Robustness of Complex Networks: Reaching Consensus Despite Adversarial Agents

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Standard Consensus Dynamics

- Network: n nodes $\{x_1, x_2, \dots, x_n\}$, edge set E
- Each node x_i starts with a real number $x_i[0]$
- **Linear averaging dynamics**:

$$x_i[k+1] = w_{ii} x_i[k] + \sum_{j \in nbr(i)} w_{ij} x_j[k]$$

- As long as the network is **connected:**

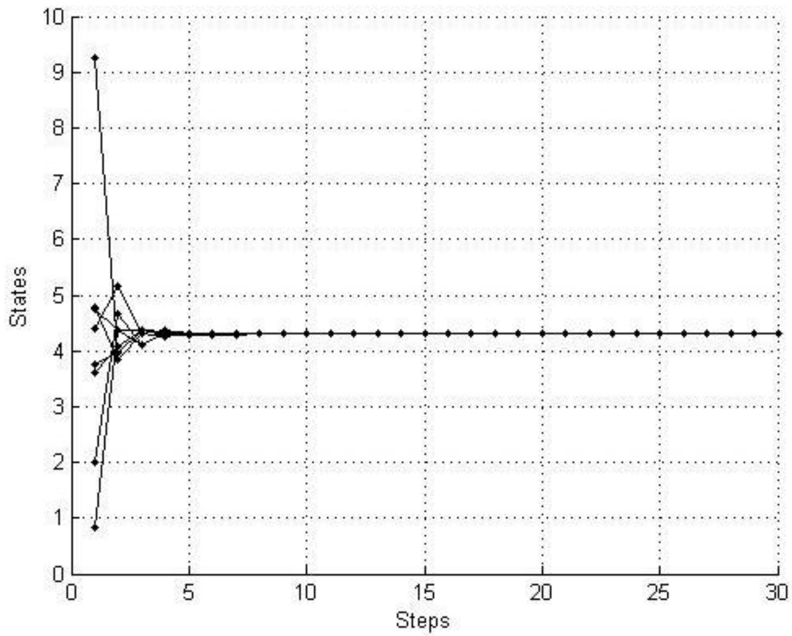
$$\lim_{k \to \infty} x_i[k] = \sum_{i=1}^{n} \alpha_i x_i[0], \qquad \forall i \in \{1,2,\dots,n\}$$

  - The coefficients α_i are nonnegative and sum to 1

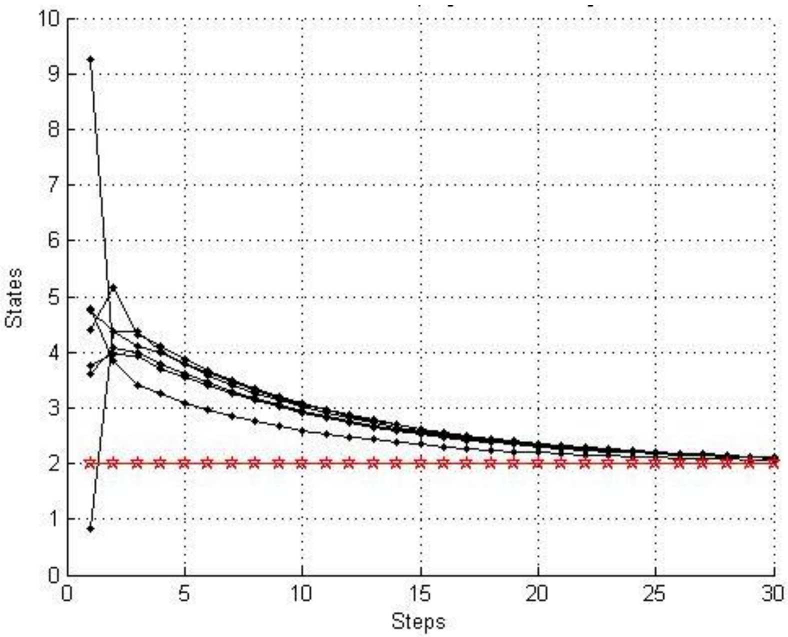
Potential for Adversarial Behavior

- What happens if some nodes **don't follow** the averaging dynamics?
- Example: suppose some node keeps its value **constant**

No adversaries



One stubborn adversary



Resilient Consensus Objective

- Node set partitioned into two sets: $\boldsymbol{N}$ **(normal nodes)** and $\boldsymbol{A}$ **(adversarial nodes)**
  - Sets N and A are unknown to normal nodes
  - Adversarial nodes are allowed to update their states arbitrarily
  - Normal nodes follow whatever dynamics we propose

- Consider the following (relaxed) objective:

  "All normal nodes should asymptotically reach consensus on some value that is between the smallest and largest initial values of the normal nodes"

- Adversarial nodes should not be able to bias the consensus value **excessively**

Local Filtering

- Natural strategy: Each normal node is "suspicious" of extreme values in its neighborhood
- Mechanism:
  - At each time-step k, each node x_i receives values from its neighbors
  - x_i **removes the F highest and F lowest values** in its neighborhood, updates its state as a convex combination of remaining values

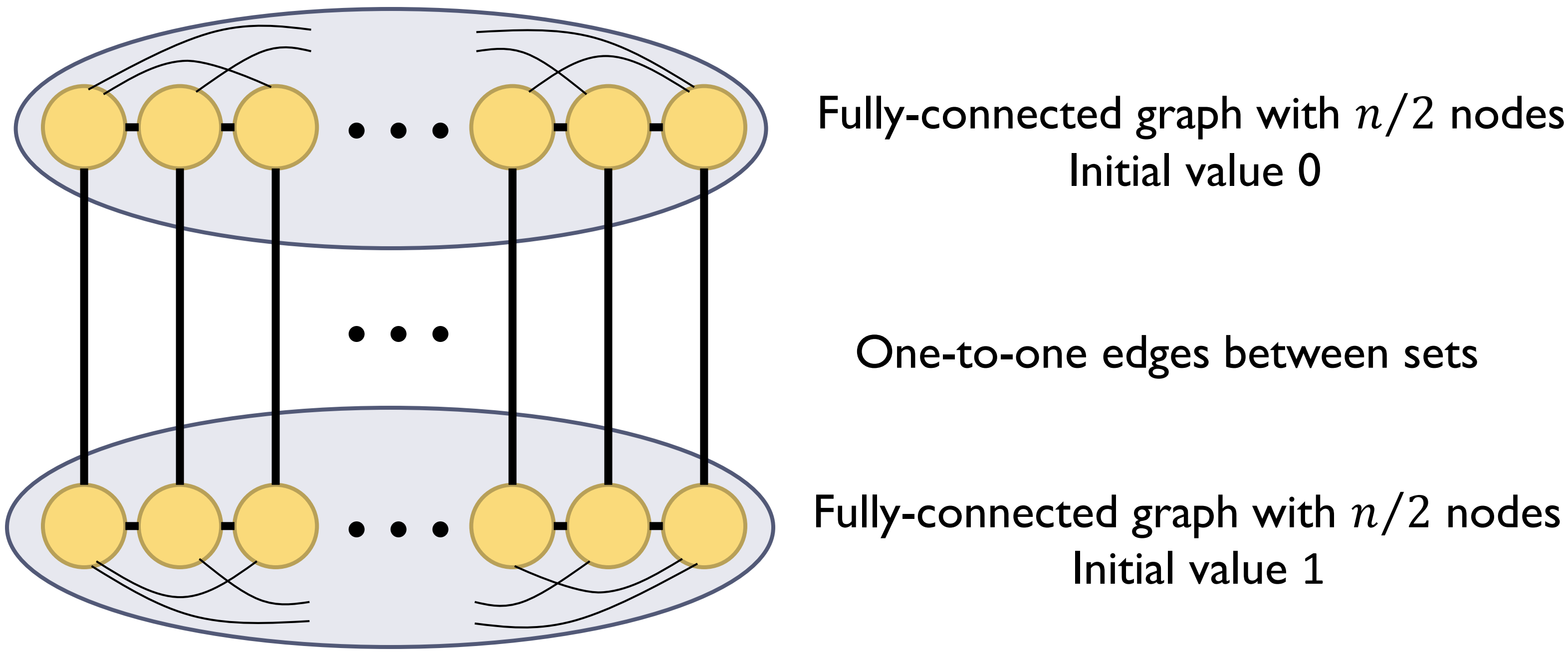
$$x_i[k+1] = w_{ii} x_i[k] + \sum_{j \in \widetilde{nbr(i)}} w_{ij} x_j[k]$$

Neighbors after removing extreme values

  - F is a parameter indicating level of suspicion

Convergence

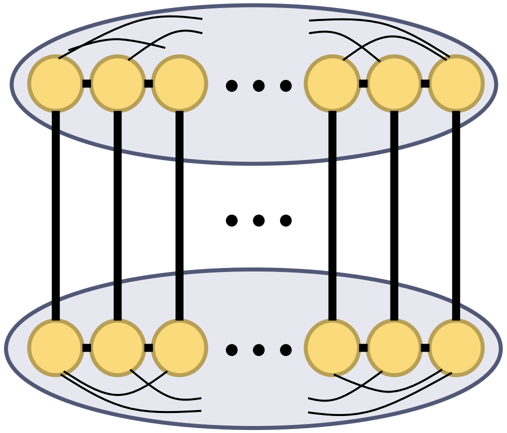
- Traditional graph metrics not useful to characterize convergence



- Connectivity of graph is $n/2$, but **no node ever uses a value from opposite set**

Insufficiency of Connectivity as a Metric

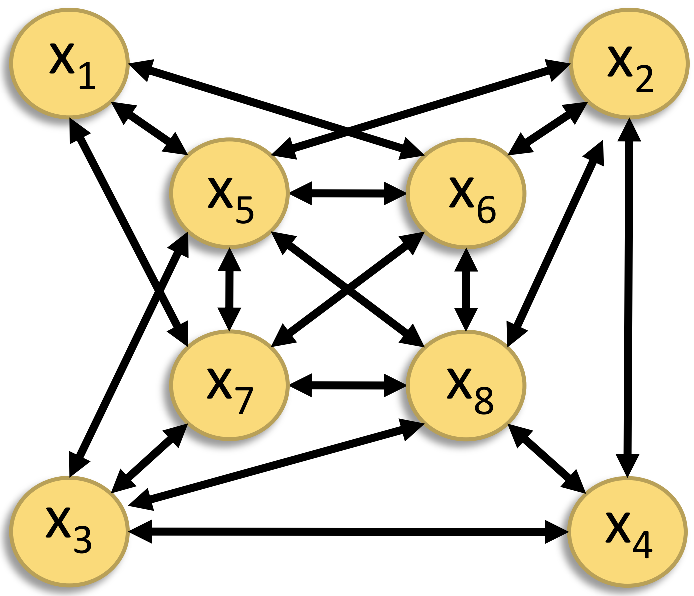
- **Connectivity is no longer a sufficient metric** to characterize behavior of purely local filtering mechanism
  - Graph contains sets where no node in any set has **enough neighbors** outside the set
  - i.e., all outside information is filtered away by each node



- **Need a new topological property** to characterize conditions under which local filtering will succeed

Robust Graphs

- We introduce the following definitions
  - A set S is ***r*-reachable** if it has a node that has at least r neighbors outside the set



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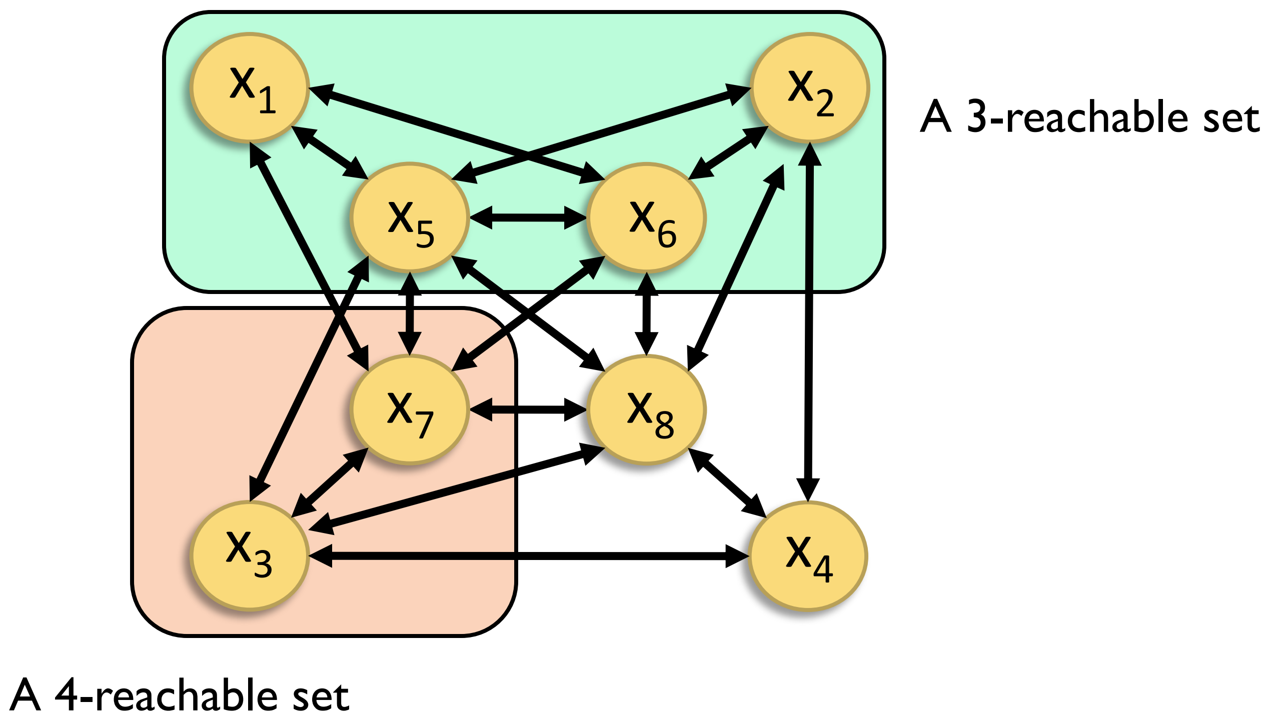
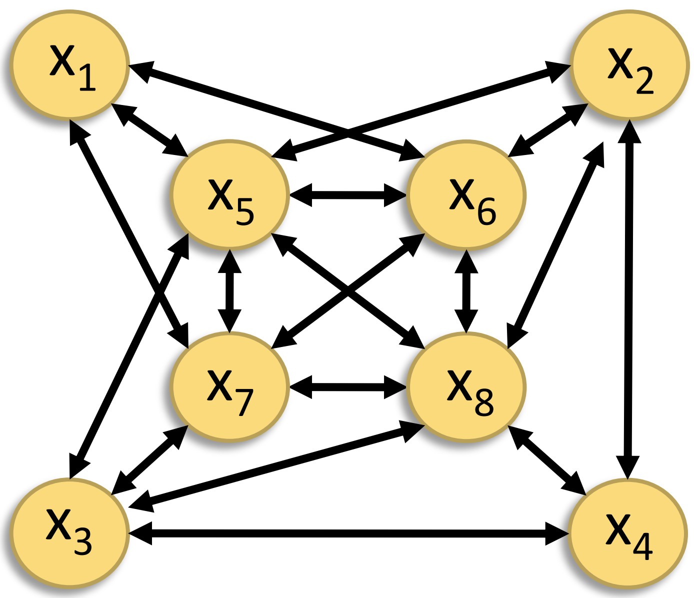


Robust Graphs

- A graph is ***r*-robust** if for any two disjoint subsets, at least one of the sets is r-reachable



3-robust graph:
Pick any two subsets of nodes, at least one is 3-reachable

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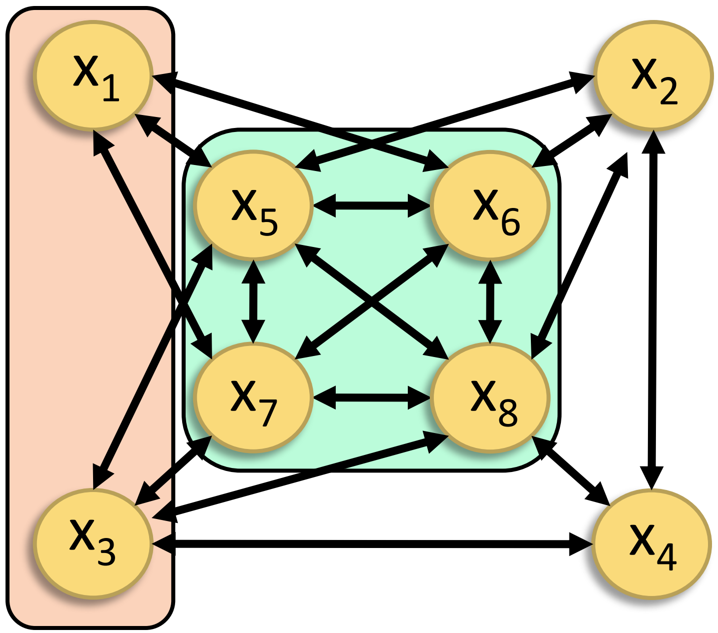

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The Role of Robustness in Convergence

- **Main result:** If there are at most F adversarial nodes

Graph is **(2F+1)-robust**	⇒	Normal nodes will reach consensus despite actions of adversarial nodes

- **Robustness is the key metric** for purely local filtering/diffusion mechanisms

- Recall: Can construct graphs that have very high connectivity ($n/2$), but that are only 1-robust

- Question: What is the robustness of "complex networks"?
  - Will purely local filtering mechanisms work on these networks?

Zhang and Sundaram, ACC 2012; LeBlanc, Zhang, Koutsoukos and Sundaram, IEEE JSAC 2013; Vaidya et al., PODC 2012

Erdos-Renyi Graphs

- Erdos-Renyi graphs $G(n, p(n))$: Define

$$p(n) = \frac{\ln(n) + (r-1)\ln\ln(n) + c(n)}{n}$$

If $c(n) \to -\infty$ as $n \to \infty$	$\Rightarrow$	$G(n, p(n))$ almost surely has **min degree, connectivity and robustness less than r** as $n \to \infty$

If $c(n) \to \infty$ as $n \to \infty$	$\Rightarrow$	$G(n, p(n))$ almost surely has **min degree, connectivity and robustness at least r** as $n \to \infty$

Phase Transition for Erdos-Renyi Graphs

- **Threshold function:**

$$t(n) = \frac{\ln(n) + (r-1)\ln\ln(n)}{n}$$

  - ER graph experiences a phase transition for r-min degree, r-connectivity and r-robustness at this threshold

- There is a "triple jump" at this threshold [Zhang & Sundaram, CDC 2012]

- "Double jump" for min degree and connectivity known since [Erdos & Renyi, 1961]

Geometric Random Graphs

- For 1-d geometric graphs, we show:

> If graph is $\left(\frac{3}{2}r\right)$-connected, then it is at least r-robust

- **Key point:** highly connected 1-d geometric random graphs are also highly robust

Preferential Attachment Networks

- One option to model graphs that grow over time: **Preferential Attachment process**

- Start with a small group of nodes
- At each time-step, a new node comes in and attaches to r existing nodes (Barabasi-Albert model)
  - Key point: prefer to attach to nodes that have a large degree
  - Produces a power law network

- If initial network is r-robust, we show:

Resulting Power-Law graph is r-connected and r-robust

Thanks!

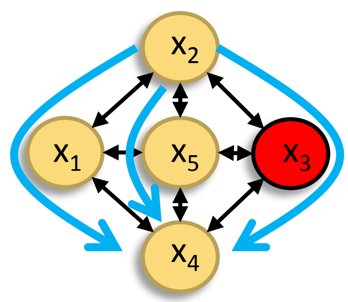
(Come see poster for more details!)

Connectivity as a Metric for Robustness

- Traditional result: In fixed networks with **up to F adversaries:**

Network has at least 2F+1	⟺	Any two nodes can reliably exchange initial values despite actions of F adversarial nodes

  - Note: adversaries allowed to update their states **arbitrarily**



- Requires normal nodes to know the entire network to route/decode information to/from other nodes