Pair-copula constructions - even more flexible than copulas

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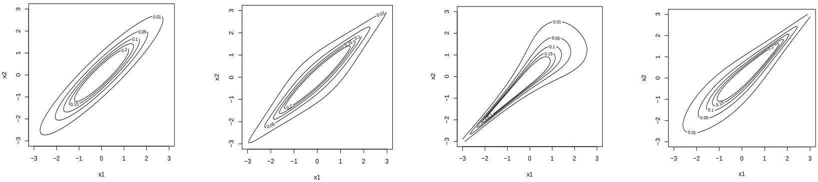
Non-Gaussian Multivariate Statistical Models and their Applications,
Calgary, May 19-24, 2013

Motivation

- While there is a multitude of bivariate copula, the class of multivariate copulae is still quite restricted.
- Hence, if the dependency structures of different pairs of variables in a multivariate problem are very different, not even the copula approach will allow for the construction of an appropriate model.
- In this talk we will describe an extension to the state-of-the-art theory of copulas, modelling multivariate data using a so-called pair-copula construction (PCC).



Overview

Copula...

Theorem (Sklar 1959)

Sklar's theorem states that every multivariate distribution F with marginals $F_1(x_1), ..., F_d(x_d)$ can be written as:

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$$

for some d-dimensional copula C.

Moreover, for an absolutely continuous joint distribution F with strictly increasing continuous marginal distribution functions $F_1, ...F_d$ it holds that

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \cdot \left[\prod_{i=1}^{d} f_i(x_i) \right]$$

for some d-dimensional copula density c.

Pair-copula constructions (I)

- For two random variables X_1 and X_2 we have

$$f(x_1|x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1)$$

- Further, for three random variables X_1, X_2 and X_3 we have

$$f(x_1|x_2, x_3) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot f_{1|2}(x_1|x_2)$$

- It follows that for every j we have

$$f(x|\mathbf{v}) = c_{xv_j;\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j})$$

Pair-copula constructions (II)

By combining the two results

$$f(x_1, \ldots x_d) = f_d(x_d) \cdot f(x_{d-1}|x_d) \cdots f(x_1|x_2, \ldots x_d)$$

and

$$f(x|\mathbf{v}) = c_{xv_j;\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j})$$

we may derive a decomposition of $f(x_1, \ldots x_d)$ that only consists of marginal distributions and bivariate copulae.

We denote a such decomposition a pair copula construction (PCC)

Joe (1996) was the first to give a probabilistic construction of multivariate distribution functions based on pair-copulas, while Aas et al. (2009) were the first to set the PCC in an inferential context.

PCC in three dimensions

A pair-copula construction of a three-dimensional density is given by

$$\begin{aligned} f(x_1, x_2, x_3) &= f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \\ &\cdot \quad c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \\ &\cdot \quad c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \end{aligned}$$

Special case: Trivariate normal distribution

If the marginal distributions are standard normal and c_{12}, c_{23} and $c_{13;2}$ are bivariate Gaussian copula densities, the resulting distribution is trivariate normal.

PCC in five dimensions

A possible pair-copula construction of a five-dimensional density is given by

$$
\begin{aligned}
& f(x_1, x_2, x_3, x_4, x_5) \\
= \quad & f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4) \cdot f(x_5) \\
\cdot \quad & c_{12}\left(F(x_1), F(x_2)\right) \cdot c_{23}\left(F(x_2), F(x_3)\right) \cdot c_{34}\left(F(x_3), F(x_4)\right) \cdot c_{45}\left(F(x_4), F(x_5)\right) \\
\cdot \quad & c_{13;2}\left(F(x_1|x_2), F(x_3|x_2)\right) \cdot c_{24;3}\left(F(x_2|x_3), F(x_4|x_3)\right) \cdot c_{35;4}\left(F(x_3|x_4), F(x_5|x_4)\right) \\
\cdot \quad & c_{14;23}\left(F(x_1|x_2, x_3), F(x_4|x_2, x_3)\right) \cdot c_{25;34}\left(F(x_2|x_3, x_4), F(x_5|x_3, x_4)\right) \\
\cdot \quad & c_{15;234}\left(F(x_1|x_2, x_4, x_3), F(x_5|x_2, x_4, x_3)\right).
\end{aligned}
$$

There are as many as 480 different such constructions in the five-dimensional case, 23,040 in the 6-dimensional case and 2,580,480 in the 7-dimensional case......

Regular vines

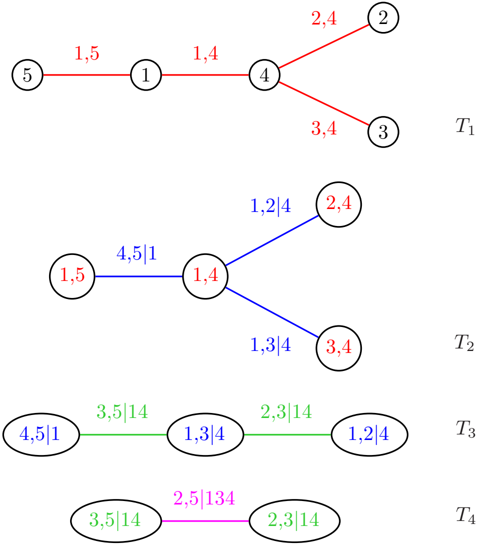
- Hence, for high-dimensional distributions, there are a significant number of possible pair-copula constructions.
- To help organising them, Bedford and Cooke (2001) introduced graphical models denoted regular vines (R-vines).

Regular vine (Bedford and Cooke 2002)

A regular vine is a sequence of $d-1$ linked trees where:

- Tree T_1 is a tree on nodes 1 to d.
- Tree T_j has $d+1-j$ nodes and $d-j$ edges.
- Edges in tree T_j become nodes in tree T_{j+1}.
- **Proximity condition:** Two nodes in tree T_{j+1} can be joined by an edge only if the corresponding edges in tree T_j share a node.

Example in five dimensions

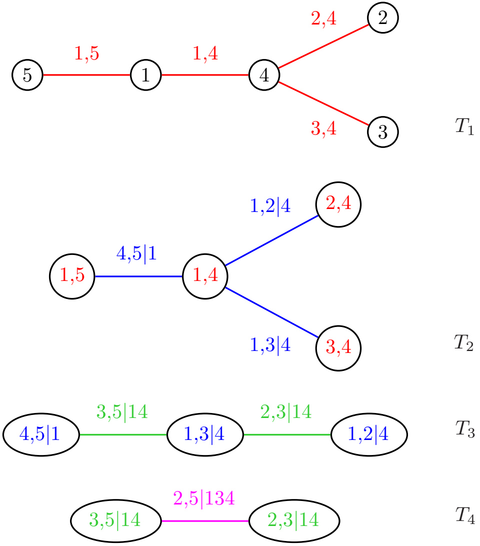


Density

$$f = f_1 \cdot f_2 \cdot f_3 \cdot f_4$$

- $\cdot\ c_{14} \cdot c_{15} \cdot c_{24} \cdot c_{34}$
- $\cdot\ c_{12;4} \cdot c_{13;4} \cdot c_{45;1}$
- $\cdot\ c_{23;14} \cdot c_{35;14}$
- $\cdot\ c_{25;134}$

Matrix representation



Matrix

Morales-Napoles (2008) shows how a lower triangular matrix may be used to store a regular vine.

$$M = \begin{pmatrix} 5 & & & & \\ 2 & 2 & & & \\ 3 & 3 & 3 & & \\ 4 & 1 & 1 & 1 & \\ 1 & 4 & 4 & 4 & 3 \end{pmatrix}$$

Special cases: C and D-vines

C-vine: Each tree has a unique node connected to $d-j$ edges.

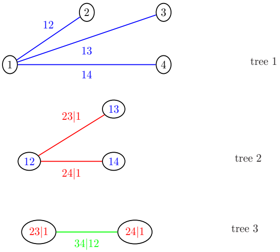
$$\begin{aligned} f_{1234} &= f_1 \cdot f_2 \cdot f_3 \cdot f_4 \\ &\cdot\ c_{12} \cdot c_{13} \cdot c_{14} \\ &\cdot\ c_{23;1} \cdot c_{24;1} \\ &\cdot\ c_{34;12} \end{aligned}$$

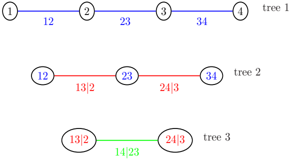
Useful for ordering by importance

D-vine: No node is connected to more than 2 edges.

$$\begin{aligned} f_{1234} &= f_1 \cdot f_2 \cdot f_3 \cdot f_4 \\ &\cdot\ c_{12} \cdot c_{23} \cdot c_{34} \\ &\cdot\ c_{13;2} \cdot c_{24;3} \\ &\cdot\ c_{14;23} \end{aligned}$$

Useful for temporal ordering of variables





General density expressions

- C-vine (Aas et al. 2009)

$$f(x_1, \dots x_d) = \left[\prod_{k=1}^{d} f(x_k) \right] \times \left[\prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{j,j+i;1,\dots,j-1} \right]$$

- D-vine (Aas et al. 2009)

$$f(x_1, \dots x_d) = \left[\prod_{k=1}^{d} f(x_k) \right] \times \left[\prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j;i+1,\dots,i+j-1} \right]$$

- Regular vine (Dißmann et al. 2013)

$$f(x_1, ..., x_d) = \left[\prod_{k=1}^{d} f_k(x_k) \right] \times \left[\prod_{j=d-1}^{1} \prod_{i=d}^{j+1} c_{m_{j,j}, m_{i,j}; m_{i+1,j}, \dots, m_{n,j}} \right]$$

Here, $m_{i,j}$ refers to element (i, j) in the matrix representation of the R-vine.

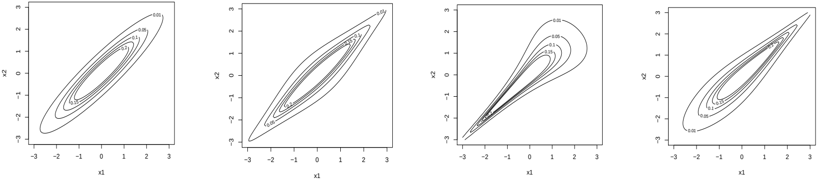
Conditional distribution functions

- The conditional distributions needed as copula arguments at level j are obtained as partial derivatives of the copulae at level $j-1$.
- This is due to the following result from Joe (1996) stating that under regularity conditions, we have

$$F(x|\mathbf{v}) = \frac{\partial C_{xv_j;\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})}.$$

Building blocs

- The resulting multivariate distribution will be valid even if the bivariate copulae involved in the pair-copula construction are of different type.
- One may for instance combine the following types of pair-copulae
  - Gaussian (no tail dependence)
  - Clayton (lower tail dependence)
  - Gumbel (upper tail dependence)
  - Student (upper and lower tail dependence)



How can we estimate and model select PCCs ?

Three problems: (Czado et al. (2013))

1. How to estimate the pair copula parameters for a given vine tree structure and the pair copula families for each edge?
2. How to select the pair copula families and estimate the corresponding parameters for a given vine tree structure?
3. How to select and estimate all components of a regular vine?

Problem 1: Parameter estimation for given tree structure and copula families

- **Sequential estimation:**
  - Parameters are sequentially estimated starting from the top tree until the last (Aas et al. (2009), Czado et al. (2012)).
  - Asymptotic theory available (Hobæk Haff (2012),Hobæk Haff (2013)), however standard error estimates are difficult to compute.
  - Can be used as starting values for maximum likelihood.
- **Maximum likelihood estimation:**
  - Asymptotically efficient under regularity conditions, estimated standard errors numerically challenging (Stoeber and Schepsmeier (2012))
  - Uncertainty in value-at-risk (high quantiles) is difficult to assess.
- **Bayesian estimation:**
  - Posterior is tractable using Markov Chain Monte Carlo (Min and Czado (2011) for D-vines and Gruber (2011) for R-vines)
  - Prior beliefs can be incorporated and credible intervals allow to assess uncertainty for all quantities.

How does sequential and ML estimation work ?

Parameters: $\Theta = (\theta_{12}, \theta_{23}, \theta_{13;2})$

Copula observations: $\{(u_{1t}, u_{2t}, u_{3t}), t = 1, \cdots, T\}$

Sequential estimates:

- Estimate θ_{12} from $\{(u_{1t}, u_{2t}), t = 1, \cdots, T\}$
- Estimate θ_{23} from $\{(u_{2t}, u_{3t}), t = 1, \cdots, T\}$.
- Define pseudo observations

$$\hat{u}_{1|2t} := F(u_{1t}|u_{2t}, \hat{\theta}_{12}) \text{ and } \hat{u}_{3|2t} := F(u_{2t}|u_{3t}, \hat{\theta}_{23})$$

Finally estimate $\theta_{13;2}$ from $\{(\hat{u}_{1|2t}, \hat{u}_{3|2t}), t = 1, \cdots, T\}$.

Maximum likelihood

$$\begin{aligned} L(\Theta|x) &= \sum_{t=1}^{T} [\log c_{12}(u_{1t}, u_{2t}|\theta_{12}) + \log c_{23}(u_{2t}, u_{3t}|\theta_{23}) \\ &+ \log c_{13;2}(F(u_{1t}|u_{2t}, \theta_{12}), F(u_{2t}|u_{3t}, \theta_{23})|\theta_{13;2})] \end{aligned}$$

Problem 2: Joint estimation of pair copula families and parameters

- **Classical approach:**
  - Restrict to a set of bivariate pair copula families and use AIC or Vuong test to select family
  - Check for truncation possibilities (Brechmann et al. (2012)) by using independence copulas in higher trees
- **Bayesian approach:**
  - Reversible jump (RJ) MCMC (Min and Czado (2011))
  - MCMC with model indicators (Smith et al. (2010)) choosing between an independence copula and a fixed copula family.

Only one more problem to go ...

sequential treewise approach
(see Dißmann et al. (2013))

How does this treewise selection of R-vines work?

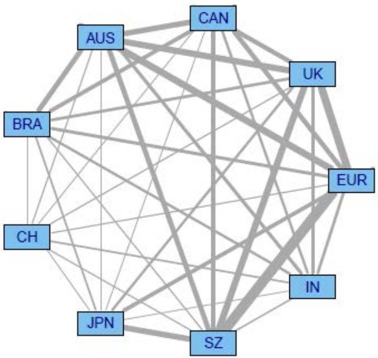
Idea: Capture strong pairwise dependencies first

For Tree $\ell = 1, \dots, d-1$

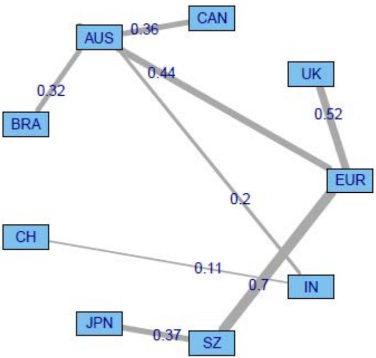
1. Calculate an empirical dependence measure $\hat{\delta}_{jk|D}$ for all variable pairs $\{jk|D\}$ ($\rightarrow$ edge weights: Kendall's τ, tail dependence coefficients) allowed by the proximity condition (D is empty for Tree 1).
2. Select the tree on all nodes that maximizes the sum of absolute empirical dependencies ($\rightarrow$ maximum spanning tree)
   Choose independence copula if possible.
3. For each selected edge $\{j, k\}$ ($\{j, k\}|D$) in Tree 1 (in Tree $\ell > 1$), select a copula and estimate the corresponding parameter(s).
4. Then transform to pseudo observations $F_{j|k \cup D}(u_{ij}|\mathbf{u}_{i,k\cup D}, \hat{\theta}_{j,k;D})$ and $F_{k|j \cup D}(u_{ik}|\mathbf{u}_{i,j\cup D}, \hat{\theta}_{j,k;D})$, $i = 1, ..., n$.

How does this look like for Tree 1?

(1) Pairwise dependencies.



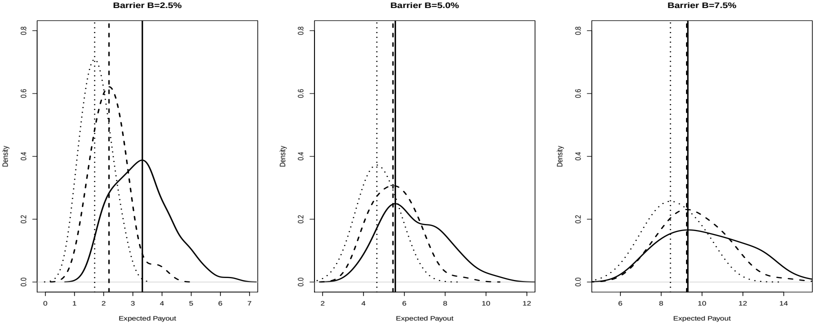
(2) Maximum dependence tree.



Czado, Jeske, and Hofmann (2013) compare sequential selection strategies

Sequential Bayesian model selection of regular vine copulas (Gruber and Czado 2013)

- Tree by tree selection to reduce search space
- Reversible jump MCMC to select tree, pair copulas and parameters jointly
- dynamic barrier payouts based on basket of 9 Dow Jones stocks



solid: bootstrapped observed, dashed: regular vine, dotted: t copula

Market risk model for largest Norwegian bank, DNB:

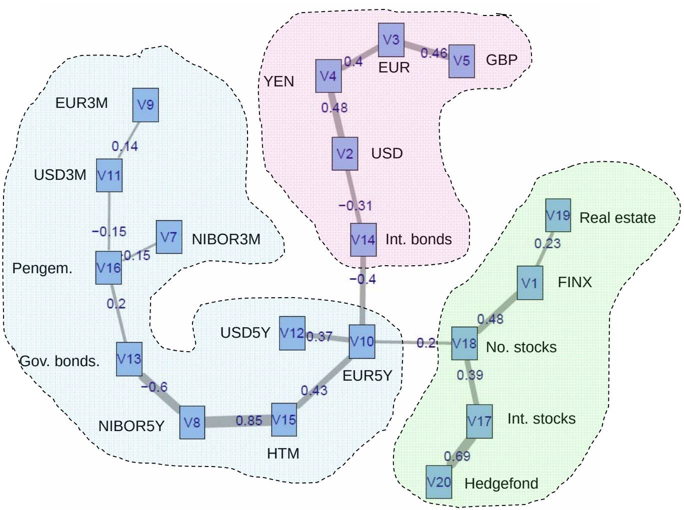
- 19 financial variables that constitute the market portfolio of DNB.
- Daily log returns from March 2003 to March 2008 (1107 obs.) are used.

ID	description	ID	description
V1	Norwegian Financial Index	V12	5-year US Government Rate
V2	USD-NOK exchange rate	V13	Norwegian bond index (BRIX)
V3	EURO-NOK exchange rate	V14	Citigroup World Government Bond Index (WGBI)
V4	YEN-NOK exchange rate		
V5	GBP-NOK exchange rate	V15	Norwegian 6-year Swap Rate
V7	3-month Norwegian Inter Bank Offered Rate	V16	ST2X - Government Bond Index (fix modified duration of 0.5 years)
V8	Norwegian 5-year Swap Rate	V17	Morgan Stanley World Index (MSCI)
V9	3-month Euro Interbank Offered Rate	V18	OSEBX - Oslo Stock Exchange main index
V10	5-year German Government Rate	V19	Oslo Stock Exchange Real Estate Index
V11	3-month US Libor Rate	V20	S&P Hedge Fund Index

Modelling procedure :

- Fit appropriate ARMA-GARCH models for log-return time series.
- Fit an R-vine as well as a multivariate Student-t copula for comparison to standardized residuals
- Pair-copulas are selected from a range of 11 bivariate families using AIC: Independence copula, Gaussian, t, Clayton, rotated Clayton (90), Gumbel, rotated Gumbel (90), Frank, Joe, Clayton-Gumbel (BB1), Joe-Clayton (BB7).

First tree of R-vine:



Results:

Copula	Log likelihood	No. of param.	AIC
R-vine	6390.75	92	-12597.50
Student-t	6324.98	172	-12305.96

Number of parameters:

Note that the number of parameters to be estimated for a 19-dimensional R-vine usually is at least d(d-1)/2. The reason why the number in the table is 92 and not 171 is that a large amount of the pair-copulae in this R-vine are identified as the independence copula, using the bivariate independence test based on Kendall's tau as described in Genest and Favre (2007) .

Truncation(I):

- The number of parameters in an R-vine grows quadratically with the dimension.
- Hence, it would be useful to be able to reduce the model complexity.
- In Brechmann et al. (2012) we have studied the problem of determining whether an R-vine may be truncated.
- By a truncated R-vine at level K, we mean an R-vine with all pair-copulae with conditioning set larger than or equal to K set to independence copulae.
- We fit one tree at a time and use the likelihood ratio test of Vuong (1989) to determine whether an additional tree provides a significant gain in the model fit.

Results:

Copula	Log likelihood	No. of param.	AIC
R-vine	6390.75	92	-12597.50
Student-t	6324.98	172	-12305.96
6-level R-vine	6274.47	77	-12394.94
4-level R-vine	6234.05	68	-12332.10

Conclusion:

We conclude from this that the most important dependencies in this data set are actually captured in the first four to six trees, meaning that the corresponding R-vine copula may be truncated at level 6, or even at level 4, depending on the desired level of parsimonity (and of course at the expense of accuracy).

Recent advances for vines

- Simplified and non simplified vines
- Time varying regular vines
- Discrete and discrete/continuous vines
- Non Gaussian DAG's using pair copula constructions
- Vines with non parametric pair copulas: Haff and Segers (2013), Kauermann and Schellhase (2013)
- Acceleration of MCMC algorithms: Schmidl et al. (2013)

Simplified and non-simplified vines

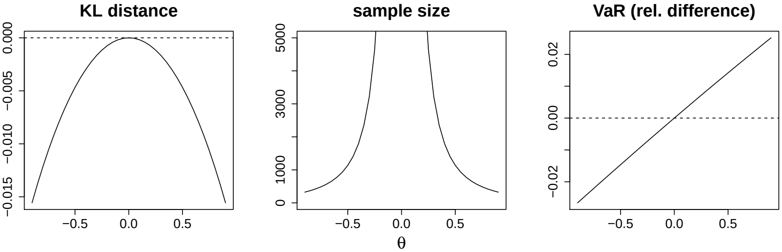
Simplifying assumption

Pair copulas depend on their conditioning value only through their conditional distributions (Haff, Aas, and Frigessi 2010)

- **Simplified vine copulas**:
  - multivariate Gauss copula
  - multivariate t-copula only one arising from arising from scale mixture of normals (Stöber et al. 2012)
  - multivariate Clayton is the only one among the Archimedean copulas (Stöber et al. 2012)
- **Non-simplified vines**:
  - Acar et al. (2012) use a smoothing approach to deal with non simplified vines in 3 dimensions based on Acar et al. (2011).
  - Occurs when considering one factor models: $X_j = Z_0 + X_j$ for $j = 1, ..., d$.

Effects of simplifying assumption

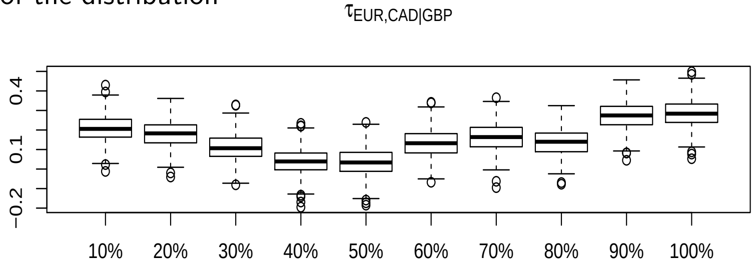
Trivariate extension of FGM copula (Stöber 2013)



- left: Kulback Leibler distance
- middle: sample size needed to distinguish the models in a LR test
- right: relative difference in Value-at-Risk

Violation of simplifying assumption might indicate time varying dependence

Conditional Kendalls τ rank correlation between the USD/EUR and USD/CAD return exchange rate conditional on USD/GBP being in a given decile of the distribution

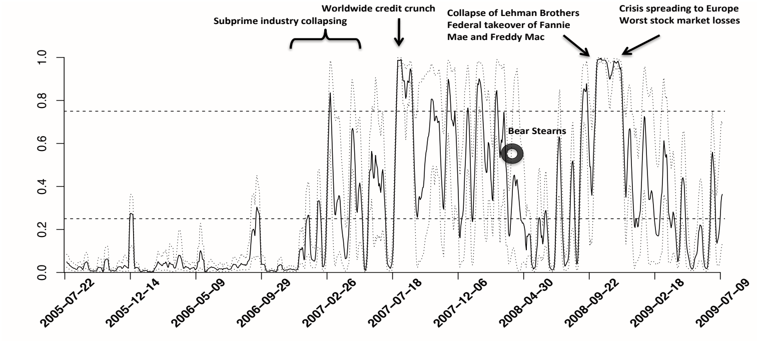


The boxplots are obtained using a non-parametric bootstrap. The analysis in Stöber and Czado (2013) shows that data has time varying dependence.

Time varying regular vines

- **AR(1) copula dynamics**
  - Bayesian bivariate analysis: Almeida and Czado (2011)
  - Multivariate analysis: Almeida et al. (2012)
- **Regime switching**
  - C-vine, copula parameters only, EM: Chollete et al. (2008)
  - R-vine and copula parameter switches, EM, MCMC: Stöber and Czado (2013)
  - Marginal and copula switches: Stöber (2013)

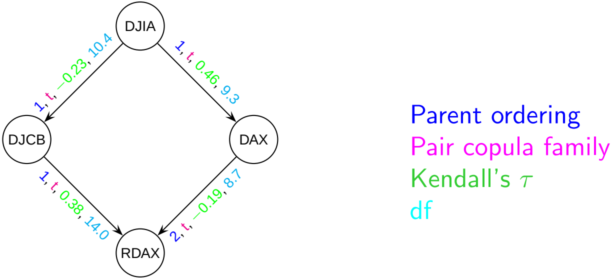
Smoothed probabilities of being in non-Gaussian regime



The solid lines correspond to Bayesian MCMC estimates, the dotted lines to 90% CIs (Stöber 2013)

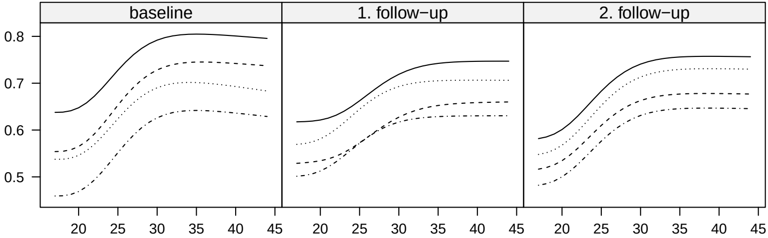
PCC based network models

- **Bayesian belief networks**: They were first considered by Hanea et al. (2006).
- **Pair-copula Bayesian networks (PCBN)**
  - Bauer et al. (2012) used a PCC construction to build Non Gaussian DAG models.
  - Bauer (2013) and Bauer and Czado (2012) give general algorithms to estimate PCBN and provides a PC algorithm to construct network.



Discrete and discrete/continuous vines

- **Discrete vines** Panagiotelis et al. (2012) construct an efficient PCC using D-vines based on the distribution function
- **Discrete/continuous vines** Stöber et al. (2012) and Stöber (2013) extend to cover discrete/continuous variables and allow for regular vines
  - solid (diabetes, hypertension),
  - dashed (diabetes, no hypertension)
  - dotted (no diabetes, hypertension)
  - dash-dotted (no diabetes, no hypertersion)



Selected Applications

- Financial risk management:
  - Euro Stoxx 50 (Brechmann and Czado 2012)
  - Systemic risk simulation (Brechmann et al. 2013)
  - Operational risk: (Brechmann et al. 2013)
  - Multivariate options: (Gruber and Czado 2013)
  - Realized volatility: (Vaz de Melo Mendes and Accioly 2013)
- Hydrology: (Gräler et al. 2013)
- Data mining: (Lopez-Paz et al. 2013)
- Health: comorbidity (Stöber et al. 2012)
- Environmental Science:(Gräler and Pebesma 2011) (Pachali 2011)

What have we learned?

- Standard multivariate copulas are less flexible, while PCC's such as C-, D- and R-vines are much more flexible.
- Sequential and MLE parameter estimation of C-, D- and R-vines are available in **R packages** `CDVine` and `VineCopula`.
- Sequential and full Bayesian estimation and Bayesian model selection of vine trees and copula families for regular vines available, but need further testing and development
- Pair copula constructions can be extended to mixed continuous and discrete data.
- Vine copulas are useful for financial risk management

What needs to be done?

- Non-parametric pair copulas, spatial vines, vines for data mining
- More applications in finance, insurance ...

Vine resource page:
`www-m4.ma.tum.de/forschung/vine-copula-models`

Vine workshop book: Kurowicka and Joe (2011)

Next vine workshop: Jan. 3/4 2014, Peking, China (?)

Thanks to our collaborators (A. Frigessi, I. Hobæk Haff, A. Min, E. Brechmann, C. Almeida, A. Bauer, T. Klein, M. Hofmann, H. Manner, C. Bernard, J. Dißmann, H. Joe, A. Panagiotelis, M. Smith, J. Stöber, U. Schepsmeier, D. Kurowicka, L. Gruber, N. Krämer...)

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