Traces of Sobolev functions

Luboš Pick (Charles University, Prague)

BIRS, July 14, 2016

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and more

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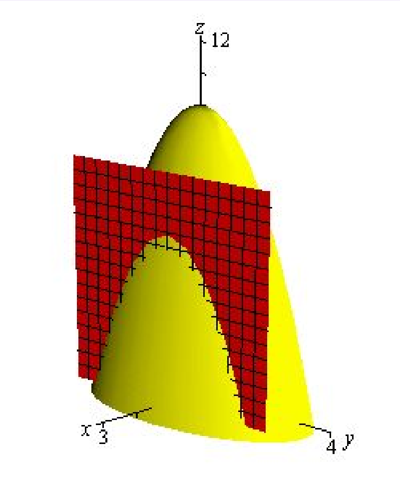

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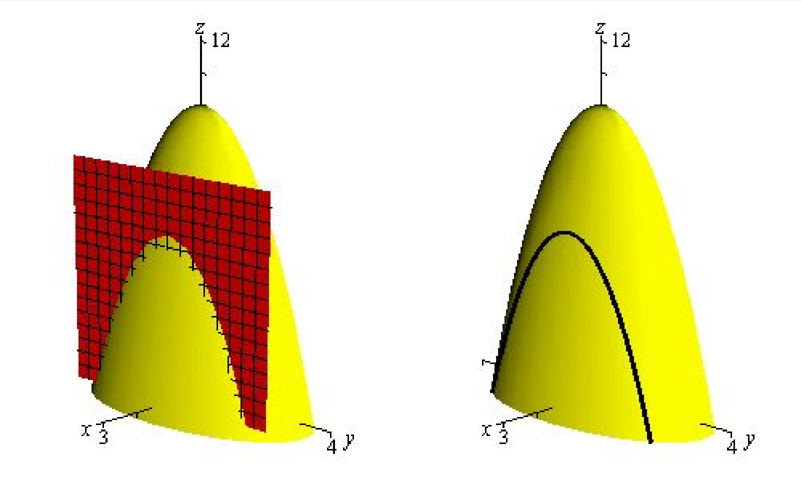

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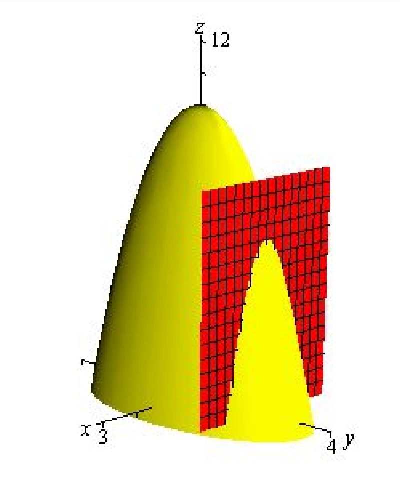

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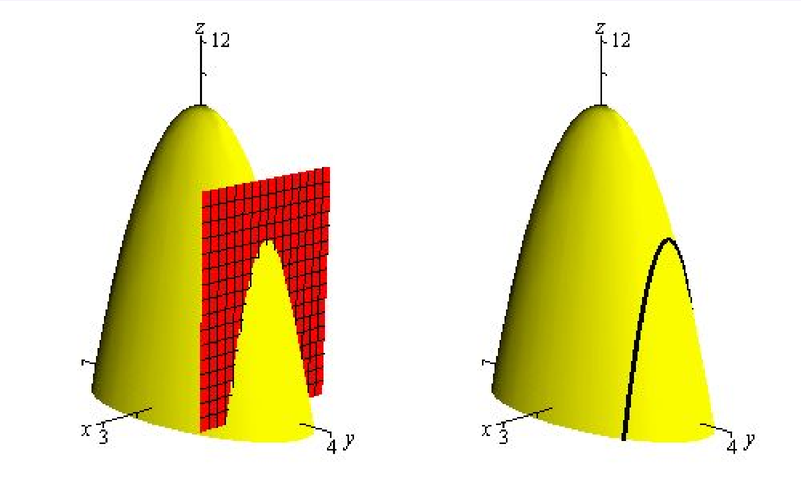

Observation. In elementary calculus, trace means just **restriction** of a function to a certain subset.

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In the principle field of application for traces, however, this concept does not work.

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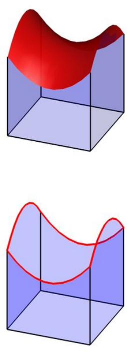

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So, restriction is useless.

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We also denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Trace operator

Remark. The **trace operator** enables one to extend the notion of **restriction** of a function to the boundary to generalized functions in a Sobolev space.

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We call this (rather incorrectly) a **trace embedding**.

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What are we going to study

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The basic question: Given some data on u, what can we say about $\operatorname{Tr} u$?

Elementary observations

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Our main interest concerns functions u in a **Sobolev space**.

Part 1: boundary traces

Classical trace embeddings

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is well defined on $W^{m,p}(\Omega)$, and

$$Tr : W^{m,p}(\Omega) \to \begin{cases} L^{\frac{p(n-1)}{n-mp}}(\partial\Omega) & \textbf{if } p < \frac{n}{m}, \\ L^{\infty}(\partial\Omega) & \textbf{if } p > \frac{n}{m}. \end{cases}$$

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while the **Sobolev boundary-trace embedding** states that

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- Lebesgue norms (heavily).

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For the case $p = \frac{n}{m}$, the classical approach gives only

$$Tr : W^{m,p}(\Omega) \to L^q(\partial\Omega) \text{ for every } q < \infty,$$

which is quite unsatisfactory.

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$$Tr : W^{m,\frac{n}{m}}(\Omega) \to \exp L^{\frac{n}{n-m}}(\partial\Omega).$$

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In fact, none of the available methods seems to cover the whole range of situations of interest in applications.

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Hence the demand: Find a unified flexible approach to trace inequalities fitting sufficiently general class of function spaces.

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- Brothers, Ziemer, *J. Reine Angew. Math.* 1988
- Kawohl, *Springer, Berlin* 1985
- Talenti, *Ann. Mat. Pura Appl.* 1976
- Aubin, *J. Diff. Geom.* 1976
- Moser, *Indiana Univ Math. J.* 1971

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Moreover, we work in a fairly general environment of rearrangement-invariant spaces.

Function spaces involved

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(P5) $\int_0^1 f(x)\,dx \leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If $\|\cdot\|_{X(0,1)}$ is a function norm and, in addition,

(P6) $$\|f\|_{X(0,1)} = \|g\|_{X(0,1)} \text{ whenever } f^* = g^*,$$

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Here, f^* is the *non-increasing rearrangement of f*, defined as

$$f^*(s) = \sup\{\lambda \geq 0 : \nu(\{t \in (0,1); |f(t)| > \lambda\}) > s\}, \quad s \in (0,1).$$

Associate norm

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- Lebesgue spaces
- Lorentz spaces
- Orlicz spaces
- Zygmund classes

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- Hölder continuity spaces
- Sobolev spaces
- Morrey spaces
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$$L^{p,1} \subsetneq L^{p,q} \subsetneq L^{p,r} \subsetneq L^{p,\infty}.$$

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$$Tr : W^{m,p}(\Omega) \to \begin{cases} L^{\frac{p(n-1)}{n-mp},p}(\partial\Omega) & \text{if } p < \frac{n}{m}, \\ L^{\infty,\frac{n}{m};-1}(\partial\Omega) & \text{if } p = \frac{n}{m}, \\ L^{\infty}(\partial\Omega) & \text{if } p > \frac{n}{m}. \end{cases}$$

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$$\left(\int_0^1 \left(\frac{u^*(s)}{\log(\frac{2}{s})} \right)^{\frac{n}{m}} \frac{ds}{s} \right)^{\frac{m}{n}}.$$

Part 2: traces on linear subspaces of smaller dimensions

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Assume that $d \in \mathbb{N}$ is such that $1 \leq d \leq n$.

We denote by Ω_d be the (non empty) intersection of Ω with a d-dimensional affine subspace of $\mathbb{R}^n$.

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- well defined on Ω_d,
- measurable with respect to the d-dimensional measure,
- integrable to some power q depending on p, d, m, n.

Observation: increasing the values of m, p causes u to be *more regular*, hence allows *smaller* values of d and *larger* values of q.

Boundary traces

As we have seen, boundary traces constitute an important special case with $d = n - 1$.

Warning

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For a general d, even the very **existence** of a trace is a problem.

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Moreover,

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Note: The case $m \geq n$ is not interesting since then any function in $W^{m,p}(\Omega)$ is continuous.

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Our first main aim: to fill this gap.

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Then any function from $W^m L^{\frac{n-d}{m},1}(\Omega)$ admits a trace on Ω_d.

Moreover, $L^{\frac{n-d}{m},1}(\Omega)$ is the largest rearrangement-invariant space with this property.

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It is a *lovely* result, but unfortunately *not true*.

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Our next aim: to answer this question.

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Because this and only this assumption guarantees that

- every function in $W^{m,1}(\Omega)$ has a trace on Ω_d,
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Hence it was still there, but invisible.

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Moreover, $X^m_{d,n}(\Omega_d)$ is the *smallest* rearrangement-invariant space with this property.

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Then the following are equivalent:

$$\mathrm{Tr} : W^m X(\Omega) \to L^\infty(\Omega_d),$$

$$\left\| t^{-1+\frac{m}{n}} \right\|_{X'(0,1)} < \infty.$$

The reduction principle

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$$\left\| \int_{t^{\frac{n}{d}}}^1 g(s) s^{-1+\frac{m}{n}} \, ds \right\|_{Y(0,1)} \le C \|g\|_{X(0,1)}.$$

Comments on the reduction principle

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- Cianchi, Edmunds, Gurka, *Math. Nachr.* 1996,
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Note: A remarkable feature of this approach is that any composition of Sobolev and trace embeddings with optimal targets ends up again having optimal target.

The stability theorem

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Let $k, h, d, l \in \mathbb{N}$.

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Let $X(\Omega)$ be a rearrangement-invariant space. Then

$$(X_{l,n}^{k})_{d,l}^{h}(\Omega_d) = X_{d,n}^{k+h}(\Omega_d).$$

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$$W^{n,1}(\Omega) \xrightarrow{\operatorname{Tr}} W^{1,d}(\Omega_d) \to \exp L^{\frac{d}{d-1}}(\Omega_d).$$

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Thank you for your attention, this is all.