Entropy Theory in Dynamics

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1 Overview of Dynamical Entropy Theory

Entropy has long been used in dynamics as a tool to classify and probe dynamical systems. In the beginning, its use was restricted to single automorphisms of a measure space. To be precise, if (X, μ) denotes a probability space, $T : X \to X$ an automorphism and $\mathcal{P}$ a partition of X, then we define

$$H_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log(\mu(P)),$$

$$h_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n} T^{-i} \mathcal{P} \right),$$

$$h_\mu(T) := \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

The first quantity above is the **Shannon entropy** of $\mathcal{P}$; it represents the amount of information gained by learning which part of $\mathcal{P}$ contains a random element $x \in X$. For example, suppose that while studying a system a measurement is made every minute. In this case, the state of the system is represented by $x \in X$. The time evolution is represented by T and the measurement is represented by the partition $\mathcal{P}$: so knowing the which part of $\mathcal{P}$ contains x is the same as making a measurement. The Shannon entropy $H_\mu(\mathcal{P})$ quantifies the amount of information gained on average from making a single measurement. The entropy rate $h_\mu(T, \mathcal{P})$ quantifies the average amount of information gained per unit time and $h_\mu(T)$ represented the total amount of information per unit time in the system.

Kolmogov's deep insight of 1958 [8] was that the entropy $h_\mu(T)$ is an isomorphism invariant. He used it to classify Bernoulli shifts which are an especially simple type of system in which, at each moment of time, the experiment consists of rolling a "die" and recording the value. The die used here could have many sides and it is not necessary that all sides have equal weight. Formally, this means that $X = K^{\mathbb{Z}}$ for some Borel space K, $\mu = \kappa^{\mathbb{Z}}$ for some Borel measure κ on K and the time evolution T is defined by $T(x)_n = x_{n-1}$. The entropy of this system equals the Shannon entropy of the base.

In 1970, Ornstein proved the converse: if two Bernoulli shifts have the same entropy then they are isomorphic, completing the classification of Bernoulli shifts [9]. At the same time, he introduced a variety of powerful tools for determining whether a given automorphism is isomorphic to a Bernoulli shift and for streamlining previous important results such as Sinai's Factor Theorem and Krieger's Generator Theorem.

All of the above results were extended to the class of amenable groups in the 1970's and 1980's. However, they could not be extended to non-amenable groups. In fact, it is an immediate consequence of the definitions that entropy cannot increase under a factor map and that the entropy of a Bernoulli is the Shannon entropy of the base space. These statements were directly contradicted in the case of free groups by an explicit example due to Ornstein-Weiss [10]. This convinced many researchers that entropy theory could not be extended to non-amenable group actions.

In 2010, Bowen introduced sofic entropy theory [1] which applies to the large class of sofic groups that includes many non-amenable groups. In fact, it is unknown whether or not all groups are sofic however, all amenable groups and linear groups are sofic. Sofic entropy is not monotone under factors in general, but it does classify Bernoulli shifts. Since this intial discovery there have been many developments including the introduction of a topological counterpart by Kerr-Li [7], sofic pressure by Chung [5] and computations of sofic entropy for algebraic actions by many authors [2, 7, 4] including a complete result for principal algebraic actions by Hayes [6].

A group Γ is **sofic** if it admits a sequence $\sigma_n : \Gamma \to \mathrm{Sym}(V_n)$ of maps that asymptotically behave like homomorphisms giving free actions on the finite sets V_n. In the simplest case in which the action can be described by: K is a finite set, Γ acts on K^Γ by shifts: $(gx)(f) = x(g^{-1}f)$ and μ is a shift-invariant probability measure on K^Γ, the sofic entropy is the exponential rate of growth of the number of **microstates** for the action. To be precise, a microstate is an element of K^{V_n} which exhibits the same local statistics up to a small error as the measure μ. Apriori, the sofic entropy can depend on the choice of sequence (which is called a **sofic approximation**). Moreover, it can even be $-\infty$ if there are no microstates. There are explicit examples of this phenomenon.

In a series of papers, B. Seward has been developing Rokhlin entropy theory [15, 12, 13, 14]. This entropy admits a deceptively simple definition in the case of essentially free ergodic actions: it is the infimum of the Shannon entropies $H_\mu(\mathcal{P})$ over all **generating** partitions $\mathcal{P}$; where partition is generating if the smallest Γ-invariant sigma-algebra containing it is the entire Borel sigma-algebra (up to measure zero). This means that the measuring device, represented by $\mathcal{P}$, is complete in the sense that if we observe the system through $\mathcal{P}$ for all "time" then we will learn everything there is to know about the state of the system.

It is an easy consequence of the definitions that Rokhlin entropy is an upper bound for sofic entropy. However, there are no other known lower bounds for Rokhlin entropy. This leads to the open question: if a system has nonnegative sofic entropy, does its sofic entropy equal the Rokhlin entropy? If so, this would imply that sofic entropy does not depend on the choice of sofic approximation (conditioned on being nonnegative). This problem is open even for the much-studied case of Gibbs measures on regular trees.

2 Recent Developments and Open Problems

B. Seward finished the remaining case in extending Ornstein's Isomorphism Theorem. So we now know that every countably infinite group satisfies the following fundamental property: if two Bernoulli shifts over Γ have the same base space entropy then they are isomorphic. Previously this was known in all cases except for the case of 2-atom base spaces [3].

In work in progress, L. Bowen has proven that when Γ is non-amenable, then all Bernoulli shifts over Γ factor onto each other. This is a corollary to the following extension of the Gaboriau-Lyons Theorem: all Bernoulli shifts over Γ satisfy the conclusions of the measurable von Neumann-Day conjecture (that its orbits can be measurably partitioned into non-amenable trees). The details were presented for the first time at the workshop. These first two results solve long-standing fundamental problems. It now makes sense to pursue a more detailed picture. For example, is it possible to classify factors of Bernoulli shifts up to measure-conjugacy? This problem and its many sub-problems were discussed at the workshop. For example, partial progress towards classifying Ornstein-Weiss factors was discussed.

In a series of works, B. Seward has been developing Rokhlin entropy theory [15, 12, 13, 14] with stunning results such as: a generalization of Krieger's generator theorem to arbitrary countable groups and a generalization of Sinai's Factor Theorem to arbitrary groups. The latter is so new it is still not publicly available. B. Seward talked about this result at the workshop and its many implications.

Russ Lyons talked on factors of Bernoulli shifts. The main problem seems to be to develop tools for determining which seems systems are factors of Bernoulli shifts. Partial progress and many open problems were discussed. Lyons' second talk was on determinantal measures and their ergodic theory on sofic groups.

Ben Hayes spoke about a new version of relative entropy in which one considers the entropy of the target relative to the source (instead of the other way around which is classical). This plays a central role in a new result: that Pinsker factors of algebraic actions are algebraic and therefore many algebraic actions, such as principal algebraic actions satisfying some mild conditions, have completely positive entropy. This is a great achievement; only a year ago such a result seemed completely out of reach. Algebraic actions form a large and diverse class of actions that are easily defined for any countable group. They have been the subject of intense research for decades. However, it is only with the advent of the new entropy theory that researchers have been able to explore these actions in the case that the acting group is non-amenable.

Peter Burton talked about a new result connecting uniform mixing and completely positive entropy (CPE). The context is this: in [11] Rudolph and Weiss proved that CPE and uniform mixing are equivalent for actions of amenable groups. The proof introduced orbit-equivalence techiniques into entropy theory. The obvious extension of this result to non-amenable group fails already for Markov chains over the free group. Peter and Tim Austin developed a new notion of uniform mixing adapted to the sofic set-up called **uniform model mixing** and proven that this implies CPE. The converse remains an open problem.

Another new development is the study of the geometry of the space of microstates. Tim Austin used geometric ideas to introduce a variation on sofic entropy which is additive under direct products unlike sofic entropy. The reason this is an important result is that (a) additivity under direct products is fundamental, especially to the study of algebraic actions and (b) the proof pinpoints the mechanism underlying how additivity can fail. Indeed, failure of additivity is marked by a quasi-factor arising from the sofic approximation. This quasi-factor might be the key to understanding how sofic entropy can depend on the choice of sofic approximation even assuming non-negativity.

The geometry of the space of microstates also plays a central role in a new example presented by Lewis Bowen. The example is about the Weak Pinsker conjecture. In the early days of entropy theory, Pinsker conjectured that every ergodic automorphism could be decomposed as a product of a zero entropy system with a Bernoulli shift. This would be the simplest explanation for the phenomena of positive entropy. However, Ornstein proved by explicit counterexample that this false (thereby constructing the first CPE non-Bernoulli transformation). Pinsker's conjecture was subsequently weakened to: for every $\epsilon > 0$ every ergodic transformation can be written as a direct product of a Bernoulli shift with an transformation of entropy $< \epsilon$. This conjecture is still open. However, Bowen showed by explicit counterexample that its generalization to free group is false. This is because there are action of the free group with the property that the microstate space splits into exponentially many clusters, each of exponentially small size relative to the total. This is an application of more general work on Gibbs measures on trees; a field of study that actively studied by probabilists but has not been approached from an ergodic theory point of view in much detail.

Many of the open problems discussed in the workshop were of the form "prove that this action is or is not a factor of a Bernoulli shift". These kinds of problems are difficult precisely because of the lack of a well-developed counterpart to Ornstein Theory in the non-amenable setting. We discussed partial progress towards developing such a theory and answering some of the specific questions through ad hoc methods.

3 Conclusions

The meetings were overall very productive. On most days we had two 3-hour long talks. This meant that participants had the time to ask detailed questions and have frequent discussions. It was helpful that the participants have varied backgrounds so that problems were attacked from multiple viewpoints. In the end, I think we all learned a great deal from each other and deepened our awareness of the main challenges lying ahead.

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