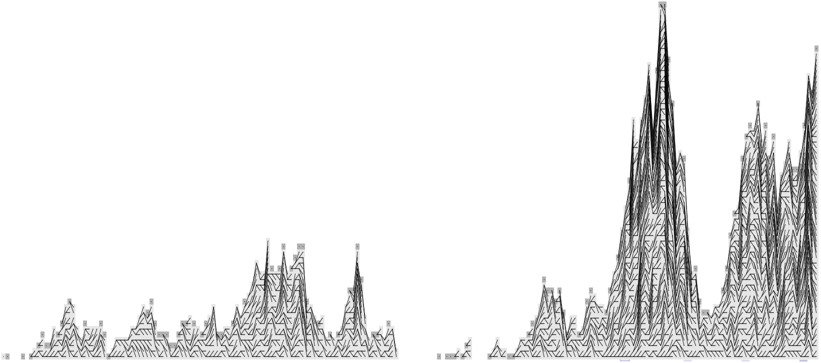

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Proposed extension:

X a SPLP, Y an independent subordinator and $x \geq 0$

$$Z_t = x + X_{\int_0^t Z_s \, ds} + Y_t.$$

An initial value problem

$$Z_t = x + X_{\int_0^t Z_s \, ds} + Y_t$$

Initial Value Problem:

Let f, g be càdlàg with $\Delta f \geq 0$, g increasing and $f(0) + g(0) \geq 0$. A function c solves IVP(f, g) if

$$c'_+ = f \circ c + g \quad \text{and} \quad c_0 = 0.$$

- f: reproduction function
- g: immigration function
- c: cumulative population
- $h = c'_+$: profile

Obvious problems

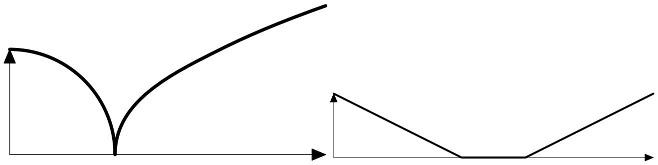
1. Existence?
2. Uniqueness?

The Lamperti transformation, existence, and uniqueness

$$c_+' = f \circ c. \qquad i = c^{-1} \quad i' = \frac{1}{f \circ c \circ i} = \frac{1}{f}!!!$$

Problem:$f(x) = \sqrt{|1-x|}$. Then there are many solutions: their derivatives are

$$\left(\frac{2-x}{2}\right)^+ \quad \text{and} \quad \begin{cases} \frac{2-x}{2} & x < 2 \\ 0 & 2 \le x \le 2+l \\ \frac{x-2-l}{2} & x \ge 2+l \end{cases}$$



Existence and uniqueness for IVP(f, g)

$$c'_+ = f \circ c + g$$

Existence and uniqueness theorem

Let f, g be càdlàg $\Delta f \geq 0$, g increasing, $f(0) + g(0) \geq 0$. There exists a non-decreasing c which satisfies IVP(f, g). If g is strictly increasing the solution is unique.

Continuity theorem

Suppose g is strictly increasing, and that $f_n \to f$ and $g_n \to g$.
Let $\sigma_n \to 0$, $t_i^n = \sigma^n i$ and define c^n by $c^n(0) = 0$ and

$$c^n(t) = c^n(t_{i-1}) + (t - t_{i-1}^n)\left[f_n \circ c^n(t_{i-1}) + g_n(t_{i-1})\right]^+ .$$

for $t \in [t_{i-1}, t_i]$. Then c^n converges to the unique solution c of IVP(f, g).

The Lamperti type representation of CBI

Theorem

Let X be a SPLP and Y and independent subordinator. For any $x \geq 0$ there exists a unique solution to

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_s.$$

Limit theorems for GWI processes

Corollary

- X^n random walk with step distribution $\mu^n_{k+1}, k \geq -1$.
- Y^n random walk with step distribution $\nu^n_k, k \geq 0$.
- $X^n_{c_n}/n \to \mu$ (μ is sP ID with Laplace exponent ψ).
- $Y^n_{d_n}/n \to \nu$ (ν corresponds to a subordinator with Laplace exponent φ).
- Z^n is GW(μ^n, ν^n), $Z^n_0 = k_n$
- $\frac{k_n d_{\frac{k_n}{x}}}{x c_{\frac{k_m}{x}}} \to c \in [0, \infty)$
- $\frac{x}{k_n} Z^n_{d_{\frac{k_n}{x}} t}$ converges weakly to $\mathrm{CBI}_x(c\psi, \varphi)$.

Anders Grimvall, *On the convergence of sequences of branching processes*, Ann. Probability **2** (1974), 1027–1045. MR: 0362529

M. Emilia Caballero, José Luis Pérez Garmendia, and Gerónimo Uribe Bravo, *A Lamperti-type representation of continuous-state branching processes with immigration*, Ann. Probab. **41** (2013), no. 3A, 1585–1627. MR: 3098685

Limit theorems for Conditioned GW processes

Theorem

- μ critical and aperiodic offspring law.
- S random walk with step distribution $\mu_{k+1}, k \geq -1$.
- S_n/a_n converges weakly to (sp) stable law of index $\alpha \in (1, 2]$.
- Z^{n,k_n} with law $\mathrm{GW}_{k_n}(\mu)$ and conditioned on

$$\sum_i Z_i^{n,k_n} = n.$$

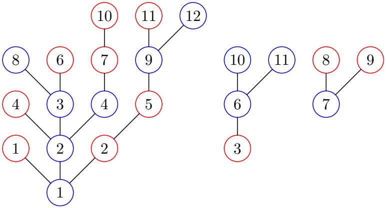
- $k_n/a_n \to l > 0$.
- F^l: first passage bridge of α stable spLp.

Then

$$\left(\frac{a_n}{n} Z_{nt}^{n,k_n}\right)_{t\geq 0} \to \text{solution of } \mathrm{IVP}\left(F^l, 0\right).$$

Jim Pitman, *The SDE solved by local times of a Brownian excursion or bridge derived from the height profile of a random tree or forest*, Ann. Probab. **27** (1999), no. 1, 261–283. MR: 1681110 (2000b:60200) M. Emilia Caballero, José Luis Pérez Garmendia, and Gerónimo Uribe Bravo, *A Lamperti-type representation of continuous-state branching processes with immigration*, Ann. Probab. **41** (2013), no. 3A, 1585–1627. MR: 3098685

Coding discrete multitype populations



- $\chi_k^{i,j}$: # children of type j of the k-th individual of type i.

$$c_n^j = c_0^j + y_n^j + \sum_i \chi_1^{i,j} + \cdots + \chi_{c_{n-1}^i}^{i,j}$$

- y_n^j: # immigrants of type j up to generation n.
- c_n^j: # individuals of type j up to generation n.

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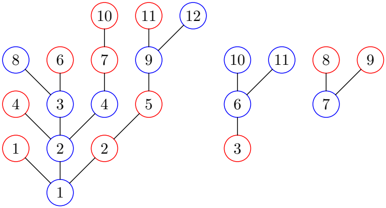

- $\chi_k^{i,j}$: # children of type j of the k-th individual of type i.
- $x_n^{j,j} = \chi_1^{j,j} + \cdots + \chi_n^{j,j} - n$.
- $i \neq j$: $x_n^{i,j} = \chi_1^{i,j} + \cdots + \chi_n^{i,j}$.
- y_n^j: # immigrants of type j up to generation n.
- c_n^j: # individuals of type j up to generation n.
- z_n^j: # individuals of type j comprising generation n.

$$c_n^j = c_0^j + y_n^j + \sum_i \chi_1^{i,j} + \cdots + \chi_{c_{n-1}^i}^{i,j}$$

$$= z_0^j + \cdots + z_n^j$$

$$z_n^j = c_0^j + \sum_i x^{i,j}\left(c_{n-1}^i\right) + y_n^j$$

A representation of Multi-type Galton-Watson processes with immigration

Consider $m+1$ independent random walks on $\mathbb{R}^m$: $X^1, \ldots, X^m$ and Y with

$$X^j = \left(X^{j,1}, \ldots, X^{j,m}\right).$$

Suppose that

- $X^{j,j}$ has jumps in $\{-1, 0, 1, \ldots\}$,
- for $i \neq j$, $X^{i,j}$ has jumps in $\{0, 1, 2, \ldots\}$ and
- $Y = \left(Y^1, \ldots, Y^m\right)$ and Y^j has jumps in $\{0, 1, 2, \ldots\}$.

Let $z = (z_1, \ldots, z_m) \in \{0, 1, 2, \ldots\}^m$.
Define $Z_0 = z$ and, recursively, $C^j_n = Z^j_0 + \cdots + Z^j_n$ and

$$Z^j_{n+1} = z_j + \sum_i X^{i,j}\left(C^i_n\right) + Y^j_{n+1}.$$

Then Z is a multi-type Galton-Watson process with immigration and all such processes can be obtained this way.

Limits of random walks associated to MGW processes

Consider $m+1$ independent random walks on $\mathbb{R}^m$: $X^1, \ldots, X^m$ and Y with

$$X^j = \left(X^{j,1}, \ldots, X^{j,m}\right).$$

Suppose that

- $X^{j,j}$ has jumps in $\{-1, 0, 1, \ldots\}$,
- for $i \neq j$, $X^{i,j}$ has jumps in $\{0, 1, 2, \ldots\}$ and

Scaling limit for fixed i

Sequence: $X^{i,n} = \left(X^{i,j,n}, 1 \leq j \leq n\right)$.

Scaling limit: $X^{i,n}_{\lfloor nt \rfloor}/a_{n,i}, t \geq 0$.

If X^i is a scaling limit: it is a Lévy processes in $\mathbb{R}^m$ such that $X^{i,i}$ is spectrally positive and $X^{i,j}$ is a subordinator for $i \neq j$.

Proposal: Construct Z such that

$$Z^1_t = z_1 + X^{1,1}_{\int_0^t Z^1_s\, ds} + X^{2,1}_{\int_0^t Z^2_s\, ds} + Y^1_t$$

$$Z^2_t = z_1 + X^{1,2}_{\int_0^t Z^1_s\, ds} + X^{2,2}_{\int_0^t Z^2_s\, ds} + Y^2_t$$

↑, *Affine processes on $\mathbb{R}^m_+ \times \mathbb{R}^n$ and multiparameter time changes*, arXiv e-prints (2015), To appear in Ann. Inst. H. Poincaré Probab.

Trees with a given degree distribution

Let $(V, E, \rho, \leq)$ be a plane tree.
Write $V = \{v_1, \ldots, v_n\}$ where $\rho = v_0 < \cdots < v_{n-1}$ and

$$\delta_i = \#\{j \geq i : \{i, j\} \in E\}.$$

Degree sequence

The degree sequence $N_0, N_1, \ldots$ is obtained by setting

$$N_i = \#\{j : \delta_j = i\}.$$

It is characterized by: $\sum_i N_i = 1 + \sum_i iN_i$.
Every such sequence arises from a plane tree.

Tree with a given degree sequence

Let $s = (N_0, N_1, \ldots)$ be a degree sequence. We will be interested in uniform trees from the set of plane trees having degree sequence s.

Examples

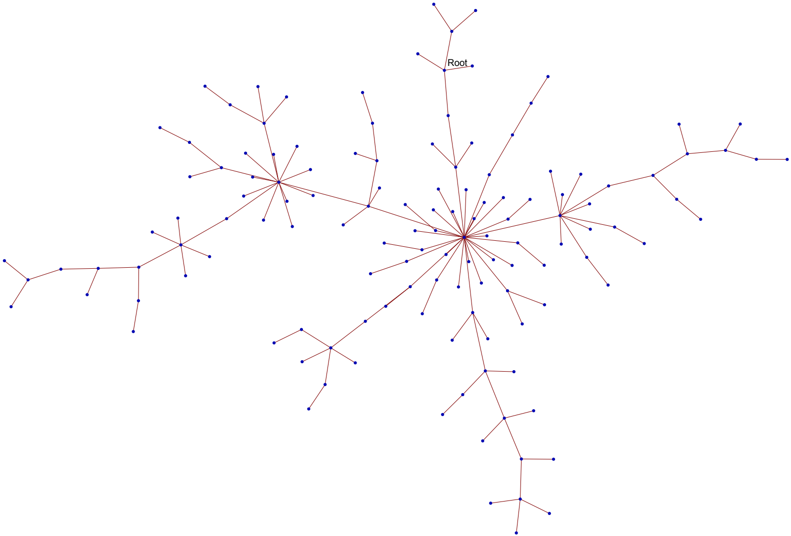

Figure: Figure by Osvaldo Angtuncio

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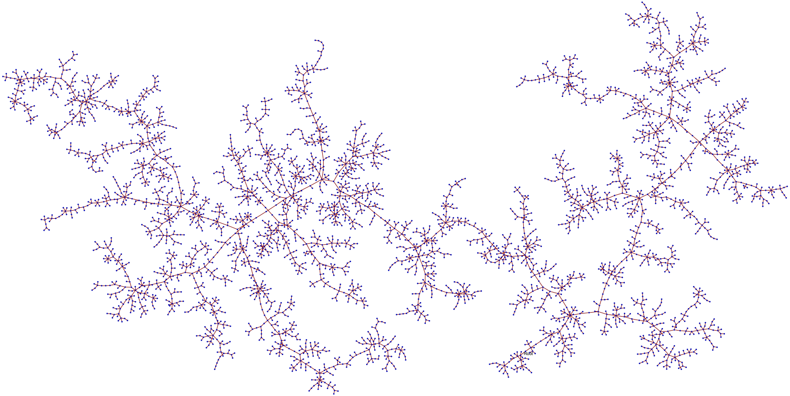

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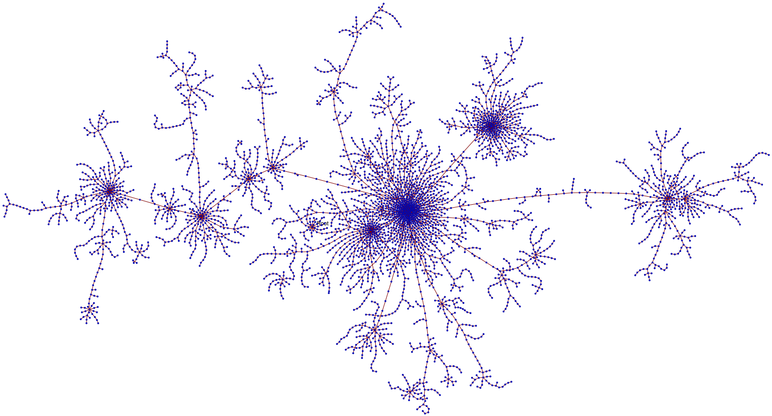

Figure: Figure by Osvaldo Angtuncio

Conditioned Galton-Watson trees

Let $\mathscr{U} = \{\emptyset\} \cup \{u_1 \cdots u_n : n \geq 1, u_i \in \mathbb{Z}_+\}$ be the set of canonical labels of plane trees.
Let $\mu = (\mu_k, k \in \mathbb{N})$ be an offspring distribution and $(\xi_u, u \in \mathscr{U})$ be iid with law μ.

Galton-Watson trees

$$\Theta = \{\emptyset\} \cup \{uj \in \mathscr{U} : j \leq \xi_u\}.$$

Conditioned Galton-Watson trees

$$\Theta_n \overset{d}{=} \Theta \text{ conditioned on having } n \text{ vertices}$$

Proposition

Θ conditioned on having degree sequence s is uniform on trees with degree sequence s.

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