Moment-free measures for the multivariate skew-t distribution

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Overview

1. Introduction: motivation and preliminary concepts
2. Location and kurtosis: the halfspace depth function and derived measures of location and kurtosis
3. Quantifying dispersion, structure of dependence and asymmetry: concordance measures, interquartile range and Yule-Bowley index
4. Final remarks: open issues and further research

Introduction

Motivation

The main features of a multivariate probability distribution are:

- location;
- dispersion and structure of dependence;
- asymmetry;
- kurtosis.

Measures are usually based on moments, e.g. Mardia (1970, 1974), Srivastava (1984) and Malkovitch et al. (1973).

Question

How can we summarise the main features of a multivariate distribution if the moments are not defined?

Heavy-tailed distributions

What is a heavy-tailed distribution? No shared answer (Gumbel, 1962; Bryson, 1974; Mantel, 1976).

- **Student's t distribution** (Student, 1908): $X \sim N(0,1) \perp \chi^2_\nu$

$$Y = \frac{X}{\sqrt{\frac{\chi^2_\nu}{\nu}}}, \nu > 0.$$

- **Slash distribution** (Rogers and Tukey, 1972): $X \sim N(0,1) \perp U \sim U(0,1)$

$$Y = \frac{X}{U^{1/\nu}}, \nu > 0.$$

Skew-t distribution and its canonical form

- **Skew-t distribution** (Azzalini and Capitanio, 2003):
  $X \sim \mathrm{SN}(0, \Omega, \alpha) \perp \chi^2_\nu$

$$Y = \xi + \frac{X}{\sqrt{\frac{\chi^2_\nu}{\nu}}} \sim \mathrm{ST}(\xi, \Omega, \alpha, \nu), \nu > 0. \quad (1)$$

- **Canonical skew-t** (Capitanio, 2012):
  $Y^* \sim \mathrm{ST}(0, I, (\alpha_*, 0, \ldots, 0)^\top, \nu)$; we write
  $Y^* \sim \mathrm{CST}(\alpha_*, \nu)$.

Remark: for any $Y \sim \mathrm{ST}(\xi, \Omega, \alpha, \nu)$ there is an affine transformation such that $AY + b = Y^*$.

Location and Kurtosis

The halfspace depth function (Tukey, 1975)

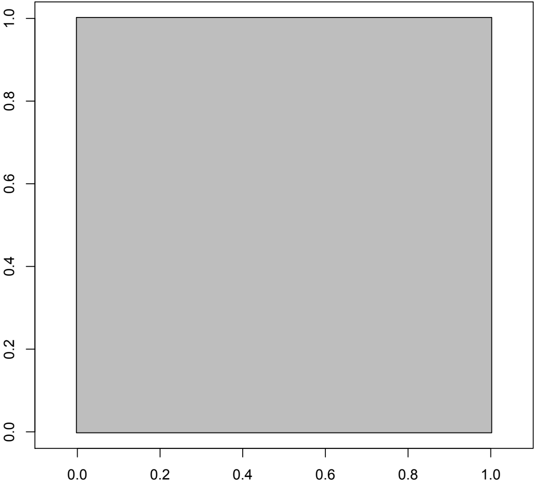
Let $X \sim \mathrm{ST}(\xi, \Omega, \alpha, \nu)$; the halfspace depth (HD) of $x \in \mathbb{R}^d$ with respect to the distribution of X is

$$\mathrm{HD}_X(x) = \min_{\|u\|=1} P_X(u^\top X < u^\top x) \tag{2}$$

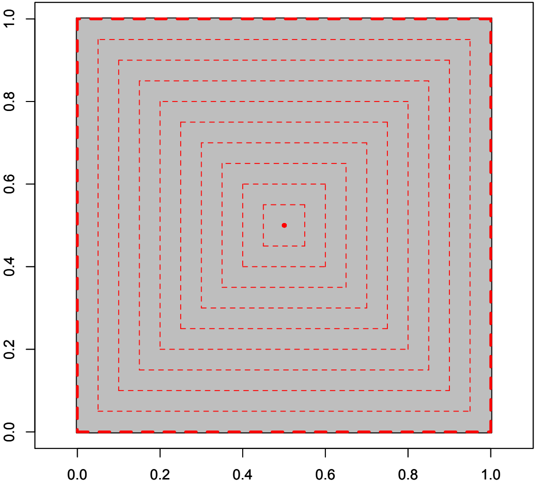
Some properties (Zuo and Serfling, 2000):

- affine invariance: $\mathrm{HD}_{AX+b}(Ax + b) = \mathrm{HD}_X(x)$ (use the canonical form!);
- convex contours (Small, 1987);
- vanishing at infinity: $\lim_{\|x\|\to\infty} \mathrm{HD}_X(x) = 0$.

Example: the uniform distribution



Example: the uniform distribution



The halfspace median and radial symmetry

The halfpsace median is defined as the global maximum of HD, i.e.

$$M_X = \arg\max_{x \in \mathbb{R}^d} \mathrm{HD}_X(x) \tag{3}$$

Main properties:

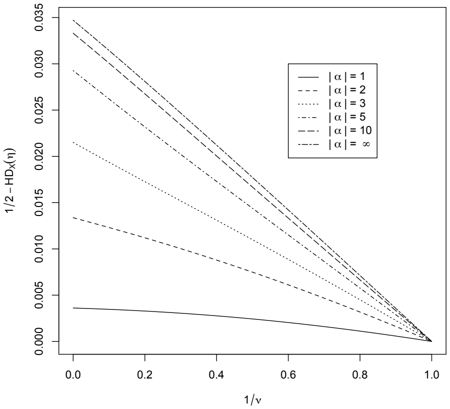
- affine equivariance: $M_{AX+b} = AM_X + b$;
- correspondence with the center of radial symmetry: $\mathrm{HD}_X(M_X) = 1/2$ if and only if

$$(X - M_X)/\|X - M_X\| \overset{d}{=} -(X - M_X)/\|X - M_X\|,$$

  i.e. X is *radially symmetric* about M_X (Dutta et al., 2011).

ST and its departure from radial symmetry (1)

Let $X \sim \mathrm{CST}(\alpha, \nu)$ and η be its componentwise median.



ST and its departure from radial symmetry (2)

Theorem (1)

Let $X \sim \mathrm{ST}(\xi, \Omega, \alpha, 1)$*, then* X *is radially symmetric about*

$$M_X = \xi + \omega\delta \tag{4}$$

where $\omega = \mathrm{diag}(\omega_{11}, \ldots, \omega_{dd})^{1/2}$, $\delta = \overline{\Omega}\alpha/(1 + \alpha^\top \overline{\Omega}\alpha)^{1/2}$ *and* $\overline{\Omega} = \omega^{-1}\Omega\omega^{-1}$.

ST and its departure from radial symmetry (3)

Theorem (2)

Let $X \sim \mathrm{ST}(\xi, \Omega, \alpha, \nu)$*, then* HD *is maximized along the direction* $\omega\delta$ *at*

$$M_X = \xi + \frac{m_*}{\delta_*}\omega\delta, \tag{5}$$

where $m_* = \arg\max_{x \in \mathbb{R}} \mathrm{HD}_{X^*}((x, 0, \ldots, 0)^\top)$*,*
$X^* \sim \mathrm{CST}(\alpha_*, \nu)$*,* $\delta^* = \alpha_*/(1+\alpha_*^2)^{1/2}$ *and* $\alpha_* = (\alpha^\top \overline{\Omega} \alpha)^{1/2}$*.*

A multivariate measure of kurtosis based on HD (1)

Wang and Serfling (2005) introduce the following measure of multivariate kurtosis:

$$\kappa_X(p) = \frac{V_X\left(\frac{1}{2}-\frac{p}{2}\right)+V_X\left(\frac{1}{2}+\frac{p}{2}\right)-2V_X\left(\frac{1}{2}\right)}{V_X\left(\frac{1}{2}+\frac{p}{2}\right)-V_X\left(\frac{1}{2}-\frac{p}{2}\right)}, p \in [0,1), \qquad (6)$$

where $V_X(r)$ is the volume of the region

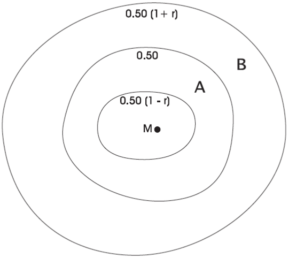
$$C_r = \{x \in \mathbb{R}^d : \mathrm{HD}_X(x) \geq c_r\}$$

and $c_r \in (0, 1/2)$ is such that $P_X(C_r) = r$.

Remark: κ_X is affine invariant (use the canonical form!).

A multivariate measure of kurtosis based on HD (2)

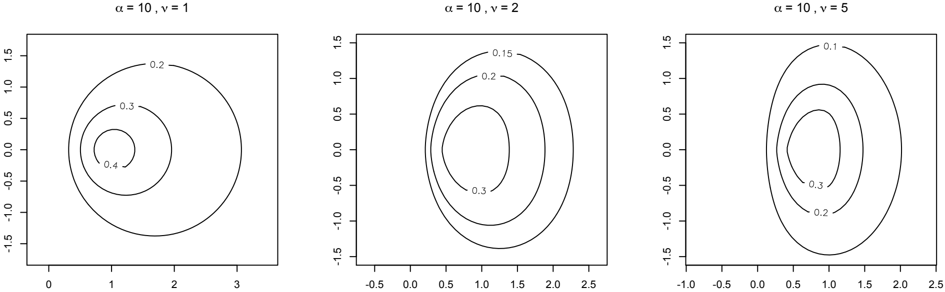
$$k_X = \frac{\text{volume}(A) - \text{volume}(B)}{\text{volume}(A) + \text{volume}(B)} \in (-1, 1)$$



- $k_X = 0 \Rightarrow$ uniform distribution
- $k_X > 0 \Rightarrow$ peaked distribution
- $k_X < 0 \Rightarrow$ bowl-shaped distribution

The HD contours of the ST distribution (1)

$$X \sim \mathrm{CST}(\alpha, \nu)$$



The HD contours of the ST distribution (2)

Theorem (3)

Let $X \sim \mathrm{CST}_d(\alpha, 1)$. *Then the* r*-th* HD *contour is circular with center* $(s(r), 0, \ldots, 0)^\top$ *and radius* $t(r)$*, where*

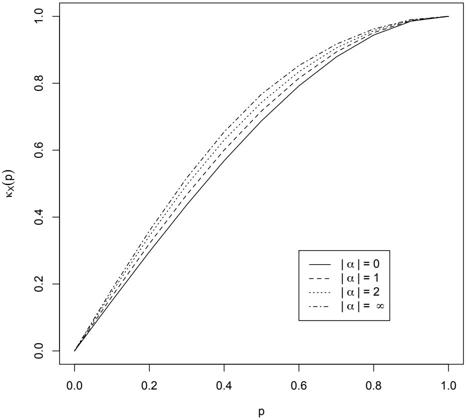
$$s(r) = \frac{\alpha}{\sqrt{1+\alpha^2}} \sec\left\{\left(\frac{1}{2} - r\right)\pi\right\}$$

and

$$t(r) = \tan\left\{\left(\frac{1}{2} - r\right)\pi\right\}.$$

Example: bivariate Skew-Cauchy

$$X \sim \mathrm{CST}(\alpha, 1)$$



Quantifying dispersion, dependence and asymmetry (some proposals)

Concordance measures and interquartile range

A surrogate of the covariance function for the (i, j)-th pair of components can be defined as follows

$$c_{ij} = IQ_i IQ_j r_{ij}, \tag{7}$$

where IQ_i is the interquartile range for the i-th component and r_{ij} is a concordance measure, such as Kendall's τ and Spearman's ρ.

Advantage:

- Equivariant under marginal linear transformation.

Disadvantages:

- The resulting scatter matrix is not affine equivariant.
- Computational issues for low values of ν.

Yule-Bowley's index of asymmetry

Let F_X^{-1} be the quantile function of the random variable X. The Yule (1911) - Bowley (1920) index is

$$b_X = \frac{F^{-1}(3/4) + F^{-1}(1/4) - 2F^{-1}(1/2)}{F^{-1}(3/4) - F^{-1}(1/4)} \in (-1, 1)$$

- $b_X > 0 \Rightarrow$ right-skewness
- $b_X = 0 \Rightarrow$ symmetry
- $b_X < 0 \Rightarrow$ left-skewness

Advantage

- Simple to interpret and compute.

Disadvantage

- Marginal measure of asymmetry.

Final remarks

Further research

Need for further research on:

- approximate methods for computing the multivariate measure of kurtosis proposed by Wang and Serfling (2005);
- efficient computation of concordance measures;
- other vector-valued measures of asymmetry;
- the behaviour of the measures of skewness, dispersion and dependence under non-singular linear transformations.

Thanks for the attention