Fluid limits for Bandwidth-Sharing Networks with Rate Constraints

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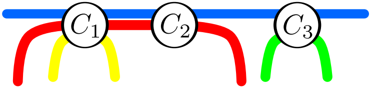
BS: Congestion at flow level

Massouliè & Roberts (1998, 1999)

- J links
  of capacities $C_1, \ldots, C_J$
- I routes / flow classes
- $A_{ji} = \mathbb{I}\{\text{link } j \in \text{route } i\}$
- Multiple flows on a route are served at the same rate
- **Rate constraint:** m_i — maximum service rate on route i



$$A = \begin{array}{c|cccc} & \text{blue} & \text{red} & \text{yellow} & \text{green} \\ \hline C_1 & 1 & 1 & 1 & 0 \\ C_2 & 1 & 1 & 0 & 0 \\ C_3 & 1 & 0 & 0 & 1 \end{array}$$

BS protocol

Given population z, $\Lambda_i(z)$:= bandwidth allocated to route i

$= z_i \times$ service rate on route i

$\Lambda(z)$ is the unique solution to

maximize $\sum_{i \leq I} z_i U_i(\Lambda_i / z_i)$ subject to $A\Lambda \leq C,$ $\Lambda_i \leq m_i z_i,\ i \leq I,$

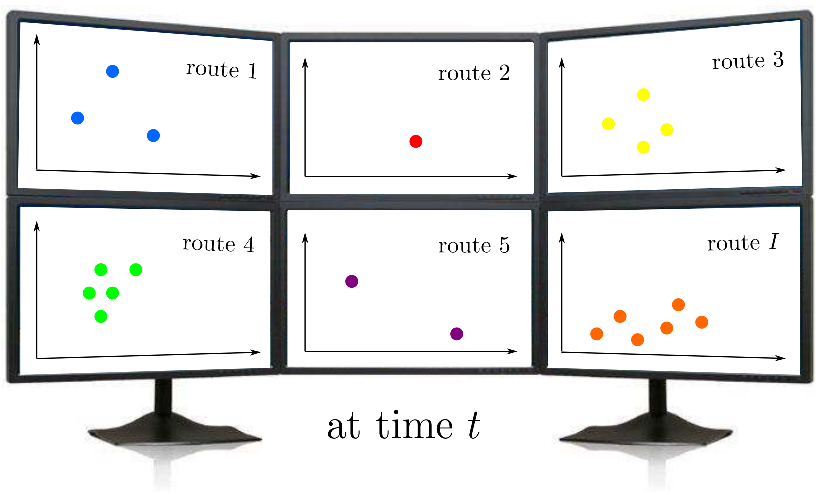
$U_i(\cdot)$, strictly increasing, concave, twice differentiable.

Stochastic model

for $i \leq I$,

- arrivals to route i: counting process $E_i(\cdot)$
- sizes of class i flows: i.i.d. copies of $B_i > 0$ a.s.
- patience times of class i flows: i.i.d. copies of $D_i > 0$ a.s.
- $\mathbb{E}B_i < \infty$, $\mathbb{E}D_i < \infty$
- $D_i \geq B_i/m_i$ a.s.
- $Z(t) = (Z_1(t), \ldots, Z_I(t))$ population vector at time t.

Measure valued state descriptor



Large capacity scaling

consider a sequence of BS networks $A^R = A$, $R \to \infty$

grow to ∞:	of a fixed order:
• link capacities $C^R = RC$	• rate constraints $m^R = m$
• arrival rates $E_i^R(\cdot)/R \Rightarrow \{\nu_i t; t \geq 0\}$	• flow sizes and patience times

Theorem (Convergence to fluid limit): $Z^R(\cdot)/R \Rightarrow z(\cdot)$

Unique fixed point of fluid limit

$z(\cdot)$ is constant if $z = z(0)$ satisfies

$$z_i = \eta_i \, \mathbb{E} \min \left\{ \frac{B_i}{\Lambda_i(z)/z_i}, D_i \right\}, \quad i \leq I$$

Theorem

Recall that $D_i/B_i \geq 1/m_i$, $i \leq I$.

If $1/m_i$ is the left most point of the support of D_i/B_i, $i \leq I$, then the fixed point is unique **and can be computed in polynomial time**.

Proof: combine fixed point equations with KKT conditions of $\Lambda(z)$: **Educated guess of optimization problem of which Λ is the solution.**