Scaling laws and SINR Random Graphs

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Inroduction

- Scaling as a device
- "Good scalings"
- Scaling in wireless networks- good and bad
- Many user scalings- mean field
- Broadcast channels
- Multiple Access Channels
- SINR Random Graphs
- Gupta-Kumar and Beyond
- Concluding remarks

Scaling as a device

Scaling is a mathematical device that allows us to characterize the behavior of mathematical models that retain the essential macroscopic facets of the model.

Typically scaling has come to mean some parameter becoming "large" or going to infinity that involves characterizing the asymptotic behavior.

This talk will focus on the probabilistic characterization of models arising in wireless networks.

Examples:

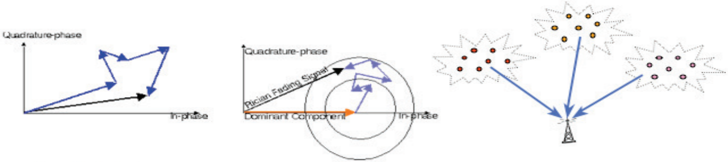
- Space and time scaling- two approaches
  - SLLN approach characterizing a mean or average behavior - useful for stability and convergence
  - CLT approach that is useful to characterize performance
- Large population models- to provide tractability and take advantage of averaging and mixing
  - Take advantage of independence- convolution of measures
  - Take advantage of dynamics of interaction- propagation of chaos

Broadcast and Multiple Access channels

- In broadcast channels (BC), one transmitter sends information to many users.
- In multiple-access channels (MAC), many users transmit information to one receiver.
- In wireless ad-hoc networks, several transmitters communicate to several receivers simultaneously and there is no infra-structure.

Fading Channel Models

- Rayleigh fading: Multiple indirect paths without no distinct dominant path
- Rician fading: There is a dominant component (Line-Of-Sight) in addition to indirect paths
- Nakagami fading: Different clusters of reflected waves with relatively large delay spreads, each cluster's signal is Rayleigh distributed



Rayleigh fading — Rician fading — Nakagami fading

Rician and Nakagami CDF

- Rician and Nakagami fading channels which are often used to model wireless communication channels. For example, in Rician fading channels, ; as a result $|h_i| \sim Rice(1, \mu)$ and $h_i \sim \mathcal{CN}(\mu, 2)$ with the distribution function $|h_i|^2 \sim \mathcal{NC}\chi_2^2(\mu^2)$

$$F_{\mathcal{NC}\chi_2^2}(x; 2, \mu^2) = \sum_{j=0}^{\infty} e^{-\mu^2/2} \frac{(\mu^2/2)^j}{j!} \frac{\gamma(j+1, x/2)}{\Gamma(j+1)}$$

- In Nakagami fading channels, $|h_i| \sim Nakagami(\mu, \omega)$ and the distribution function of $|h_i|^2$ is given by

$$F(x; \mu, \omega) = \frac{\gamma(\mu, \frac{\mu}{\omega} x)}{\Gamma(\mu)}$$

Rate constrained users

- Opportunistic schemes adapt allocated power to variations of the user's channel.
- Some users/links experiencing bad channel conditions may have very low transmission rate which is not sufficient for some applications and waste system resources.
- Maintaining a minimum rate is not always possible for all users because of the transmitted-power constraint.
- Proposed solution: Maximize the number of simultaneously active users/links maintaining the minimum-rate constraint

How many active users/links can be supported in a multi-user wireless system with a minimum-rate constraint?

BC Channel Model

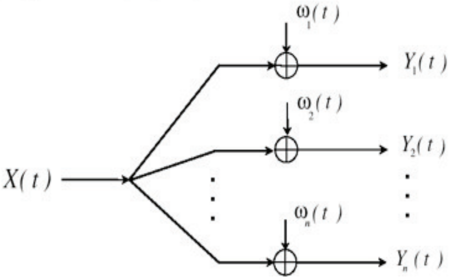
- Consider the following broadcast channel

$$Y_i(t) = h_i X(t) + Z_i(t), \quad i = 1, 2, \ldots, n,$$

- It is assumed that channel gains are constant during each time block. Hence, the equivalent channel can be written as

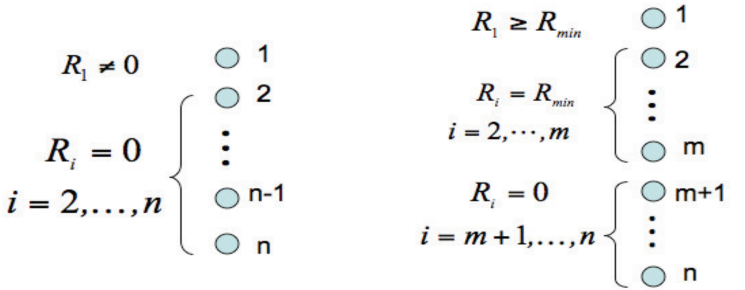
$$Y_i'(t) = X(t) + Z_i(t)/h_i, \quad i = 1, 2, \ldots, n$$

$$\omega_i(t) = Z_i(t)/h_i$$



Structure of Solution

Trade-off between maximizing the sum rate and maximizing the number of active users maintaining a minimum rate



Maximizing the sum rate

Maximizing the number of active users with a minimum rate

Algorithm

Proposed power allocation maximizing the number of active receivers

m: number of active receivers
n: total number of receivers
P: total transmitted power

N_i: the i[th] user's noise variance
P_i: the i[th] user's signal power
R_{min}: the minimum rate

$$\begin{cases} \max\{m\} \\ \ln\left(1 + \frac{P_1}{N_1}\right) \geq R_{\min} \\ \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) = R_{\min}, \quad i = 2, \ldots, m \\ \sum_{i=1}^{m} P_i = P \end{cases}$$

Solution-contd.

- The solution depends on $P_i = h_i P$ where h_i the channel gain is random.
- Solution is sample path or realization dependent
- However: if we ask the question; what is the typical size of m?

Answer can be obtained from scaling, i.e. when n is large a typical behavior can be estimated with great accuracy (concentration of measure).

Scaling result

- Assume channel gains $h_i, i = 1,\ldots,n$ are independent realizations of the complex Gaussian distribution. $|h_i|^2; i = 1,\ldots,n$ are independent realizations of the gamma distribution. In addition, $N_1 \leq \ldots \leq N_n$.
- **Theorem 1:** Under the assumption of independent Rayleigh fading channels for different receivers and for any $\varepsilon > 0$, the maximum number of active receivers is bounded as:

$$\mathbb{P}\left(\lfloor \nu(n) - \epsilon \rfloor \leq m \leq \nu(n) + \epsilon\right) \rightarrow 1$$

$$\nu(n) \triangleq \frac{1}{R_{min}} \ln\left(\frac{P}{\sigma^2} \ln n\right)$$

as $n \rightarrow \infty$, and $\lfloor x \rfloor$ denotes the maximum integer no greater than x .

BC Different channel models

Rayleigh fading

$$\nu(n) \triangleq \ln(P \ln n)/R_{\min}$$

Rician fading

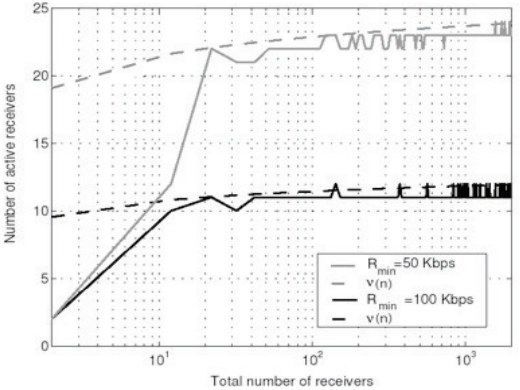
$$\nu_1(n) \triangleq \ln(2P \ln n)/R_{\min}$$

Nakagami fading

$$\nu_2(n) \triangleq \ln(\frac{\omega}{\mu} P \ln n)/R_{\min}$$

Simulation results

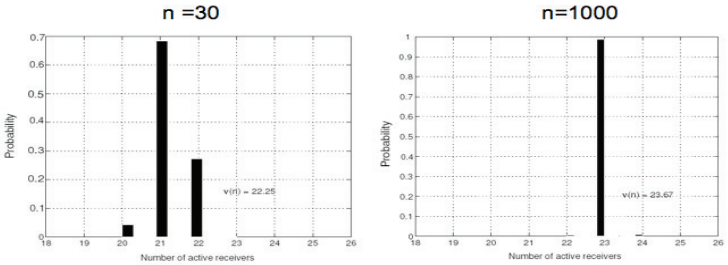
Consider a system with $|h_i|^2 \sim \chi_2^2$, $\sigma^2 = 1$, SNR=40 dB at the transmitter, and bandwidth of 50K sample/sec.



Concentration effects

Simulation Results

- Sample paths will become more concentrated as “n” increases.



Asymptotic Sum Capacity of BC

- According to theorem 1 ($\sigma^2 = 1$), the total throughput scales at least as

$$\nu(n) R_{\min} = \ln(P \ln n)$$

- Remark: The maximum total throughput (when total transmitted power is allocated to the best receiver) is upper bounded as

$$\ln(1 + \beta P \ln n)$$

with $\beta > 1$ can be arbitrarily close to one.

This follows from the fact that capacity is maximized if all power is given to the best user.

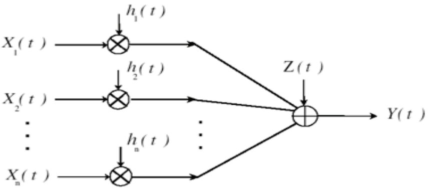
MAC Version

MAC Channel Model

- Consider the following MAC channel

$$Y(t) = \sum_{i=1}^{n} h_i X_i(t) + Z(t)$$

- Assume $|h_1| \leq \ldots \leq |h_n|$



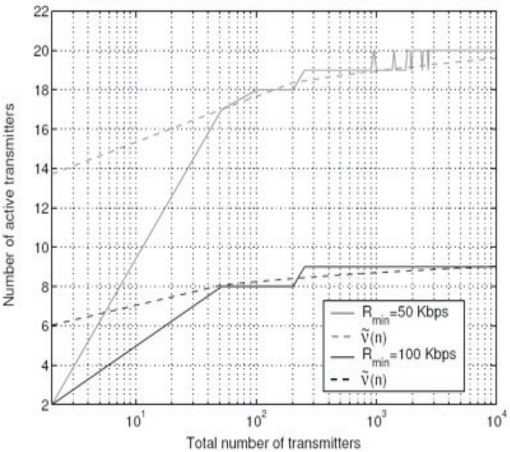
- Assume channel gains $h_i, i = 1,\ldots,n$ are independent realizations of the complex Gaussian distribution. $|h_i|^2; i = 1,\ldots,n$ are independent realizations of the gamma distribution.

- **Theorem 2:** Under the assumption of independent Rayleigh fading channels for different transmitters and for any $\varepsilon > 0$, the maximum number of active transmitters is bounded as:

$$\mathbb{P}\left(\lfloor\tilde{\nu}(n) - \epsilon\rfloor \leq M_n \leq \tilde{\nu}(n) + \epsilon\right) \to 1$$

$$\tilde{\nu}(n) = \frac{1}{R_{min}} \ln\left(\frac{P\tilde{\nu}(n)}{\sigma^2} \ln n\right)$$

Consider a system with $|h_i|^2 \sim \chi_2^2$, $\sigma^2 = 1$, SNR=20 dB at each transmitter, and bandwidth of 50K sample/sec.



Combining BC and MAC Results

- Theorem 1 and theorem 2 can be combined together and form a single theorem. Based on the results obtained for BC and MAC, the maximum number of active users in multi-user Rayleigh fading channels is arbitrarily close to

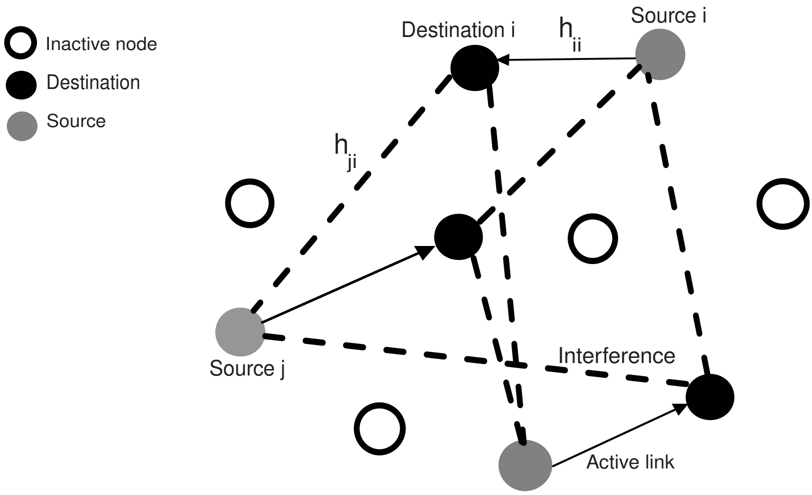
$$\ln\left(\frac{P_{total}}{\sigma^2}\ln n\right)/R_{min}$$

with probability approaching one as $n \to \infty$, where P_{total} denotes total transmitted power in the system.

Discussion

- Scaling allows us to study typical behavior of systems.
- Larger the number of users the better the concentration **but** independence assumption is less likely to hold. Channel gains are correlated.
- Essentially related to the fact $\max\{X_1, X_2, \cdots, X_n\}$ where X_i are i.i.d. exponential (light tailed) concentrates at $\ln n$

Number of Simultaneously Transmitting Nodes



Problem Formulation

$$Y_i(t) = h_{ii}(t)\, X_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{m} h_{ji}(t)\, X_j(t) + Z_i(t)$$

where $h_{ii}(t)$ denotes the link fading channel between transmitter i and receiver i for $t_i \in \{1, \ldots, m\}$ and m refers to the number of active transmitters that are simultaneously transmitting, $h_{ji}(t)$ represents an interference channel for receiver i, and $Z_i(t) \sim \mathcal{CN}(0, \sigma^2)$ represents background noise at node i.

The achievable rate (in nats) of link i can be thus be written as

$$R_i \leq \ln \left(1 + \frac{P\left|h_{ii}\right|^2}{\sigma^2 + \sum\limits_{\substack{j=1 \\ j \neq i}}^{m} P\left|h_{ji}\right|^2} \right)$$

Problem Formulation

Problem Formulation

Define:

$$\mathcal{A} = \{Links\ i : \log(SINR_i > R_{min}\}$$

we want to solve

$$\begin{array}{l} \max_{\mathcal{A} \subset \mathbb{N}_n} |\mathcal{A}| \\ \text{s.t. } R_i > R_{min} \text{ for } i \in \mathcal{A} \end{array} \quad (2.1)$$

i.e., #of maximum links supporting the minimum rate R_{min}

Main Result

Theorem 1

Under the assumption of independent Rayleigh fading channels for different source-destination pairs with channel gains $h_{t_i r_j} \sim \mathcal{CN}(0,1)$*;* $i, j = 1, \ldots, n$*, and the typical behavior of the maximum number active links,* M_n*, determined by* R_{min} *is bounded as*

$$\lim_{n \to \infty} \mathbb{P}\left(\lfloor \beta_1(n) \rfloor \leq M_n \leq \beta_2(n)\right) = 1,$$

where

$$\beta_1(n) = \left(\frac{c_1 \ln(n)}{e^{R_{\min}}}\right)^2$$

$$\beta_2(n) = \left(\frac{c_2 \ln(n)}{e^{R_{\min}}}\right)^2$$

and $0 < c_1 < 1$ *and* $1 < c_2 < \infty$ *are any arbitrary constants.*

Then the idea is to show that on defining the r.v. X_n as:

$$X_n \triangleq \ln \left(1 + \frac{P\left|h_{ii}\right|^2 \mathbf{1}_{[|h_{ii}|^2 \geq h_0]}}{\sigma^2 + \sum_{j \in \mathsf{A}, j \neq i} P\left|h_{ji}\right|^2 \mathbf{1}_{[|h_{jj}|^2 \geq h_0]}} \right)$$

where $\mathbf{1}_A(\omega)$ denotes the indicator function of event A (i.e. equals one if $\omega \in A$).

For any large integer m and $\varepsilon > 0$ small, define event V_n as

$$\mathcal{V}_n \triangleq \left\{ \omega : \left| \frac{1}{m-1} \sum_{\substack{k=1 \\ k \neq i}}^{m} (\left|h_{ki}\right|^2 \mathbf{1}_{[|h_{kk}|^2 \geq h_0]}) - \mathsf{E}\left(\left|h_{11}\right|^2\right) p_0 \right| < \varepsilon \right\}.$$

$$X_n = X_n \mathbf{1}_{\mathcal{V}_n} + X_n \mathbf{1}_{\mathcal{V}_n^c}$$

On V_n we have the required result and we can show that $X_n \mathcal{V}_n^c \to 0$ as $n \to \infty$

Idea of Proof

Definitions

- Let $h_0 = m(n)^{\frac{1}{2}} e^{R_{min}}$ where $m(n)$ is an integer function of network size n and

$$\xi_i = \begin{cases} 1 & \text{if} \ |h_{ii}|^2 \geq h_0 \\ 0 & \text{if} \ |h_{ii}|^2 < h_0 \end{cases}$$

  then $M_n = \sum_{i=1}^{n} \xi_i$ counts the number of *good* links.
- Let also $p_0 = \mathbb{P}(\xi_i = 1)$ then

$$\begin{aligned} p_0 &= \exp(-h_0) \\ &\sim n^{-c_1} \end{aligned}$$

  for $m(n) \sim \beta_1(n) = \left(c_1 \log(n)/e^{R_{min}}\right)^2$

Proof

Proof of First Part

Using Chernoff bound for a binomial $B(n, p_0)$ rv:

$$\lim_{n\to\infty} \mathbb{P}(M_n \geq m) \geq \lim_{n\to\infty} \left(1 - \exp\left(-\frac{(np_0 - m + 1)^2}{2np_0}\right)\right) = 1 \quad (2.2)$$

note that we have

- $np_0 \sim n^{1-c_1}$ where $c_1 < 1$

Let A_n denote the set of transmitters that transmit simultaneously with i.

Proof

Stochastic Rate

Define the *stochastic rate* $X_{i,n}$ of link i in the network of n communication links by

$$X_{i,n} = \ln\left(1 + \frac{P|h_{ii}|^2 \mathbf{1}_{[|h_{ii}|^2 > h_0]}}{\sigma^2 + \sum_{j \in \mathcal{A}_n, j \neq i} |h_{ji}|^2 \mathbf{1}_{[|h_{jj}|^2 > h_0]}}\right) \tag{2.3}$$

we want to show that $X_{i,n} > R_{min}$ for all $i \in \mathcal{A}_n$.
Let $\mathcal{V}^{\varepsilon}_{i,n} \subset \Omega$ such SLLN holds.
Then by Cramer's theorem

$$\mathbf{P}(\mathcal{V}_n^c) \leq e^{-nI(\varepsilon)}$$

Lemma 2

The rate function of the random variable $|hii|^2 \mathbf{1}_{[|h_{ji}|^2 > h_0]}$ *is determined by*

$$I(\epsilon) = (\sqrt{\epsilon} - \sqrt{p_0})^2 \tag{2.4}$$

for x *satisfying* $p_0(1 - \sqrt{p_0/\epsilon}) \ll \sqrt{p_0/\epsilon}$ *where* $p_0 = \mathbb{P}(|hji|^2 > h_0)$

corollary

For $\epsilon \sim (\ln(n)^{-1})$ which satisfies the constraint, we have

$$I(\epsilon) \sim 1/\log(n) = m^{-0.5}$$

For

- $\omega \in \mathcal{V}_{i,n}^{\epsilon}$
- $i \in \mathcal{A}$

stochastic rate of link i

$$\begin{aligned} X_{i,n}(\omega) &= \ln\left(1 + \frac{P|h_{ii}|^2 \mathbf{1}_{[|h_{ii}|^2 > h_0]}}{\sigma^2 + \sum_{j \in \mathcal{A}, j \neq i} |h_{ji}|^2 \mathbf{1}_{[|h_{ji}|^2 > h_0]}}\right) \\ &\overset{a}{\geq} \ln\left(1 + \frac{Ph_0}{\sigma^2 + P(m-1)(p_0 + \epsilon)}\right) \\ &= \ln\left(1 + \frac{Ph_0}{\sigma^2 + P(m-1)(e^{-h_0} + \epsilon)}\right) \\ &\overset{b}{\sim} \ln\left(1 + \frac{Pm^{0.5} e^{R_{min}}}{P(m-1)m^{-0.5}}\right) \\ &\sim \ln\left(1 + e^{R_{min}}\right) \geq R_{min} \end{aligned}$$

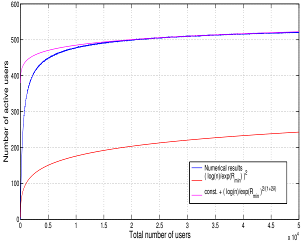
A direct consequence of this along with Cramer's theorem is

$$\mathbb{P}(X_{i,n} < R_{min}) < \mathbb{P}((\mathcal{V}^{\epsilon}_{i,n})^c) < e^{-(m-1)I(\epsilon)} \qquad (2.5)$$

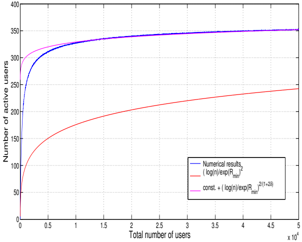
Therefore

$$\begin{aligned}
\mathbb{P}(X_{i,n} < R_{min}, \text{ for some } i \in \mathcal{A}) &\leq \sum_{i \in \mathcal{A}} \mathbb{P}(X_{i,n} < R_{min}) \\
&\overset{a}{\sim} \ln(n)^2 \mathbb{P}(X_{i_0,n} < R_{min}, i_0 \in \mathcal{A}) \\
&\overset{b}{\sim} \ln(n)^2 e^{-(m-1)I(\epsilon)} \\
&\overset{c}{\sim} \ln(n)^2 e^{-(\ln(n)^2)/\ln(n)} \\
&\sim \ln(n)^2/n \to 0
\end{aligned}$$

Simulation Results



(a) Active links for $R_{min} = 100kbps$



(b) Active links for $R_{min} = 150kbps$

Figure: Number of active links vs. total number for different minimum rates

Random Networks with Pathloss

Here the channel gains for $j \neq i$ are given by:

$$h_{ij} = X_{ij} D_{ij}^{-\alpha}$$

where X_{ij} is $\exp(1)$ and D_{ij} is a light tailed r.v. (tail is exponential). For example if nodes are poisson distributed nearest neighbours are exponentially distributed, 2-nd nearest are Erlang (2), etc.,

In this case we can show the following:

Let h, X, D be random variables having the same distribution as h_{ij}, X_{ij} and D_{ij} respectively. We will show that

$$\mathbb{P}\left(h > z\right) \sim \frac{c_1}{z^{1/\alpha}}$$

for some constant c_1 dependent only on the network parameters α and λ and to be determined shortly.

In other words h_{ij} is heavy tailed **Pareto** like.

Main Result

Theorem 3

For the pathloss model with randomly placed nodes and Rayleigh fading:

$$\mathbb{P}(M_n = O(n^{\frac{1}{3}})) \sim 1 \ as \ n \ \to \infty$$

Key challenge: SLLN does not apply for heavy-tailed r.v's.
Moreover $\max\{X_1, X_2, \ldots, , X_n\} = n^{-\frac{1}{\alpha}}$.

Idea of proof

Even if X is not integrable, $\exists p <$ s.t. $|X|^p$ is integrable.

Theorem 4

Suppose that $p \in (0,2)$ and $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. random variables. The SLLN for heavytailed r.v's:

$$n^{-1/p}(S_n - an) \to 0, \text{ a.s.} \tag{2.6}$$

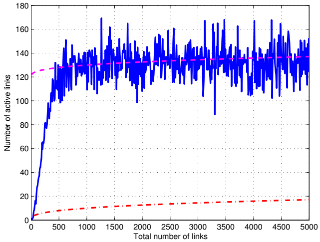
holds for some real constant a if and only if $E|X|^p < \infty$. If $\{X_n\}$ obeys the SLLN then we can choose

$$a = 0 \qquad if \qquad p < 1 \tag{2.7}$$

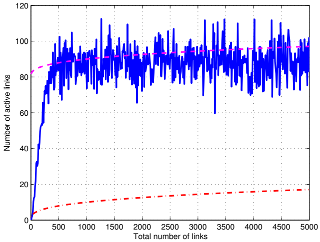
Principle of single large value

$$\mathbb{P}(X_1 + X_2 + \cdots + X_n > x) \sim \mathbb{P}(\max_{1 \le k \le n} > x) \sim n(1 - F(x))^n$$

Simulation Results



(a) Active links vs. total for $R_{min} = 100kbps$



(b) Active links vs. total for $R_{min} = 150kbps$

Figure: Number of active links vs. total number for different minimum rates

Discussion

- Scaling allows us to study typical behavior of systems.
- Larger the number of users the better the concentration **but** independence assumption is less likely to hold. Channel gains are correlated.
- Essentially related to the fact $\max\{X_1, X_2, \cdots, X_n\}$ where X_i are i.i.d. exponential (light tailed) concentrates at $\ln n$

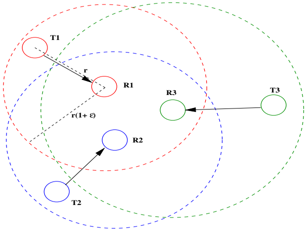
Caution !: Scaling needs to be treated with care! Static Wireless Networks

- Random ad hoc network model [Gupta and Kumar, IEEE Trans IT 00]
  - Consider n nodes distributed within a disk of unit area
  - Average distance between source-destination of a bit is $O(1)$ - independent of n
  - Perfect links, muti-hop communication
  - Point-to-point communication - i.e., network layer notion of capacity
- **Main Result:** The per-node capacity scales as $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$ under the Protocol Model

Protocol Model

- Protocol Model of Gupta and Kumar (IEEE IT 2000) Node i can communicate directly with node j at a rate of W bits per second at time t, provided for all nodes k that are simultaneously transmitting, we have

$$d(X_t^k, X_t^j) \geq (1+\epsilon) d(X_t^i, X_t^j)$$



Model behavior

- n nodes are uniformly distributed over a given area.
- Density $\frac{n}{Area}$.
- Scaling involves $n \to \infty$ implying nodes get closer and closer.
- For large n pathloss does not play a role since $d_{ij}(n) \to 0$.

Static Wireless Networks (Contd.)

- Why does per-node capacity scale as $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$?
- The following two factors reduce the per-node throughput:
  - Multiple relaying: Each packet on an average goes through $\Theta(1/r_n)$ hops, where r_n is the average per-hop distance.
  - Interference: $\Theta(nr_n^2)$ nodes in the vicinity of the receiver must remain silent during the transmission.
- Overall throughput is thus $\Theta(1/nr_n)$
- r_n must at least be $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$ for a.s. connectivity

Static ad hoc networks - lossy links

- In G-K model: links are ideal - either each transmission is successful w.p.1 or infinite retransmissions
- In reality link success prob. is an increasing function of SINR, i.e. $p = f(SINR)$ with $p \to 1$ as $SINR \to \infty$.
- Main result [Mhatre, Mazumdar, and Rosenberg (IEEE IT 2009)]
- Finite retransmissions, SINR $< \infty \implies$ Capacity $\sim O(\frac{1}{n})$
- K_n re-transmissions/decrease in spatial reuse by a factor of K_n ($K_n \to \infty$) $\implies$ Capacity $\sim O(\frac{1}{K_n \sqrt{n \log n}})$

Why is this so?

- Typical number of hops = $\Theta\left(\sqrt{\frac{n}{\log n}}\right)$
- If p is prob of success on a link, then end-to-end $p^{\sqrt{\frac{n}{\log n}}} = O\left(\sqrt{\frac{\log n}{n}}\right)$ as $n \to \infty$
- Hence combining: we obtain $O\left(\frac{1}{n}\right)$

Main difficulty: to show there are a sufficient number of links on which loss probability is positive.

Discussion

What result should we use to guide us in a practical setting? It turns out in these type of scaling results there is no robustness because the limit model does not inherit the macroscopic properties of the physical model.
Further problems:

- Results do not hold when node placement is non-uniform (Han+ Makowski JSAC 2009)
- There are no monotonicity results and so is the result an asymptotic equivalence, lower or upperbound on the capacity?
- Clearly one would not expect to see G-K behavior in practical systems as any ARQ procedure only allows a finite number of retransmissions. Also in reality the hops are bounded so the other result is also questionable.

Concluding Remarks

- Discussed different types of scaling, when it works and when it is just a formal procedure.
- Scaling works well when there is an underlying invariance principle at play- heavy traffic, many users involving measure convolution or independence and concentration
- Less useful when there are no attendant monotonicity results as in G-K setting or even random mobility models (for example BM is very different from random waypoint).
- We need to move beyond scaling and try to show model continuity and monotonicity.

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