The Rigidity of Plane Grids and Some New Extensions
Joe Aiken¹, Sandra Gregov²
Department of Mathematics and Statistics, York University, Ontario, M3K 1P3
Supervised by Walter Whiteley³

Prior Work with Square Grids

THE PROBLEM

Methods for determining the rigidity of square grids

Combinatorics: Necessary Condition: For a square m x n grid need (m + n - 1) cross braces. Not sufficient (B, C).

Rigidity Matrix: A framework of squares and cross braces is rigid if and only if its associated \(\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \) rigidity matrix has rank 2. With an m x n grid and (m+n-1) cross braces this means if and only if the associated rows are independent.

Zone Graphs: A braced square grid is rigid if and only if its associated abstract zone graph is connected. \(A'\) and \(B'\) are the abstract zone graphs of \(A\) and \(B\), respectively.

PROBLEM: How to determine the rigidity of a general planar framework formed by decomposing a disc into triangles and parallelograms, with cross braces?

OUR RESULT

We extend previous work on the rigidity of rectangular grids to a general planar framework we call a 'cellular leaf' and test its rigidity through the connectivity of its associated zone graph and zone star graph.

PRELIMINARIES

DEFINITIONS

A cellular leaf is a 'disc' decomposed into triangles and parallelograms, with the possibility of cross braces in the parallelograms, and with vertices \(V\), edges \(E\) and cells \(F\), plus an exterior cell \(F_0\).

A zone graph is an abstract graph representing the constraints of each zone in a cellular leaf.

A zone star graph is a graph where each zone is represented by both its length and direction such that there remains triangle consistency and the cross braces are the same vectors as in the Leaf.

A flexible motion is a continuous motion that preserves all designated lengths but not congruence.

A deformable motion is a continuous motion that preserves all designated directions, but not translation or similarity.

METHODS

A LEAF AND ITS ZONE GRAPH

LEMMA 1: Given a Cellular Leaf \(G(V,E,F)\), \(p\), then there exists an abstract zone graph and a consistent zone star graph.

PARTITION EDGES

LEMMA 2: Given a Cellular Leaf \(G(V,E,F)\), \(p\), there exists a spanning graph \(G=(V,E)\) and a dual graph \(G^*=(F^*,E)\), where \(F^* = F + F_0\).

INDUCTIVELY BUILDING THE LEAF

LEMMA 3: Given a consistent zone star graph and the abstract dual graph \(G^*=(F^*,E)\), there is a Leaf Structure preserving all directions and lengths of zones and diagonals.

PROOF: Build spanning tree from zone vectors. Select any of the interior cells, \(F\). In completing a cell, there are two possible cases.

Case 1 - Adding an edge to complete a triangle (A).

Case 2 - Adding an edge to complete a parallelogram (B). Inductively, we continue until all of the interior cell's \(F\) are completed, never selecting the exterior cell, \(F_0\).