ON THE MINIMIZERS OF TRACE INEQUALITIES IN BV

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Geometric and Analytic Inequalities
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Joint papers with A. Cianchi, C. Nitsch and C. Trombetti

Isoperimetric and functional inequalities

- Isoperimetric inequalities for physical quantities like the lowest principal frequency of vibrating clamped membranes, the electrostatic capacity or the torsional rigidity
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All the above quantities decrease or increase under Schwarz symmetrization. A key ingredient in finding sharp bounds is the classical isoperimetric property of the ball.

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The reduction to the case of the ball, when possible, is related to isoperimetric inequalities involving the relative perimeter.

Sobolev-Poincaré inequality in $\mathbb{R}^2$

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$$\|Du\|(K) \geq C(K)\|u - \bar{u}\|_2, \qquad u \in BV(K),$$

where $\|Du\|(K)$ is the total variation of u in K, $\bar{u}$ is the mean value of u on K and the best constant $C(K)$ is given by

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$$C(K) = |K|^{1/2} \inf_{\substack{G \subset K \\ 0<|G|<|K|}} \frac{Per(G;K)}{\sqrt{|G|\,|K \setminus G|}}.$$

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The constant $C(K)$ can be related to the *relative isoperimetric constant*

$$\gamma(K) = \inf_{\substack{G \subset K \\ 0<|G|<|K|}} \frac{Per(G;K)}{(\min\{|G|,|K \setminus G|\})^{1/2}}.$$

Sobolev-Poincaré inequality in $\mathbb{R}^2$

Theorem

If K is a convex set in $\mathbb{R}^2$, we have

$$\gamma(K) \leq \gamma(K^\sharp).$$

where $K^\sharp$ is the disc such that $|K^\sharp| = |K|$. Equality holds if and only if K is a disc.

[ESPOSITO - V.F. - KAWOHL - NITSCH - TROMBETTI (2012)]

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From the above theorem the following inequality follows:

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Poincaré trace inequalities in *BV*

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If the boundary $\partial\Omega$ of Ω is smooth, then a linear operator is defined on the space $BV(\Omega)$ of functions of bounded variation in Ω, which associates with any function $u \in BV(\Omega)$ its (suitably defined) boundary trace $\widetilde{u} \in L^1(\partial\Omega)$.

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There exists a constant C, depending on Ω, such that

$$\inf_{c\in\mathbb{R}} \|\widetilde{u} - c\|_{L^1(\partial\Omega)} \leq C(\Omega)\|Du\|(\Omega)$$

for every $u \in BV(\Omega)$.

[MAZ'YA (2011)]

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for every $u \in BV(\Omega)$.

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We are interested in the minimization of $C(\Omega)$.

Poincaré trace inequalities in *BV*

We observe that the previous inequality is the case $p = 1$ (in *BV* setting) of the following one

$$\inf_{c\in\mathbb{R}} \|\widetilde{u} - c\|_{L^p(\partial\Omega)} \le C_p(\Omega)\|Du\|_{L^p(\Omega)} \tag{1}$$

where the extremal functions are solutions to the Stekloff eigenvalue problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ C_p(\Omega)|Du|^{p-2}\dfrac{\partial u}{\partial \nu} = |u|^{p-2}u & \text{on } \partial\Omega. \end{cases}$$

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The problem of minimizing the constant $C_p(\Omega)$ in (1) has been solved only for $p = 2$. The well known Weinstock-Brock inequality asserts that the (unique) minimizer among sets with fixed measure is the ball.

[WEINSTOCK (1954)], [BROCK (2001)]

Poincaré trace inequalities in *BV*

A property of L^1 norms ensures that the infimum

$$\inf_{c\in\mathbb{R}} \|\widetilde{u} - c\|_{L^1(\partial\Omega)}$$

is attained when c agrees with the median of $\widetilde{u}$ on $\partial\Omega$, given by

$$\mathrm{med}_{\partial\Omega}\widetilde{u} = \sup\{t \in \mathbb{R} : \mathcal{H}^{n-1}(\{\widetilde{u} > t\}) > \mathcal{H}^{n-1}(\partial\Omega)/2\}$$

[CIANCHI - PICK (2003)]

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Thus, the above trace inequality is equivalent to

$$\|\widetilde{u} - \mathrm{med}_{\partial\Omega}\widetilde{u}\|_{L^1(\partial\Omega)} \leq C_{\mathrm{med}}(\Omega)\|Du\|(\Omega)$$

for every $u \in BV(\Omega)$, where $C_{\mathrm{med}}(\Omega)$ denotes the optimal – smallest possible – constant which renders the inequality true.

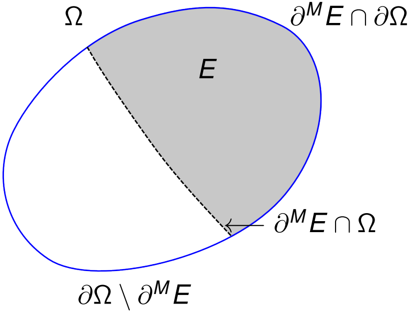
Poincaré trace inequalities in BV

The constant $C_{\mathrm{med}}(\Omega)$ can be characterized as a genuinely geometric quantity associated with Ω, namely,

$$C_{\mathrm{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)\,,\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)},$$

where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure and $\partial^M E$ denotes the essential boundary of E.

[Maz'ya (2011)]



Some related problems

In studying the quality of transportation networks like waterways, railroad systems, or urban street systems one introduces the *dilation* of the network which is defined as C_{med}.

[EBBERS-BAUMANN - GRÜNE - KLEIN (2006)]

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In studying the quality of transportation networks like waterways, railroad systems, or urban street systems one introduces the *dilation* of the network which is defined as C_{med}.

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Also the definition of *distortion* of a curve is related to C_{med}, and it turns out to be useful in determining the thickest curve of prescribed length in a knot class. Such curves are of interest to chemists and biologists modeling polymers and DNA.

[KUSNER - SULLIVAN (1998)]

Poincaré trace inequalities in *BV*

A stronger version of Poincaré trace inequality holds, when $\mathrm{med}_{\partial\Omega}\widetilde{u}$ is replaced with the mean value $\widetilde{u}_{\partial\Omega}$ of $\widetilde{u}$ over $\partial\Omega$, defined as

$$\widetilde{u}_{\partial\Omega} = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} \widetilde{u}\, d\mathcal{H}^{n-1}(x)\,.$$

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The relevant inequality reads

$$\|\widetilde{u} - \widetilde{u}_{\partial\Omega}\|_{L^1(\partial\Omega)} \leq C_{\mathrm{mv}}(\Omega)\|Du\|(\Omega)$$

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Both $C_{\mathrm{med}}(\Omega)$ and $C_{\mathrm{mv}}(\Omega)$ are invariant under dilations of Ω, and hence they only depend on the shape of Ω, but not on its size.

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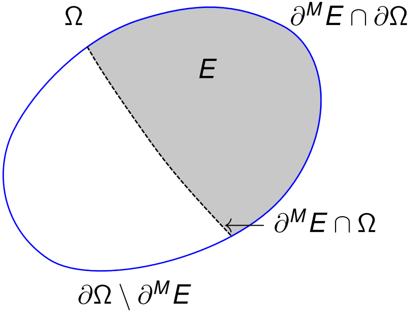
The constant $C_{\mathrm{mv}}(\Omega)$ can be characterized as a genuinely geometric quantity associated with Ω, namely,

$$C_{\mathrm{mv}}(\Omega) = \frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \sup_{E\subset\Omega} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)\ \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)},$$

where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure.

[Cianchi (2012)]



Poincaré trace inequalities in *BV*

Theorem ([CIANCHI - V.F. - NITSCH - TROMBETTI, Crelle (to appear)])

We have:

$$C_{med}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}. \tag{2}$$

Moreover, equality holds in (2) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*

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Remark. The constant which appears in (2) coincides with

$$\frac{n\omega_n}{2\omega_{n-1}},$$

where $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The supremum which defines $C_{\text{med}}(\Omega)$ is attained at a half-ball in this case.

[BOKOWSKI - SPERNER (1979)], [ESCOBAR (1999)], [MAZ'YA (2011)]

Poincaré trace inequalities in *BV*

Theorem ([Cianchi - V.F. - Nitsch - Trombetti, Crelle (to appear)])

If $n \geq 3$*, then*

$$C_{mv}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}, \tag{3}$$

and the equality holds in (3) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*

If $n = 2$*, then*

$$C_{mv}(\Omega) \geq 2, \tag{4}$$

and the equality holds in (4) *if* Ω *is a disc. However there exist open sets* Ω*, that are not equivalent to a disc, for which equality yet holds in* (4)*.*

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Theorem ([CIANCHI - V.F. - NITSCH - TROMBETTI, Crelle (to appear)])

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and the equality holds in (4) *if Ω is a disc. However there exist open sets Ω, that are not equivalent to a disc, for which equality yet holds in* (4).

Remark. Also in this case the lower bounds which appear in (3) and (4) coincide with the values of C_{mv} computed on a ball. When $n \geq 3$, C_{mv} is attained at a half-ball, when $n = 2$, C_{mv} is attained in the limit, considering any sequence of circular segments whose measure converges to 0 (on the half-circle the ratio is $\pi/2$).

[CIANCHI (2012)]

An example

Let us consider a stadium-shaped domain

$S_{R,d} =$ convex hull of two discs of equal radii R, with centers at distance d,

with semi-perimeter $p = d + \pi R$.

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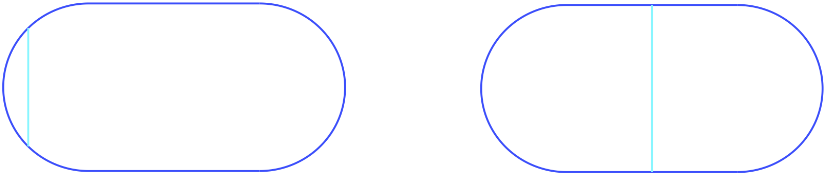
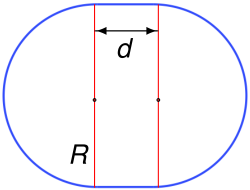


An example

If $d \leq (4-\pi)R$

$$C_{\mathrm{mv}}(S_{R,d}) = 2.$$



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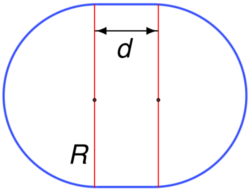
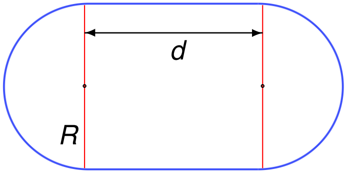


If $d > (4-\pi)R$

$$C_{\mathrm{mv}}(S_{R,d}) = \frac{d+\pi R}{2R} > 2.$$



Further Poincaré trace inequalities in *BV*

We have considered also two "unconventional" Poincaré trace inequalities where the mean value and the median of $\widetilde{u}$ on $\partial\Omega$ is substituted by the mean value and the median of u on Ω.

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Let us denote by $K_{mv}(\Omega)$ the optimal constant in the inequality

$$\|\widetilde{u} - mv_\Omega(u)\|_{L^1(\partial\Omega)} \leq K_{mv}(\Omega)\|Du\|(\Omega) \tag{5}$$

for $u \in BV(\Omega)$.

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for $u \in BV(\Omega)$.

Our first result asserts that $K_{mv}(\Omega)$ agrees with the isoperimetric constant

$$H_{mv}(\Omega) = \sup_{E\subset\Omega} \frac{|E|\,\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E) + |\Omega \setminus E|\,\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{|\Omega|\,\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{6}$$

Further Poincaré trace inequalities in *BV*

Theorem ([Cianchi - V.F. - Nitsch - Trombetti, preprint])

Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \geq 2$. Then

$$K_{mv}(\Omega) = H_{mv}(\Omega). \tag{7}$$

Equality holds in (5) *for some nonconstant function u if and only if the supremum is attained in* (6) *for some set E. In particular, if E is an extremal set in* (6)*, then the function $a\chi_E + b$ is an extremal function in* (5) *for every $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$.*

Further Poincaré trace inequalities in *BV*

Analogously, let $K_{med}(\Omega)$ be the optimal constant in the inequality

$$\|\widetilde{u} - med_\Omega(u)\|_{L^1(\partial\Omega)} \leq K_{med}(\Omega)\|Du\|(\Omega) \tag{8}$$

for $u \in BV(\Omega)$. The isoperimetric constant which now comes into play is defined as

$$H_{med}(\Omega) = \sup_{\substack{E \subset \Omega \\ |E| \leq |\Omega|/2}} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{9}$$

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Theorem ([CIANCHI - V.F. - NITSCH - TROMBETTI, preprint])

Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \ge 2$. Then

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Equality holds in (8) *for some nonconstant function u if and only if the supremum is attained in* (9) *for some set E. In particular, if E is an extremal in* (9)*, then the function $a\chi_E + b$ is an extremal in* (8) *for every $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$.*

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We have started to calculate the constants $K_{mv}(\Omega)$ and $K_{med}(\Omega)$ when $\Omega = B$ is a ball.

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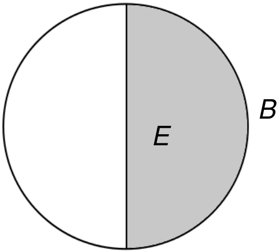
Theorem ([CIANCHI - V.F. - NITSCH - TROMBETTI, preprint])

Let $n \geq 2$. Then

$$H_{mv}(B) = \frac{n\omega_n}{2\omega_{n-1}}.$$

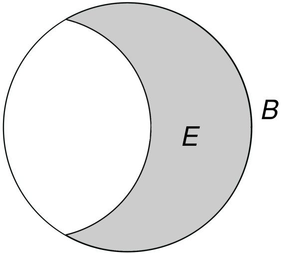
Half-balls are extremal sets for $K_{mv}(B)$.



Further Poincaré trace inequalities in *BV*

Theorem ([Cianchi - V.F. - Nitsch - Trombetti, preprint])

Let $n \geq 2$. Then there exists a half-moon shaped set E which is extremal for $H_{med}(B)$.



An approach to the proof of the first result

Theorem

We have:

$$C_{med}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}. \tag{11}$$

Moreover, equality holds in (11) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*

An approach to the proof of the first result

Suppose Ω is convex. We have to estimate from below the quantity

$$C_{\text{med}}(\Omega) = \sup_{E\subset\Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E\cap\partial\Omega)\,,\mathcal{H}^{n-1}(\partial\Omega\setminus\partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E\cap\Omega)}.$$

We denote by H_ν the half-space, with boundary having normal vector ν, such that

$$\mathcal{H}^{n-1}(\partial^M(H_\nu\cap\Omega)\cap\partial\Omega) = \frac{Per(\Omega)}{2},$$

and we put

$$h(\nu) = \mathcal{H}^{n-1}(\partial^M(H_\nu\cap\Omega)\cap\Omega).$$

We can use $E = H_\nu\cap\Omega$ in the ratio above to get

$$C_{\text{med}}(\Omega) \geq \frac{Per(\Omega)}{2} \frac{1}{\displaystyle\min_{\substack{H \text{ half-space}\\ \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\partial\Omega)=\frac{Per(\Omega)}{2}}} \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\Omega)}$$

An approach to the proof of the first result

On the other hand, using Cauchy formula, we have

$$\min_{\substack{H \text{ half-space} \\ \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\partial\Omega)=\frac{Per(\Omega)}{2}}} \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\Omega) \leq \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} h(\nu)\,d\nu \leq$$

$$\leq \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu\Omega)\,d\nu = \frac{1}{n\omega_n}\omega_{n-1}Per(\Omega),$$

then

$$C_{\text{med}}(\Omega) \geq \frac{n\omega_n}{2\omega_{n-1}}$$

and the proof of the inequality is complete.

An approach to the proof of the first result

If equality holds in the previous inequality, that is,

$$C_{\text{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)} = \frac{n\omega_n}{2\omega_{n-1}},$$

all the inequalities used above hold as equalities and we have

$$\mathcal{H}^{n-1}(\partial^M(H_\nu \cap \Omega) \cap \Omega) = \mathcal{H}^{n-1}(\Pi_\nu(\Omega)) = Per(\Omega)\frac{\omega_{n-1}}{n\omega_n}, \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

It follows that Ω is, in fact, strictly convex. Indeed, assume, by contradiction, that there exists a straight line intersecting $\partial\Omega$ in a whole segment Σ.

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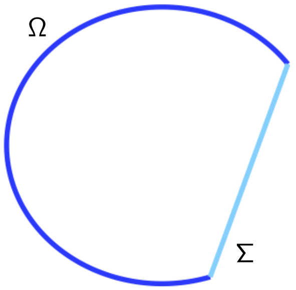

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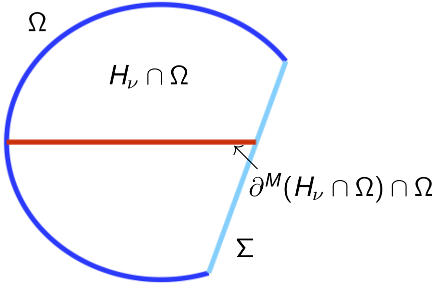
It follows that Ω is, in fact, strictly convex. Indeed, assume, by contradiction, that there exists a straight line intersecting $\partial\Omega$ in a whole segment Σ. It results

$$\mathcal{H}^{n-1}(\partial^M(H_\nu \cap \Omega) \cap \Omega) < \mathcal{H}^{n-1}(\Pi_\nu(\Omega)).$$



An approach to the proof of the first result

By the strict convexity of Ω, we have:

$$\mathcal{H}^{n-1}(I_\nu(\Omega)) = \mathcal{H}^{n-1}(\partial\Omega \cap H_\nu) = Per(\Omega)/2, \qquad \forall \nu \in \mathbb{S}^{n-1},$$

where $I_\nu(\Omega)$ denotes the *illuminated portion* of Ω. In particular,

$$\mathcal{H}^{n-1}(I_\nu(\Omega)) = \mathcal{H}^{n-1}(I_{-\nu}(\Omega)), \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

The above property implies that Ω is centrally symmetric.

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The above property implies that Ω is centrally symmetric.

Finally, on calling B the ball with the same perimeter as Ω, we infer that

$$\mathcal{H}^{n-1}(\Pi_\nu(\Omega)) = \mathcal{H}^{n-1}(\Pi_\nu(B)), \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

Hence, we conclude that Ω is a ball.

(The last two assertions come from well known results about convex bodies [GROEMER (1996)])

The general case

If we do not suppose that Ω is convex the Cauchy surface area formula cannot be used and a weaker version of it is needed.

The general case

If we do not suppose that Ω is convex the Cauchy surface area formula cannot be used and a weaker version of it is needed.

Theorem ([Federer (1969)], [Cianchi - V.F. - Nitsch - Trombetti, Crelle (to appear)])

Let G be a set of finite perimeter and finite Lebesgue measure in $\mathbb{R}^n$. Then

$$Per(G) = \frac{1}{2\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \left(\int_{\nu^\perp} \mathcal{H}^0((\partial^M G)_z^\nu)\, d\mathcal{H}^{n-1}(z) \right) d\mathcal{H}^{n-1}(\nu), \qquad (12)$$

where we use the notation $E_z^\nu = \{r \in \mathbb{R} : z + r\nu \in E\}$. In particular,

$$Per(G) \geq \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu(G)^+)\, d\nu\,, \qquad (13)$$

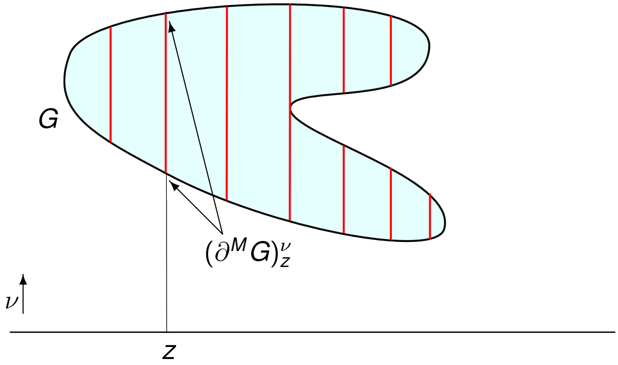
where $\Pi_\nu(E)^+ = \{z \in \nu^\perp : \mathcal{L}^1(E_z^\nu) > 0\}$. Moreover, the following facts are equivalent:

(i) The equality holds in (13);

(ii) G is equivalent to a convex set, up to sets of Lebesgue measure zero;

(iii) The set G^1 of points of density 1 with respect to G is convex.

The general case



$$\int_{\nu^\perp} \mathcal{H}^0((\partial^M G)^\nu_z)\, d\mathcal{H}^{n-1}(z) \geq 2\mathcal{H}^{n-1}(\Pi_\nu(G)^+).$$

An approach to the proof of the second result

Theorem

If $n \geq 3$*, then*

$$C_{mv}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}, \tag{14}$$

and the equality holds in (14) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*
If $n = 2$*, then*

$$C_{mv}(\Omega) \geq 2, \tag{15}$$

and the equality holds in (15) *if* Ω *is a disc. However there exist open sets* Ω*, that are not equivalent to a disc, for which equality yet holds in* (15)*.*

An approach to the proof of the second result

When $n \geq 3$, we have observed that

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and the inequality is proved. The assertion concerning the case of equality follows as well.

An approach to the proof of the second result

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$$C_{\mathrm{mv}}(\Omega) \geq C_{\mathrm{med}}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \tag{16}$$

and the inequality is proved. The assertion concerning the case of equality follows as well.

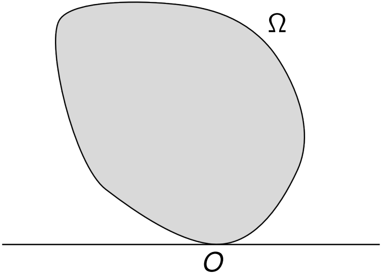
When $n = 2$, inequality (16) still holds true, but the right-hand side does not coincide with the constant C_{mv} computed on a ball. Indeed,

$$\sqrt{\pi}\,\frac{n}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\Bigg|_{n=2} = \frac{\pi}{2} < 2.$$

An approach to the proof of the second result

In order to prove that, when $n = 2$, $C_{\mathrm{mv}}(\Omega) \geq 2$ we consider a reference frame such that the origin O belongs to the boundary of Ω and

$$\Omega \subset \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$



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Given $\varepsilon > 0$, consider the open set

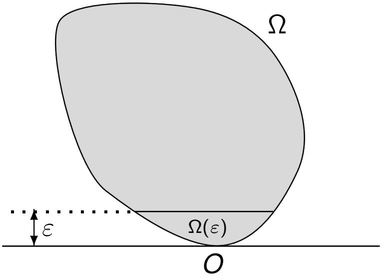
$$\Omega(\varepsilon) = \{(x, y) \in \Omega : y < \varepsilon\}.$$

We have:

$$\mathcal{H}^1(\partial^M\Omega(\varepsilon) \cap \Omega) \leq \mathcal{H}^1(\partial^M\Omega(\varepsilon) \cap \partial\Omega)$$

and

$$\lim_{\varepsilon \to 0^+} \mathcal{H}^1(\partial\Omega \setminus \partial^M\Omega(\varepsilon)) = Per(\Omega).$$



An approach to the proof of the second result

It follows:

$$\begin{aligned} C_{\mathrm{mv}}(\Omega) &\geq \lim_{\varepsilon\to 0^+} \frac{2}{\mathcal{H}^1(\partial\Omega)} \frac{\mathcal{H}^1(\partial\Omega(\varepsilon)\cap\partial\Omega)\,\mathcal{H}^1(\partial\Omega\setminus\partial\Omega(\varepsilon))}{\mathcal{H}^1(\partial\Omega(\varepsilon)\cap\Omega)} \\ &\geq \lim_{\varepsilon\to 0^+} \frac{2\,\mathcal{H}^1(\partial\Omega\setminus\partial\Omega(\varepsilon))}{\mathcal{H}^1(\partial\Omega)} = 2\,. \end{aligned}$$