From Ginzburg-Landau Equations to n-harmonic maps

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- $\Omega \subset \mathbb{R}^n$ a bounded smooth domain
  $g : \partial\Omega \to \mathbb{S}^{n-1}$ a smooth prescribed map
  $d = deg(g, \partial\Omega, \mathbb{S}^{n-1})$ degree of g

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- **n-dimensional Ginzburg-Landau energy functional**

$$E_\varepsilon(u, \Omega) = \int_\Omega \left(\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n}(1 - |u|^2)^2 \right) dx \tag{1}$$

where
$\varepsilon > 0$ a small parameter
$u \in W_g^{1,n}(\Omega, \mathbb{R}^n) = \{w \in W^{1,n}(\Omega, \mathbb{R}^n) : w|_{\partial\Omega} = g\}.$

- **Euler-Lagrange equation**

$$\begin{cases} -div\left(|\nabla u_\varepsilon|^{n-2}\,\nabla u_\varepsilon\right) & = & \frac{1}{\varepsilon^n}\left(1-|u_\varepsilon|^2\right)u_\varepsilon & \text{in } \Omega \\ u_\varepsilon & = & g & \text{on } \partial\Omega\ . \end{cases}$$

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- **Motivation.**
  $n = 2$, **Supraconductivity**, etc...
- **Reference** :
  Brezis, Bethuel, Hélein, Rivière, F. Lin, Struwe, Serfaty, Sandier ... etc

Known results in 2-dimension

Euler-Lagrange equation

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)\,u_\varepsilon & \text{in } \Omega \\ u_\varepsilon = g & \text{on } \partial\Omega\ . \end{cases}$$

harmonic map :
$u : U \subset \mathbb{R}^2 \to \mathbb{S}^m$ is an *harmonic map* if

$$-\triangle u = |\nabla u|^2 u \ \text{ dans } \mathcal{D}'(U)$$

or equivalently

$$div(u \wedge \nabla u) = 0 \ \text{ dans } \mathcal{D}'(U)$$

Theorem (Bethuel-Brezis-Hélein 1994)

Ω star-shaped, $d \neq 0$, then $\exists \varepsilon_k \to 0$, exactly $|d|$ distinct points $a_1, a_2, \cdots, a_{|d|}$, and a harmonic map $u_* \in \mathbf{C}^\infty(\Omega \setminus \{a_1, a_2, \cdots, a_{|d|}\})$ with boundary value g such that

$$u_{\varepsilon_n} \to u_* \quad \text{in} \quad \mathbf{C}^k_{loc}(\Omega \setminus \cup_i \{a_i\}) \cap \mathbf{C}^{1,\alpha}_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}) \quad \forall \alpha < 1.$$

In addition, each singularity has degree $sign(d)$.

Renormalized energy

Given $b = (b_1, b_2, \cdots, b_{|d|})$ of distinct points in Ω, its **renormalized energy** is defined as

$$W(b,d,g) := -\pi \sum_{i\neq j} \ln|b_i - b_j| + \frac{1}{2}\int_{\partial\Omega} \Phi(g \times g_\tau) - \pi \sum_{i=1}^{|d|} R(b_i)$$

Here

- $$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{|d|} \delta_{b_i} & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = g \times g_\tau & \text{on } \partial\Omega \end{cases}$$
  with ν unit normal, τ unit tangent vector and
- $R(x) = \Phi(x) - \sum_{i=1}^{|d|} \ln|x - b_i|$.

Theorem (Bethuel-Brezis-Hélein 1994)

- Configuration $\cup_i \{a_i\}$ minimizes $b \to W(b, d, g)$.

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- **Vanishing gradient property**
  Near each singularity a_j,

$$u_*(z) = \frac{z - a_j}{|z - a_j|} \mathrm{e}^{iH_j(z)},$$

where H_j is a real harmonic function such that

$$\nabla H_j(a_j) = 0.$$

Known results for minimizing sequences when $n \geq 3$

Assume u_ε minimizer and $d \geq 0$

- $E_\varepsilon(u_\varepsilon, \Omega) = d\kappa_n |\log \varepsilon| + O(1)$ with
  $\kappa_n = \frac{1}{n}(n-1)^{\frac{n}{2}}|\mathbb{S}^{n-1}|$
- $d = deg(g) = 0$
  $u_\varepsilon \to u_*$ in $W^{1,n}$ (Strzelecki, 96)
- $d = deg(g) \neq 0$
  $u_\varepsilon \to u_*$ in $W^{1,n}_{loc}(\bar{\Omega} \setminus \cup_{1 \leq i \leq d}\{a_i\})$(Hong, Han-Li, 96)

Here u_* n-harmonic map into sphere.

n-harmonic map :

Let U be a domain in $\mathbb{R}^n$. $u : U \to \mathbb{S}^{n-1}$ is an *n-harmonic map* if

$$-div(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u \;\text{ dans } \mathcal{D}'(U)$$

or equivalently

$$div(|\nabla u|^{n-2} u \wedge \nabla u) = 0 \;\text{ dans } \mathcal{D}'(U)$$

Renormalized energy formula

Renormalized energy for n-harmonic maps
(Hardt-Lin-Wang)
Given d distinct points $a = \{a_1, a_2, \cdots, a_d\}$ and $\delta > 0$, let $\Omega_{a,\delta} = \Omega \setminus \cup_{i=1}^{d} B_\delta(a_i)$.

$$\mathcal{W}_{a,\delta} = \left\{ w \in W^{1,n}(\Omega_{a,\delta}; \mathbb{S}^{n-1}) : w|_{\partial\Omega} = g, \deg(w, \partial B_\delta(a_i)) = 1 \forall i \right\}.$$

Renormalized energy of $a = \{a_1, a_2, \cdots, a_d\}$:

$$W_g(a) := \lim_{\delta \to 0} \left(\min_{w \in \mathcal{W}_{a,\delta}} E_n(w, \Omega_{a,\delta}) - d\kappa_n |\ln \delta| \right),$$

where

$$E_n(w, \Omega_{a,\delta}) = \int_{\Omega_{a,\delta}} \frac{|\nabla w|^n}{n} dx.$$

Theorem 1(G-Sandier-Zhang)

Let $a = \{a_i\}_{i=1}^d$ be the limit singular points of minimizing sequence, then

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) = d\kappa_n |\ln \varepsilon| + W_g(a) + d\gamma + o(1) \,\text{as}\, \varepsilon \to 0,$$

where γ is some constant independent of g. Moreover, the configuration $\{a_i\}_{i=1}^d$ minimizes W_g.

stationary n-harmonic map :

Set $\Omega_0 = \Omega \setminus \{a_1, a_2, \cdots, a_d\}$ and let $u : \Omega_0 \to \mathbb{S}^{n-1}$ be an n-harmonic map. We say u is a *stationary n-harmonic map*

- if its **stress-energy tensor**

$$T_{i,j} := |\nabla u|^{n-2} \langle \partial_i u, \partial_j u \rangle - \frac{1}{n} |\nabla u|^n \delta_{i,j}$$

  satisfies

$$\sum_i \partial_i T_{i,j} = 0 \text{ in } \Omega_0,$$

- if $\forall 1 \leq k \leq d$ and $\rho > 0$ such that for $\partial B_\rho(a_k) \subset \Omega_0$

$$\int_{\partial B_\rho(a_k)} \sum_i T_{i,j} \nu_i = 0,$$

  where $\nu = (\nu_1, \cdots, \nu_n)$ is normal to $\partial B_\rho(a_k)$.

Regularity of n-harmonic map :
It is conformally invariant problem in n dimension.
Reference :
Schoen, Uhlenbec, Evans, Bethuel, Jost, Morrey, Hélein, Rivière, Hildbrandt, Kaul, P. Yang, F. Lin, Hardt, C. Wang, L. Mou, B. Chen, A. Naber, M. Struwe, G. Mingione, Duzaar ... etc

Proposition (G-Sandier-Zhang)

$u : \Omega_0 \subset \mathbb{R}^n \to \mathbb{S}^{n-1}$ is a stationary $n-$harmonic map and $\deg(u, a_k) = 1$. Assume around a_k, one has expansion $u(x) = \mathrm{e}^{B_k(x)} \dfrac{x - a_k}{|x - a_k|}$,
where $B_k(x) \in so(n)$ is an antisymmetric matrix satisfying $B_k(a_k) = 0$ such that $x \to B_k(x)$ is C^1. Then $\sum_{i=1}^n \partial_i B_k(a_k) e_i = 0$, where $(e_1, \cdots, e_n)$ is the canonical basis in $\mathbb{R}^n$. Moreover, we have

$$u(x) = \frac{x - a_k}{|x - a_k|} + \frac{Q_k(x - a_k)}{|x - a_k|} + O(|x - a_k|^2)$$

where $Q_k(x)$ is a harmonic polynomial of degree 2. In particular, when $n = 2$, we have $B_k(x) = O(|x - a_k|^2)$.

Theorem2 (G-Sandier-Zhang)

Assume u_ε is a critical point of $\mathbf{E}_\varepsilon$ such that for some $M > 0$ independent of ε one has $\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n|\ln\varepsilon| + M$. Then $\exists$ a subsequence $\{\varepsilon\}$ tending to zero, a collection of d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, a finite subset U of $\bar{\Omega}$, and a stationary n-harmonic map
$u_* : \Omega_0 := \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$, such that
$u_\varepsilon \to u_* \quad$ strongly in $\quad \mathbf{W}^{1,n}_{loc}(\Omega_0 \setminus U, \mathbb{R}^n)$
and for any $1 \leq p < n$
$u_\varepsilon \rightharpoonup u_* \quad$ weakly in $\quad \mathbf{W}^{1,p}(\Omega, \mathbb{R}^n)$.
Furthermore, $\deg(u_*, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$, for $1 \leq j \leq d$. and any small enough $\sigma > 0$.

Remark :Jerrard gets local weak convergence in Ω_0 with upper energy bound.

Existence of non-minimizing critical points

Theorem3 (G-Sandier-Zhang)

There exists a domain $\Omega \subset \mathbb{R}^3$, a boundary map $g : \partial\Omega \to \mathbb{S}^2$, and for every small enough $\varepsilon > 0$ a non minimizing critical point u_ε of the functional $\mathbf{E}_\varepsilon(u, \Omega)$ such that the energy bound $\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_3|\ln \varepsilon| + M$.

Some basic facts

Assumptions : u_ε is a critical point of $\mathbf{E}_\varepsilon$ with upper Energy bound $\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n|\ln\varepsilon| + M$

Some facts :

- **Fact 1** :
$$|u_\varepsilon| \leq 1$$
- **Fact 2** :
Divergence Free Stress-Energy Tensor, that is,
$$\sum_{i=1}^{n} \partial_i T_{i,j}(u_\varepsilon) = 0$$
where
$$T_{i,j}(u_\varepsilon) = |\nabla u|^{n-2} \langle \partial_i u, \partial_j u \rangle - \left(\frac{1}{n}|\nabla u|^n + \frac{1}{4\varepsilon^n}(1-|u|^2)^2 \right) \delta_{i,j}$$

- **Fact 3** : **Pohozaev inequality**
  Let $D \subset \mathbb{R}^n$ be a bounded strictly star-shaped domain w.r.t. $x_0 \in D$, and $\alpha > 0$ such that $(x - x_0) \cdot \nu \geq \alpha d$ for all $x \in \partial\Omega$ and d diameter of D. Then there exists a constant C depending only on n, α such that

$$\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 + \alpha d \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2$$
$$\leq C(n,\alpha) d \int_{\partial D} \frac{1}{n} |\nabla u_\varepsilon|^{n-2} |\nabla_\tau u_\varepsilon|^2 + \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2,$$

  where $|\nabla_\tau u_\varepsilon|^2 = |\nabla u_\varepsilon|^2 - |\partial_\nu u_\varepsilon|^2$ and $C(n,\alpha) = 2 + \frac{n^2(n-1)}{2(n-2)\alpha}$.

Step 1 : Weak convergence

$\exists$ subsequence $\{u_\varepsilon\}_\varepsilon$, d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, and an $\mathbb{S}^{n-1}$-valued map $u_* : \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$ such that, as $\varepsilon \to 0$,

$$u_\varepsilon \rightharpoonup u_* \quad \text{weakly in} \quad \mathbf{W}^{1,n}(\Omega \setminus \{a_1, a_2, \cdots, a_d\}, \mathbb{R}^n)$$

and for any $1 \leq p < n$

$$u_\varepsilon \rightharpoonup u_* \quad \text{weakly in} \quad \mathbf{W}^{1,p}(\Omega, \mathbb{R}^n).$$

Moreover, $\deg(u_*, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$, for $1 \leq j \leq d$ and for small $\sigma > 0$.

Main ingredient : Upper energy bound

Step 2 : Improved convergence, u_* is n-harmonic

- $|u_\varepsilon| \longrightarrow 1$ uniformly in compact K, as $\varepsilon \longrightarrow 0$,
- $\frac{1}{\varepsilon^n}\int_K \left(1-|u_\varepsilon|^2\right)^2 + \int_K |\nabla u_\varepsilon|^{n-2}|\nabla|u_\varepsilon||^2 \longrightarrow 0$, as $\varepsilon \longrightarrow 0$,
- u_* is n-harmonic

Main ingredient : Pohozaev inequality

Step 3 : η-regularity

Define

$$e(x, r, u_\varepsilon) := \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n$$

Main interior result : Assume $\{u_\varepsilon\}_\varepsilon$ satisfy the hypothesis of Theorem 2, and that $u_\varepsilon \to u_*$ in $\Omega \setminus \{a_i\}_{1\leq i\leq d}$. Then $\exists \eta > 0$, $\alpha > 0$ such that for any compact subset K of $\Omega \setminus \{a_i\}_{1\leq i\leq d}$ there exist $\varepsilon_0 > 0, r_0 > 0$ depending on K such that if $x \in K$, $\varepsilon \in (0, \varepsilon_0)$, $r \in (0, r_0)$ and $e(x, r, u_\varepsilon) \leq \eta$ then we have

$$\|u_\varepsilon\|_{C^\alpha(B_{r/2}(x))} \leq C.$$

where C is some positive constant independent of ε.

Main boundary result : Under the same assumptions, suppose the domain Ω is C^2 and that the boundary data $g : \partial\Omega \to \mathbb{S}^{n-1}$ is C^1. Then there exist $C, \eta, \varepsilon_0, r_0 > 0$ and $\bar{\theta} \in (0,1)$ such that if $r < r_0$, if $\varepsilon < \varepsilon_0$ and if $x \in \partial\Omega$ then

$$e(x, r, u_\varepsilon) \leq \eta \quad \Longrightarrow \quad \|u_\varepsilon\|_{C^\alpha(B_r(x)\cap\Omega)} \leq C,$$

where C is independent of ε.

- **Main difficulties** : Critical problem.

- **Main difficulties** : Critical problem.
- **Strategy** : Duality between BMO and Hardy space.

Step 4 : Strong convergence

Define

$$S = \bigcap_{r>0} \left\{ x \in \bar{\Omega} \setminus \{a_1, a_2, \cdots, a_d\} \,\middle|\, \liminf_{\varepsilon \longrightarrow 0} \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n > \frac{\eta}{2} \right\}$$

η-regularity $\implies$ strong convergence outside S

Step 5 : stationarity of u_*

Divergence free Stress-Energy tensor for u_ε + Strong convergence out of S

Remark : The Hopf vibration $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a stable 3-harmonic map with finite energy in its homotopy class. As the problem is conformally invariant, this gives a 3-harmonic map with finite energy on $\mathbb{R}^3$. This is the reason for which S could contain more than d points. It is different than 2 dimension. This is related to the topological fact that the fundamental group $\pi_3(\mathbb{S}^2)$ is not trivial.

Construction of domain

$n = 3$ and $x = (x', x_3)$ with $x' \in \mathbb{R}^2$. Consider a domain $\Omega = C \cup D_+ \cup D_-$ consisting of a long cylinder $C = \{x \in \mathbb{R}^3 | |x'| \le 1, |x_3| \le L\}$ of radius 1 and length $2L$ plus two spherical caps at each end $D_+ = B(P, 1) \cap \{x_3 \ge L\}$ and $D_- = B(Q, 1) \cap \{x_3 \le -L\}$, where $P = (0, 0, L)$ and $Q = (0, 0, -L)$.

Construction of boundary map

$g : \partial\Omega \to \mathbb{S}^2$ of degree one defined on the spherical caps by

$$g(x) = \frac{x - P}{|x - P|} \text{ on } \partial D_+ \cap \partial\Omega, \qquad g(x) = \frac{x - Q}{|x - Q|} \text{ on } \partial D_- \cap \partial\Omega.$$

On the cylindrical part, choosing an $h > 0$,

$$g(x) = \sqrt{\frac{1}{1+h^2}}(x', -h) \text{ if } 1 \leq x_3 \leq L - 1$$

$$g(x) = \sqrt{\frac{1}{1+h^2}}(x', h) \text{ if } -L + 1 \leq x_3 \leq -1$$

and the boundary map interpolates between the these on the remaining part.

Assume $g \circ S = S \circ g$, where $S(x', x_3) = (x', -x_3)$ and for any $\theta \in \mathbb{R}$, identifying $\mathbb{R}^2$ with $\mathbb{C}$,

$$g \circ R_\theta = R_\theta \circ g, \quad \text{where} \quad R_\theta(z, x_3) = (e^{i\theta} z, x_3).$$

Define sobolev spaces of equivariant maps by

$$\bar{W}(\Omega,\mathbb{R}^3)=\{u\in W_g^{1,3}(\Omega,\mathbb{R}^3)\mid u\circ S=S\circ u, u\circ R_\theta=R_\theta\circ u,\forall\theta\},$$

If L is sufficiently large,

$$\min_{u\in W_g^{1,3}(\Omega,\mathbb{R}^3)}\mathbf{E}_\varepsilon(u)<\min_{u\in\bar{W}(\Omega,\mathbb{R}^3)}\mathbf{E}_\varepsilon(u)\le\kappa_3|\log\varepsilon|+C$$

Thank you for your attention !