On Hegselmann-Krause Dynamics

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Hegselmann-Krause Opinion Dynamics

- Studied by R. Hegselmann and U. Krause for modeling opinion formation within a group of interacting agents
- The model is due to Friedkin and Johnsen (1990), who proposed a more general model and studied some of its special cases
- A gossip-type version of the model was studied by Deffuant, Neau, Amblard, and Weisbuch (2000, 2002)
- From the dynamic point of view, the model is **averaging dynamic**, which had been studied in various forms and used in many other applications:
  - Control: coordination of network of robots - Bullo, Cortés and Martnez (2009)
  - Signal processing: distributed in network estimation and tracking
  - Optimization and Learning: distributed multi-agent optimization over network, distributed resource allocation, distributed learning, data-mining
  - Social/Economical Behavior: spread of influence, emergent behavior, polarization

Hegselmann-Krause Model for Opinion Dynamics

- Discrete-time averaging dynamic model for interactions of m agents:
  - The set $[m] = \{1, 2, \ldots, m\}$ represents the agents
  - The agents interact at discrete time instances, indexed by $t = 0, 1, 2, 3, \ldots$
- The dynamics is specified by an **initial opinion profile** $\{x_i(0) \in \mathbb{R}^n,\ i \in [m]\}$ and the **bounded confidence** ϵ which limits the agents' interactions
  - At time t, the opinion of agent i is given by a vector $x_i(t) \in \mathbb{R}^n$
  - The neighbors of agent i are defined in terms of the confidence ϵ, as follows

$$N_i(t) = \{j \in [m] \mid \|x_j(t) - x_i(t)\| \leq \epsilon\}$$

    where $\|\cdot\|$ is Euclidean norm (other can be used)
  - Each agent updates his opinion by averaging the opinions of his neighbors

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) \qquad \text{for } t = 0, 1, 2, \ldots$$

    where $|S|$ stands for the cardinality of a set S.

Literature Overview

- Initial study by Hegselmann and Krause 2002 (mainly simulations)[*] for a scalar case
- The stability of multi-dimensional model thoroughly studied in the thesis work of Lorenz 2007 (matrix analysis; finite time convergence established)[†]
- Convergence time for the scalar case of the order $O(m^4)$, shown by Touri 2011[‡]
- The continuous time scalar model studied by Blondel, Hendrickx, and Tsitsiklis 2009[§]
- The model can be approached from **dynamical system** point of view
- We provide an alternative stability analysis based on dynamic system tools:
  - Construct **Lyapunov function** decreasing along the trajectories
  - The construction relies on the use of **adjoint dynamics**

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[*] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," Journal of Artificial Societies and Social Simulation, vol. 5, 2002

[†] J. Lorenz, "Repeated averaging and bounded confidence modeling, analysis and simulation of continuous opinion dynamics," Ph.D. Dissertation, Universität Bremen, 2007

[‡] B. Touri "Product of random stochastic matrices and distributed averaging," Ph.D. Thesis UIUC, 2011; Asilomar 2011

[§] V. Blondel, J. Hendrickx, and J. Tsitsiklis, "On Krause's multiagent consensus model with state-dependent connectivity," IEEE Transactions on Automatic Control, vol. 54, no. 11, pp. 2586–2597, 2009

Results on Hegselmann-Krause Model

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t), \qquad N_i(t) = \{j \in [m] \mid \|x_j(t) - x_i(t)\| \leq \epsilon\}$$

- For any initial profile $\{x_i(0), i \in [m]\}$ and any confidence interval ϵ, the opinions $x_i(t)$, $i \in [m]$ converge to some limiting opinion x_i^* in a finite time, i.e., for each agent i, there exists time $T_i \geq 0$ such that

$$x_i(t) = x_i^* \qquad \text{for all } t \geq T_i$$

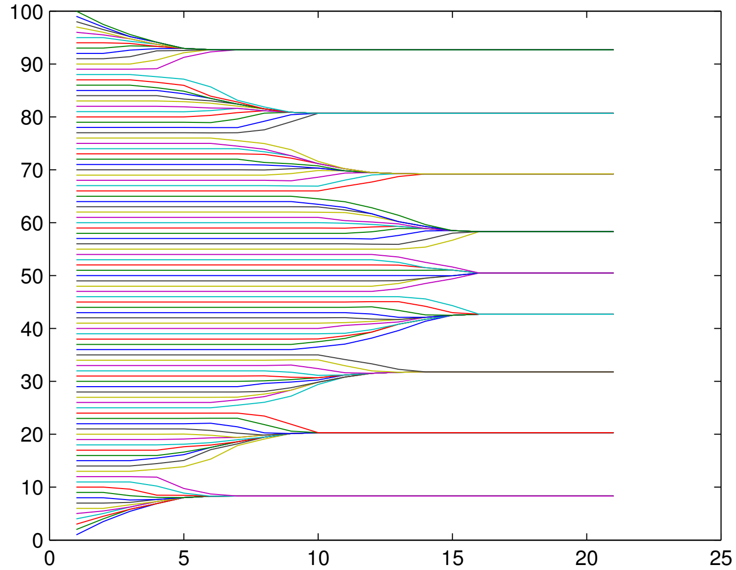
- The fragmentation of the agents typically occurs i.e., the agent groups are formed with different opinions

$$x_i^* \neq x_j^* \qquad \text{when } i \text{ and } j \text{ are not in the same group}$$
$$x_i^* = x_j^* \qquad \text{when } i \text{ and } j \text{ are in the same group}$$

- **For a scalar case**, the best known bound on the termination time is $O(m^4)$ uniformly across all initial profiles, and $O(m^2)$ for a fixed initial profile
- In multidimensional case some results show $O(m^6)$ uniformly over all initial profiles¶
- The difficulty in analyzing the dynamic is state-dependency (quasi-linear dynamics)

¶S.R. Etesami, T. Basar, A. Nedić, B. Touri "Termination Time of Multidimensional Hegselmann-Krause Opinion Dynamics," to appear in Proceedings of ACC 2013

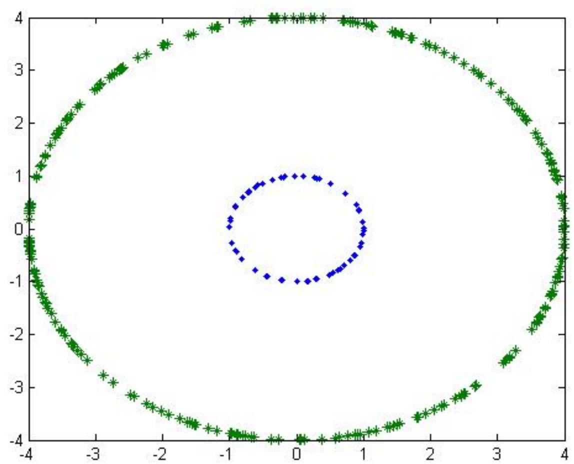
Some Illustrations of the Opinion Dynamics

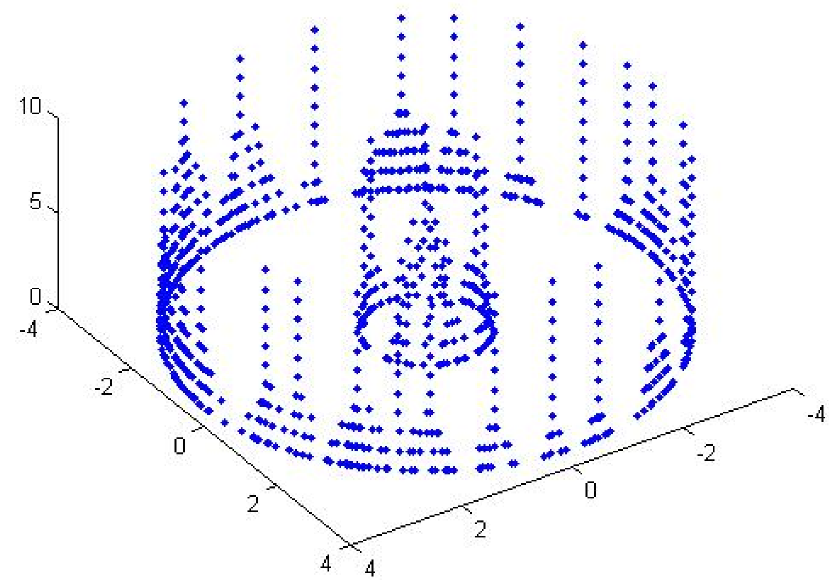


$m = 100$ and $\epsilon = 5$

- 9 groups in the limit, convergence in 16 time steps
- In a scalar case, **the order of the initial profile is preserved in time**
- The connected groups of **agents can only split** into two or more connected groups

What Complicates Multi-dimensional Case?

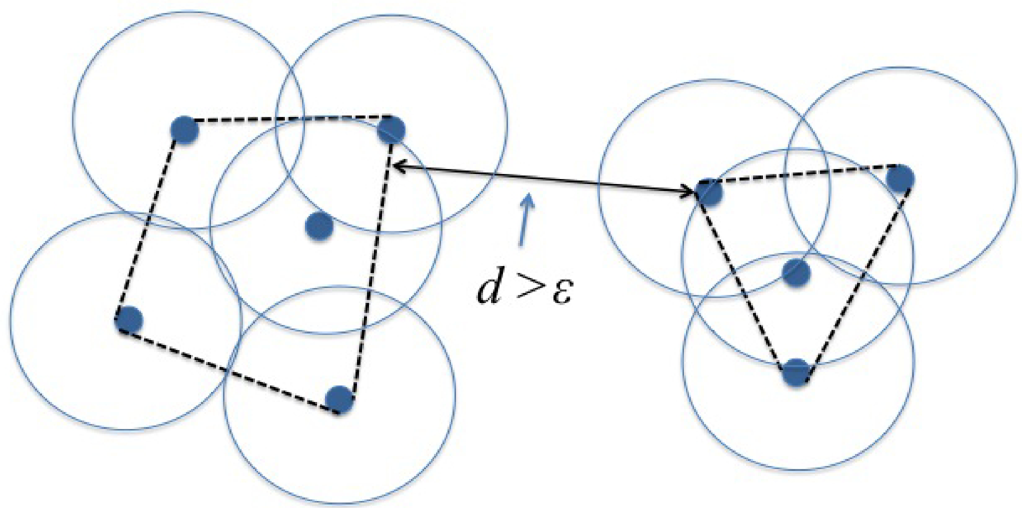




Connected groups can split and merge!

Isolated Agent Groups

Let $S_1, S_2 \subseteq [m]$ be two groups of agents.
We say that the groups S_1 and S_2 are **isolated from each other** if the convex hull of $\{x_i(t), i \in S_1\}$ and the convex hull of $\{x_j(t), j \in S_2\}$ are at the distance greater than ϵ.



An illustration of two isolated agent groups in a plane

- **Lemma:** Two isolated agent groups will remain isolated at all times
- The result allows us to focus on a **single isolated group**
- There is no need to actually ever find these groups - just a concept

Local Opinion Spread

- Local opinion classes: large and small - relatively speaking

$$L(t) = \{i \in [m] \mid \max_{j \in N_i(t)} \|x_j(t) - x_i(t)\| > \epsilon/3\}$$ **Large opinion spread**

$$S(t) = \{i \in [m] \mid \max_{j \in N_i(t)} \|x_j(t) - x_i(t)\| \leq \epsilon/3\}$$ **Small opinion spread**

- If an agent and all his neighbors have small local-opinion spread, then the agent and his neighbors will reach an agreement in the next time step.

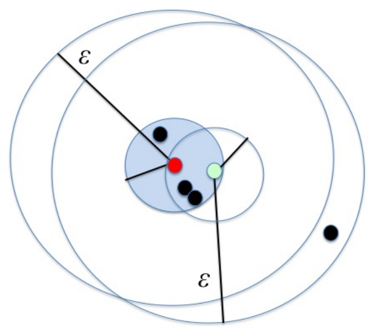
Left An agent and all his neighbors have small local-opinion spread, which leads to their agreement at next time step. **Right** An agent with a small local-opinion spread, with a neighbor whose local-opinion spread is large.

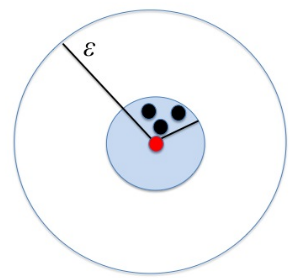
Lemma 1 *There holds for all* $t \geq 0$,

(a) $N_i(t) \subseteq N_j(t)$ *for all* $i \in S(t)$ *and all* $j \in N_i(t)$.

(b) $N_j(t) = N_i(t)$ *for all* $i \in S(t)$ *and all* $j \in N_i(t) \cap S(t)$.

(c) *For every* $i, \ell \in S(t)$, *either* $N_i(t) = N_\ell(t)$ *or* $N_i(t) \cap N_\ell(t) = \emptyset$.





Left: an agent in $S(t)$ with a neighbor in $L(t)$ - part (a);

Right: an agent in $S(t)$ with all neighbors in $S(t)$ - parts (b) and (c)

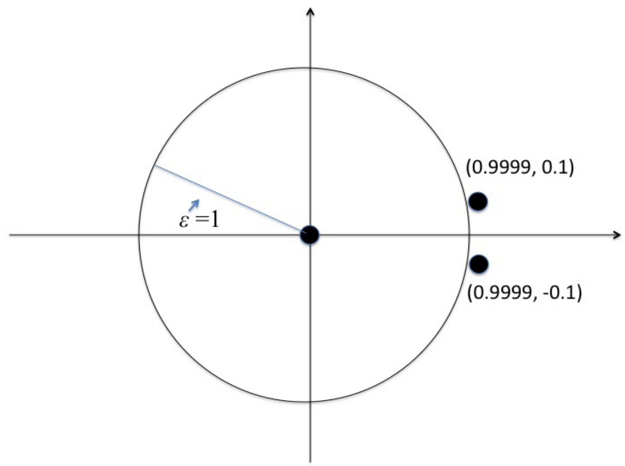
A consequence of Lemma 1(b), for $i \in S(t)$ with $N_i(t) \subseteq S(t)$, we have

$$x_i(t+1) = x_j(t+1) \qquad \text{for all } j \in N_i(t).$$

Important Relation

Agent type can also change in time.

Figure shows 3 agents, all with small opinion spread (all in $S(t)$) at time $t = 0$, but in the next time step all have large opinion spread (all in $L(t)$).



Lemma 2 [**S-agents**] *Let $i \in S(t)$ with $N_i(t) \subseteq S(t)$. Then, at time $t+1$:*

$$\text{either} \qquad N_i(t+1) = N_i(t) \qquad \text{or} \qquad N_i(t+1) \supset N_i(t).$$

In the second case: if there is an agent $\ell \in S(t)$ such that $\ell \in N_i(t+1) \setminus N_i(t)$, then $\ell \in N_j(t+1)$ for all $j \in N_i(t)$ and

$$\|x_j(t+1) - x_\ell(t+1)\| > \frac{\epsilon}{3} \qquad \text{for all } j \in N_i(t).$$

Termination Criterion

- The termination time T of the HK dynamics is defined by

$$T = \inf_{t\geq 0}\{t \mid x_i(k+1) = x_i(k) \text{ for all } k \geq t \text{ and all } i \in [m]\},$$

Proposition 1 [**Termination Criterion**]

Suppose that the time $\hat{t}$ is such that $S(\hat{t}) = [m]$ and $S(\hat{t}+1) = [m]$. Then, we have

$$N_i(\hat{t}+1) = N_i(\hat{t}) \qquad \text{for all } i \in [m],$$

and the termination time T of the Hegselmann-Krause dynamics satisfies $T \leq \hat{t}+1$.

Since $S(\hat{t}) = [m]$, by Lemma 2 (S-agents), it follows that either $N_i(\hat{t}+1) = N_i(\hat{t})$ for all $i \in [m]$, or $N_i(\hat{t}+1) \supset N_i(\hat{t})$ for some $i \in [m]$. We show that the latter case cannot occur. Specifically, every $\ell \in N_i(\hat{t}+1) \setminus N_i(\hat{t})$ must belong to the set $S(\hat{t})$ since $L(\hat{t}) = \emptyset$. Therefore, by Lemma 2 it follows that $\|x_j(\hat{t}+1) - x_\ell(\hat{t}+1)\| > \frac{\epsilon}{3}$ for all $j \in N_i(\hat{t})$. In particular, this implies $i, \ell \in L(\hat{t}+1)$ which is a contradiction since $L(\hat{t}+1) = \emptyset$. Therefore, we must have $N_i(\hat{t}+1) = N_i(\hat{t})$ for all i, which means that each agent group has reached a local group-agreement and this agreement persists at time $\hat{t}+1$ and onward.

Adjoint Dynamics

We next show that the Hegselmann-Krause dynamics has an adjoint dynamics. To do so, we first compactly represent the dynamics by defining the neighbor-interaction matrix $B(t)$ with the entries given as follows:

$$B_{ij}(t) = \begin{cases} \frac{1}{|N_i(t)|} & \text{if } j \in N_i(t), \\ 0 & \text{otherwise.} \end{cases}$$

The HK dynamics can now be written as:

$$X(t+1) = B(t)X(t) \qquad \text{for all } t \geq 0, \tag{1}$$

where $X(t)$ is the $m \times n$ matrix with the rows given by the opinion vectors $x_1(t), \ldots, x_m(t)$.

- We say that the Hegselmann-Krause dynamics has an *adjoint dynamics* if there exists a sequence of probability vectors $\{\pi(t)\} \subset \mathbb{R}^m$ such that

$$\pi'(t) = \pi'(t+1)B(t) \qquad \text{for all } t \geq 0.$$

- We say that an adjoint dynamics is uniformly bounded away from zero if there exists a scalar $p^* \in (0,1)$ such that $\pi_i(t) \geq p^*$ for all $i \in [m]$ and $t \geq 0$.

Existence of Adjoint Dynamics for HK model

To show the existence of the adjoint dynamics, we use a variant of Theorem 4.8, Touri 2011[‖]. The result requires balancedness property[**]

Theorem 2 *Let $\{A(t)\}$ be a sequence of stochastic matrices such that:*

(i) There exists a scalar $\alpha \in (0,1]$ such that for every nonempty $S \subset [m]$ and its complement $\bar{S} = [m] \setminus S$, there holds

$$\sum_{i \in S, j \in \bar{S}} A_{ij}(t) \geq \alpha \sum_{j \in \bar{S}, i \in S} A_{ji}(t) \qquad \text{for all } t \geq 0,$$

(ii) There exists a scalar $\beta \in (0,1)$ such that $A_{ii}(t) \geq \beta$ for all $i \in [m]$ and $t \geq 0$;

Then, the dynamics $z(t+1) = A(t)z(t)$, $t \geq 0$, has an adjoint dynamics $\{\pi(t)\}$ which is uniformly bounded away from zero.

- HK dynamics is driven by stochastic matrices $B(t)$ to which the Theorem applies
- **HK opinion dynamics has an adjoint dynamics which is uniformly bounded away from zero.**

‖ B. Touri "Product of random stochastic matrices and distributed averaging," Ph.D. Thesis UIUC, 2011; Springer Theses Series 2012

** J.M. Hendrickx and J.N. Tsitsiklis "Convergence of type-symmetric and cut-balanced consensus seeking systems" to appear in IEEE Transactions on Automatic Control

Lyapunov Comparison Function

We construct a Lyapunov comparison function by using the adjoint HK dynamics. The comparison function for the dynamics is a function $V(t)$, which is defined for the $m \times n$ opinion matrix $X(t)$ and $t \geq 0$:

$$V(t) = \sum_{i=1}^{m} \pi_i(t) \| x_i(t) - \pi'(t) X(t) \|^2, \tag{2}$$

where the row $X_{i,:}(t)$ is given by the opinion vector $x_i(t)$ of agent i and $\pi(t)$ is the adjoint dynamics. For this function, we have following **essential relation**.

Lemma 3 *For any* $t \geq 0$, *we have*

$$V(t+1) = V(t) - D(t)$$

where

$$D(t) = \frac{1}{2} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|^2} \sum_{i \in N_\ell(t)} \sum_{j \in N_\ell(t)} \| x_i(t) - x_j(t) \|^2.$$

- Proof relies on the strong convexity property of the Euclidean norm (and its exact second order expansion gives "=")

Finite Termination Time

Proposition 3 *The Hegselmann-Krause opinion dynamics* (1) *reaches its steady state in a finite time.*

By summing the relation of essential Lemma for $t = 0, 1, \ldots, \tau - 1$ for some $\tau \geq 1$, and by rearranging the terms, we obtain

$$V(\tau) + \frac{1}{2} \sum_{t=0}^{\tau-1} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|^2} \sum_{i \in N_\ell(t)} \sum_{j \in N_\ell(t)} \|x_i(t) - x_j(t)\|^2 = V(0).$$

Letting $\tau \to \infty$, since $V(\tau) \geq 0$ for any τ, we conclude that

$$\lim_{t \to \infty} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|^2} \sum_{i \in N_\ell(t)} \sum_{j \in N_\ell(t)} \|x_i(t) - x_j(t)\|^2 = 0.$$

By Theorem 2 (existence of adjoint dynamics) the adjoint dynamics $\{\pi(t\}$ is uniformly bounded away from zero, i.e., $\pi_\ell(t) \geq p^*$ for some $p^* > 0$, and for all $\ell \in [m]$ and $t \geq 0$. Furthermore, $|N_\ell(t)| \leq m$ for all $\ell \in [m]$ and $t \geq 0$, Therefore, it follows that for every $\ell \in [m]$,

$$\lim_{t \to \infty} \sum_{i \in N_\ell(t)} \sum_{j \in N_\ell(t)} \|x_i(t) - x_j(t)\|^2 = 0.$$

We further have

$$\sum_{i\in N_\ell(t)}\sum_{j\in N_\ell(t)}\|x_i(t)-x_j(t)\|^2 \geq \sum_{i\in N_\ell(t)}\|x_i(t)-x_\ell(t)\|^2 \geq \max_{i\in N_\ell(t)}\|x_i(t)-x_\ell(t)\|^2,$$

where the first inequality follows from the fact $\ell \in N_\ell(t)$ for all $\ell \in [m]$ and $t \geq 0$. Consequently, we obtain for every $\ell \in [m]$,

$$\lim_{t\to\infty}\max_{i\in N_\ell(t)}\|x_i(t)-x_\ell(t)\|^2 = 0.$$

Hence, for every $\ell \in [m]$, there exists a time t_ℓ such that

$$\max_{i\in N_\ell(t)}\|x_i(t)-x_\ell(t)\| \leq \frac{\epsilon}{3} \qquad \text{for all } t \geq t_\ell.$$

In view of the definition of the set $S(t)$ (small-opinion spread agents), the preceding relation implies that

$$S(t) = [m] \qquad \text{for all } t \geq \max_{\ell\in[m]} t_\ell.$$

By Proposition 1 (termination criteria), it follows that the termination time T satisfies $T \leq \max_{\ell\in[m]} t_\ell + 1$.

△ **Multidimensional HK dynamics converges in a finite time**

Upper-Bound for Convergence Time: Scalar Case

- Lyapunov function decrease:

$$V(\tau) + \frac{1}{2}\sum_{t=0}^{\tau-1}\sum_{\ell=1}^{m}\frac{\pi_\ell(t+1)}{|N_\ell(t)|^2}\sum_{i\in N_\ell(t)}\sum_{j\in N_\ell(t)}(x_i(t)-x_j(t))^2 = V(0)$$

- Provide a lower bound bound for

$$D(t) = \frac{1}{2}\sum_{\ell=1}^{m}\frac{\pi_\ell(t+1)}{|N_\ell(t)|^2}\sum_{i\in N_\ell(t)}\sum_{j\in N_\ell(t)}(x_i(t)-x_j(t))^2$$

- The L-agents have "large" opinion spread

$$D(t) \geq \frac{1}{2}\left(\frac{\epsilon}{3}\right)^2\sum_{\ell\in L(t)}\frac{\pi_\ell(t+1)}{|N_\ell(t)|} \geq \frac{1}{2m}\left(\frac{\epsilon}{3}\right)^2\sum_{\ell\in L(t)}\pi_\ell(t+1)$$

- For all $t < T-2$ we have

$$\sum_{\ell\in L(t)}^{m}\pi_\ell(t+1) \geq \frac{1}{2}\frac{1}{\max_{j\in S(t)}|N_j(t)|} \geq \frac{1}{2m}$$

- The preceding relies on the fact that only splitting occurs (i.e., some opinions collapse and are separated from the other opinions, they reach steady state)
- Results in

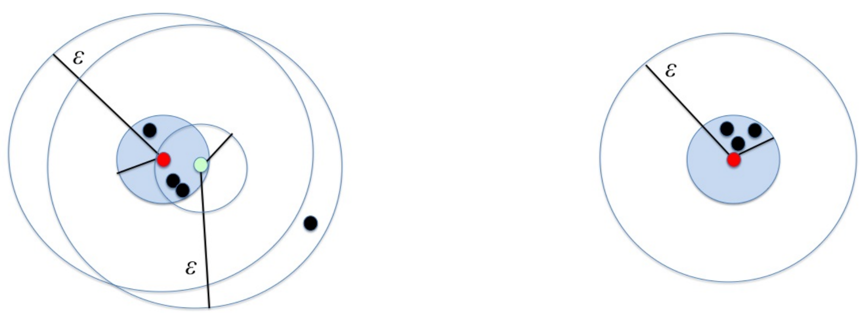
$$T-2 \leq \frac{1}{36m^2}V(0) \qquad \text{with} \qquad V(0) = \sum_{i=1}^{m}\pi_i(0)(x_i(0)-\pi'(0)x(0))^2$$

$V(0) \approx O(\frac{m^2}{\epsilon^2})$

- Fails for multi-dimensional dynamics

Lessons Learned

- Large-opinion spread agents (L-agents) are good for convergence time (fast-mixing)
  - In scalar case: small-opinion spread agents (S-agents) are good convergence time, as long as all of their neighbors are also S-agents;
  - In multi-densional case, they are not always good as they can have "dormant periods" followed by "active periods", where the periods depend on the geometry (inherited from the initial profile and the bounded confidence ϵ)
- Small-opinion spread agents with neighbors in L-agent set are the "bad" agents - they are slowing the dynamics (slow-mixing process)



Left: an agent in $S(t)$ with a neighbor in $L(t)$; **Right**: an agent in $S(t)$ with all neighbors in $S(t)$

Extensions

- Rate for multi-dimensional case - improve the current best upper bound of $O(m^6)$
- Gossip-type opinion mixing is also convergent (rate have not been addressed)
- The agents could be on a graph over which we define neighbors (based on ϵ confidence)
- Other mixing weights can be used