The global stable homotopy category

Stefan Schwede

Mathematisches Institut, Universität Bonn

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Introduction

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simultaneous and compatible actions of all compact Lie groups

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A justification for this definition is:

Theorem (Equivariant Whitehead theorem)

Every G-weak equivalence between G-CW-complexes is a G-equivariant homotopy equivalence.

The linear isometries monoid

Definition

The linear isometries monoid is

$$\mathcal{L} \;=\; \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \;=\;$$
$$\{\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^\infty, \mathbb{R}^\infty) \mid \langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^\infty\}$$

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We will drastically change the perspective on $\mathcal{L}$-spaces by introducing a much finer notion of equivalence.

Universal subgroups

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Not every compact Lie subgroup of $\mathcal{L}$ is universal.

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Proposition

(i) *Every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$.*

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More generally: Grassmannians

$Gr_n(\mathbb{R}^\infty),\ Gr_n^+(\mathbb{R}^\infty),\ Gr_n^{\mathbb{C}}(\mathbb{C} \otimes \mathbb{R}^\infty),\ Gr_n^{\mathbb{H}}(\mathbb{H} \otimes \mathbb{R}^\infty)$

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Global refinements of BO

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$$(BO)^K \simeq \prod\nolimits'_{[\lambda]:K\text{-irrep}} B(O_\lambda)$$

where

$$O_\lambda = \begin{cases} O & \text{if } \lambda \text{ is of real type,} \\ U & \text{if } \lambda \text{ is of complex type,} \\ Sp & \text{if } \lambda \text{ is of quaternionic type.} \end{cases}$$

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We define **bO** as the space of all $\mathbb{R}$-subspaces L of $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ with the following property:
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The $\mathcal{L}$-spaces BO and **bO** are not globally equivalent, and neither one is left nor right induced.

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II. Stable global homotopy theory

Definition

An orthogonal spectrum X consists of

- based $O(V)$-spaces $X(V)$, for every inner product space V
- $O(V) \times O(W)$-equivariant structure maps

$$\sigma_{V,W} \ : \ X(V) \wedge S^W \ \longrightarrow \ X(V \oplus W)$$

subject to associativity and identity conditions.

Here: $S^W = W \cup \{\infty\}$ one-point compactification

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- forget the $O(n)$-actions

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- $\pi_0^G(X)$ is the stable analog of $\pi_0(Y^G)$ for an $\mathcal{L}$-space Y

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A model category structure is available.

Examples

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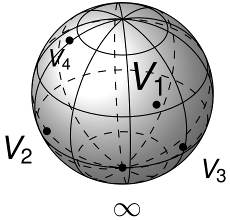

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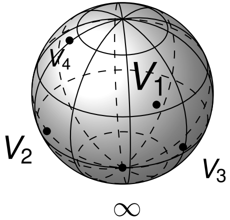

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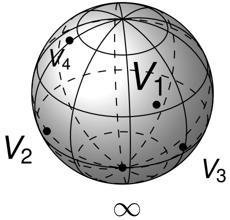

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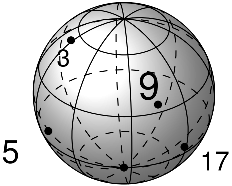
$$(H\mathbb{Z})(V) \;=\; Sp^\infty(S^V)$$

infinite symmetric product



Some global morphisms

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The morphism $\mathbb{S}_{\mathbb{Q}} \longrightarrow H\mathbb{Q}$ is a non-equivariant equivalence, but not a global equivalence.

More examples

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Suspension spectra of global spaces, part of an adjoint pair

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replacing $Gr_n(V \oplus \mathbb{R}^\infty)$ by $Gr_n(V \oplus V)$ yields an orthogonal Thom spectrum **MO** with different global homotopy type.

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- objects in $\mathcal{GH}$ represent 'genuine' cohomology theories on stacks (Gepner-Henriques, Gepner-Nikolaus)
- a t-structure is given by 'globally connective' respectively 'globally coconnective' spectra; the heart consists of all X such that

  $$\pi_n^G(X) \;=\; 0$$

  for all $n \neq 0$ and all G.

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- double coset formula

Recollement

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