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# Geometrizing Characters of Tori

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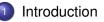
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Character Sheaves



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# Objective

- K a finite extension of  $\mathbb{Q}_p$ ,
- **T** an algebraic torus over K (e.g.  $\mathbb{G}_m$ ),
- $\ell$  a prime different from *p*.

### Goal

Construct "geometric avatars" for characters in

 $\text{Hom}(\textbf{T}(\textbf{\textit{K}}),\overline{\mathbb{Q}}_{\ell}^{\times})$  :

sheaves on some space functorially associated to T.

- Try to push characters forward along maps such as  $\textbf{T} \hookrightarrow \textbf{G};$
- Deligne-Lusztig representations ⇒ character sheaves;
- Give a new perspective on class field theory.

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# Approach

- Generalize the notion of character sheaf from connected, commutative algebraic group to commutative group schemes locally of finite type.
- 2 Associate to **T** a projective system  $\mathfrak{T}$  of commutative group schemes  $\mathfrak{T}_d$  over the residue field *k* of *K*.
- Solution Map from character sheaves on  $\mathfrak{T}$  to characters on  $\mathcal{T}(\mathcal{K})$ .

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# Character Sheaves (G connected)

Two definitions of character sheaves for a connected (commutative) algebraic groups G over k:

#### Definition

- An *ℓ*-adic local system is a constructible sheaf of Q<sub>ℓ</sub>-vector spaces on the étale site of G that becomes trivial after pulling back along a finite étale map H → G.
- A geometric character sheaf on G is an ℓ-adic local system
   *E*° on G equipped with an isomorphism m\**E*° ≅ *E*° ⊠ *E*°,
   where m: G × G → G is multiplication.

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### Character Sheaves 2 (G connected)

#### Definition

Alternatively, a character sheaf on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character  $A \to \overline{\mathbb{Q}}_{\ell}^{\times}$ , so that

- $\bigcirc H \to G \text{ is a finite étale cover,}$
- 2  $\operatorname{Fr}_q$  acts trivially on A.

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## Rationality of character sheaves

Base change to  $\overline{k}$  yields a pair  $(\overline{\mathcal{E}}^{\circ}, \operatorname{Fr}_{\mathcal{E}^{\circ}})$ , where  $\overline{\mathcal{E}}^{\circ}$  is a character sheaf on  $\overline{G}$  and  $\operatorname{Fr}_{\mathcal{E}^{\circ}} : \operatorname{Fr}_{q}^{*}\overline{\mathcal{E}}^{\circ} \xrightarrow{\sim} \overline{\mathcal{E}}^{\circ}$ .

#### Proposition

In general this functor is faithful; when G is connected, base change defines an equivalence of categories

 $\left\{\begin{array}{c} \text{character sheaves} \\ \text{on } G \end{array}\right\} \rightarrow \left\{\text{pairs } (\overline{\mathcal{E}}^{\circ}, \mathsf{Fr}_{\mathcal{E}^{\circ}})\right\}$ 

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### Characters in the connected case

Suppose (*ε*<sup>°</sup>, Fr<sub>ε°</sub>) is a character sheaf on *G*. Define a character χ<sub>ε</sub><sup>°</sup> of *G*(*k*) by

$$\chi^{\circ}_{\mathcal{E}}(\mathbf{x}) = \mathsf{Tr}(\mathsf{Fr}_{\mathcal{E}^{\circ}}, \overline{\mathcal{E}}^{\circ}_{\mathbf{x}})$$

for  $x \in G(k)$ .

 Suppose χ is a character of G(k). Define a character sheaf on G using the Lang isogeny L(x) = x<sup>-1</sup> Fr<sub>q</sub>(x),

$$1 \to G(k) \to G \xrightarrow{L} G \to 1,$$

together with the character  $\chi$  of G(k).

#### Theorem (Deligne, SGA 4.5)

The maps defined above are mutually inverse isomorphisms between character sheaves on G and  $\text{Hom}(G(k), \overline{\mathbb{Q}}_{\ell}^{\times})$ .

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# Character Sheaves (G non-connected)

#### Definition

A character sheaf on *G* is a triple  $\mathcal{E} = (\bar{\mathcal{E}}, \mu, F)$ , where

•  $\overline{\mathcal{E}}$  is a constructible  $\ell$ -adic sheaf on  $\overline{G}$ , locally constant of rank 1 on each connected component;

2 
$$\mu: m^* \overline{\mathcal{E}} \to \overline{\mathcal{E}} \boxtimes \overline{\mathcal{E}}$$
 is an isomorphism of sheaves on  $\overline{\mathcal{G}} \times \overline{\mathcal{G}}$ ;

**③** 
$$F : \operatorname{Fr}_{G}^{*} \overline{\mathcal{E}} \to \overline{\mathcal{E}}$$
 is an isomorphism of sheaves on  $\overline{G}$ .

 $\mu$  and *F* are required to satisfy various compatibility diagrams. We write CS(G) for the category of character sheaves on *G*. (

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### **Trace of Frobenius**

Write  $G(k)^*$  for Hom $(G(k), \overline{\mathbb{Q}}_{\ell}^{\times})$ . For any *G*, trace of Frobenius defines a map

$$\mathcal{C}(G)_{iso} \to G(k)^*.$$

Pullback then gives a diagram

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## Character Sheaves 2 (G non-connected)

In the non-connected case, not every character sheaf can be realized in the second manner.

#### Definition

A bounded character sheaf on G is a short exact sequence

$$1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1$$

together with a character  $A \to \overline{\mathbb{Q}}_{\ell}^{\times}$ , so that

- H → G is a finite étale cover, inducing an isomorphism on component groups
- 2  $\operatorname{Fr}_q$  acts trivially on A.

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### Extending character sheaves

#### Theorem

Every character sheaf on  $G^{\circ}$  extends to a (bounded) character sheaf on G.

#### Proof.

Suppose that  $1 \to A \to H \to G \to 1$  and  $\chi: A \to \overline{\mathbb{Q}}_{\ell}^{\times}$  defines a bounded character sheaf. Suppose that  $\operatorname{Gal}(\overline{k}/k)$  acts on H and G through the finite quotient  $\Gamma$ . Restriction to  $H^{\circ} \to G^{\circ}$  then defines a character sheaf on  $G^{\circ}$ .

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### Extending character sheaves

#### Proof.

On extension classes, this map is the first in

$$\mathsf{Ext}^1_{\mathbb{Z}[\Gamma]}(G,A) \to \mathsf{Ext}^1_{\mathbb{Z}[\Gamma]}(G^{\circ},A) \to \mathsf{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^{\circ},A).$$

Since  $\mathbb{Z}[\Gamma]$  is a product of Dedekind domains it has cohomological dimension 1 and thus  $\operatorname{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^\circ, A)$ vanishes. So  $\operatorname{Ext}^1_{\mathbb{Z}[\Gamma]}(G, A) \to \operatorname{Ext}^1_{\mathbb{Z}[\Gamma]}(G^\circ, A)$  is surjective.

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# Character Sheaves (*G* étale)

- The category of étale group schemes is equivalent to the category of groups with with Galois action.
- A character sheaf on an étale group scheme G is a collection of 1-dimensional Q
  <sub>ℓ</sub>-vectors spaces E<sub>x</sub> for x ∈ G(k̄) together with F<sub>x</sub> : E<sub>Fr(x)</sub> → E<sub>x</sub> and μ<sub>x,y</sub> : E<sub>x</sub> ⊗ E<sub>y</sub> → E<sub>x+y</sub>.

#### Proposition

Suppose that G is an étale commutative group scheme and  $G(\bar{k})$  is finitely generated. Then there is a canonical isomorphism

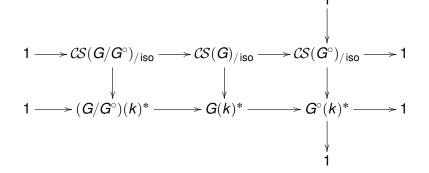
 $\mathcal{CS}(\mathbf{G})_{iso} \cong \mathrm{H}^{1}(\mathbf{W}_{k}, \mathbf{G}(\bar{k})^{*}).$ 

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### Trace of Frobenius Diagram



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# Trace of Frobenius (*G* étale)

#### Theorem

If G is an étale, commutative group scheme with  $G(\bar{k})$  finitely generated then trace of Frobenius is an isomorphism.

#### Corollary

If G is a commutative group scheme with finitely generated component group then trace of Frobenius gives an isomorphism

 $\mathcal{CS}(G)_{iso} \cong G(k)^*.$ 

### Character Sheaves

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# Surjectivity (G étale)

#### Proof.

Since  $\overline{\mathbb{Q}}_{\ell}^{\times}$  is divisible it is injective as a  $\mathbb{Z}$ -module and thus  $\operatorname{Ext}_{\mathbb{Z}}^{1}(G(\bar{k})/G(k), \overline{\mathbb{Q}}_{\ell}^{\times}) = 0$  so restriction

$$G(\bar{k})^* \to G(k)^*$$

is surjective. The map  $\mathcal{CS}(G)_{iso} \cong H^1(W_k, G(\bar{k})^*) \to G(k)^*$  is given by evaluation at Frobenius and restriction.

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# Injectivity (G étale)

#### Proof.

Suppose  $\mathcal{E}$  is in the kernel of trace-of-Frobenius, and suppose  $\phi \in G(\bar{k})^*$  is the image of Frobenius under a corresponding cocycle. By assumption  $\phi(x) = 1$  for  $x \in G(k)$ ; it suffices to construct  $\psi \in G(\bar{k})^*$  with  $\phi(x) = \frac{\psi(x)}{\psi(\operatorname{Fr}(x))}$  for all  $x \in G(\bar{k})$ .

Again, let  $\text{Gal}(\bar{k}/k)$  act through the finite quotient  $\Gamma$ . Since  $\mathbb{Z}[\Gamma]$  is a product of Dedekind domains, any  $\mathbb{Z}[\Gamma]$ -module decomposes as the direct sum of cyclic modules (generated by one element), so we may assume that  $G(\bar{k})$  is a cyclic  $\mathbb{Z}[\Gamma]$ -module with generator *y*, isomorphic to  $\mathbb{Z}[\Gamma]/I$ .

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# Injectivity (G étale)

#### Proof.

Suppose 
$$I = (Fr^d - a_{d-1} Fr^{d-1} - \cdots - a_0)$$
. Choose  $\alpha \in \overline{\mathbb{Q}}_{\ell}^{\times}$  with

$$\phi(\mathsf{Fr}^{d-1}(y)) = \frac{\prod_{i=0}^{d-1} \left( \alpha \prod_{j=0}^{i-1} \phi(\mathsf{Fr}^{j}(y)) \right)^{a_{i}}}{\alpha \prod_{j=0}^{d-2} \phi(\mathsf{Fr}^{j}(y))}$$

Define

$$\psi(\mathsf{Fr}^{i}(\mathbf{y})) = \alpha \prod_{j=0}^{i-1} \phi(\mathsf{Fr}^{j}(\mathbf{y}))$$

for  $0 \le i < d$  and extend by linearity to all of  $G(\bar{k})$ . We have  $\phi(x) = \frac{\psi(x)}{\psi(Fr(x))}$  for  $x = Fr^i(y)$  and thus combinations. And if *I* is non-principal....

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# The Néron model of a torus

- **R** ring of integers of K with uniformizer  $\pi$
- $R_d R/\pi^{d+1}R$
- $T_R$  The Néron model of T: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_{\boldsymbol{R}}(\boldsymbol{R}) = \mathbf{T}(\boldsymbol{K})$$

In the  $\mathbb{G}_m$  case the Néron model is a union of copies of  $\mathbb{G}_m/R$ , glued along the generic fiber.

 $\mathbf{T}_d - \mathbf{T}_R \times_R R_d.$ 

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# Components

• The geometric component group of  $\mathbf{T}_R$  fits in a sequence

 $1 \to \mathsf{H}^1(\mathcal{I}_{\mathcal{K}}, X^*(\mathbf{T}))^* \to \pi_0(\mathbf{T}_R) \to \mathsf{Hom}(X^*(\mathbf{T})^{\mathcal{I}_{\mathcal{K}}}, \mathbb{Z}) \to 1.$ 

- π<sub>0</sub>(**T**<sub>R</sub>) is a constant group scheme after base change to the maximal unramified extension of *K*, but Frobenius may act nontrivially.
- The sequence of commutative *R*-group schemes

$$1 \to \mathbf{T}_R^{\circ} \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1.$$

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# The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathbf{v}_d = \operatorname{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \lim_{\leftarrow} \mathfrak{T}_d.$$

 $\mathbf{\tau}$  is a commutative group scheme over k with

$$\mathbf{T}(\mathbf{k}) = \mathbf{T}(\mathbf{K}).$$

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### Character sheaves on ${f t}$

### We write $CS(\mathbf{T})$ for the projective limit of the categories $CS(\mathbf{T}_d)$ .

#### Theorem

$$T(K)^* \cong \mathcal{CS}(\mathbf{T})_{/iso}$$

and this isomorphism preserves depth.

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### Local class field theory

Suppose that L/K is a totally ramified abelian extension of local fields and we're given a character of Gal(L/K). The Artin reciprocity map gives a character of  $K^{\times}$  vanishing on  $\text{Nm}_{L/K}(L^{\times})$ . We'd like to give a different description of this map, passing through character sheaves. Let  $\mathbf{T} = \mathbb{G}_m$  and  $\boldsymbol{\tau}$  the Greenberg transform of  $\mathbf{T}_R$ .

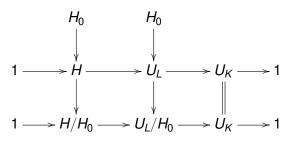
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# An Isogeny

- $U_K$  the connected Néron model of  $\mathbb{G}_m$ .
- $U_L$  the connected Néron model of  $\operatorname{Res}_{L/K} \mathbb{G}_m$ .
  - H the kernel of  $\operatorname{Nm}_{L/K}$ :  $U_L \to U_K$ .
- $H_0$  the subgroup of H generated by  $\frac{\sigma(u)}{u}$  for  $\sigma \in \text{Gal}(L/K)$  and  $u \in U_L$ .



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# A Character of $\mathcal{O}_{K}^{\times}$

The Greenberg transform is exact on commutative algebraic groups, so we get a finite étale cover of  $\mathfrak{T}^{\circ}$ . Write  $\mathfrak{T}_{L}^{\circ}$  for the Greenberg transform of  $U_{L}/H_{0}$ , and note that  $H/H_{0} \cong \text{Gal}(L/K)$ . Then the sequence

$$1 \to \operatorname{Gal}(L/K) \to \mathbf{T}_L^\circ \to \mathbf{T}^\circ \to \mathbf{1},$$

together with a character of Gal(L/K), yields a character sheaf on  $\mathfrak{C}^{\circ}$ . From this character sheaf, we can recover a character of  $\mathcal{O}_{K}^{\times}$ .

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# Local Langlands

- G connected quasi-split reductive group over K
- E splitting field of G
- $\hat{\mathbf{G}}$  dual group over  $\overline{\mathbb{Q}}_{\ell}$
- $^{L}\mathbf{G} \hat{\mathbf{G}} \rtimes \operatorname{Gal}(E/K)$ 
  - $\varphi$  a tame discrete Langlands parameter  $W_K \rightarrow {}^L \mathbf{G}$

A construction of DeBacker and Reeder produces from  $\varphi$  an unramified anisotropic torus **T** in **G** and a depth 0 character  $\chi$  of **T**(*K*). They then describe supercuspidal representations of **G**(*K*) as compact inductions of Deligne-Lusztig representations determined by **T** and  $\chi$ .

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# Geometrizing Local Langlands

In contrast to the Néron model of **T**, there's no canonical integral model of **G**. Instead there are many models, parameterized by the Bruhat-Tits building of **G**. We hope to obtain "representation sheaves" on the Greenberg transforms of these models from character sheaves on  $\mathbf{T}$  by an analogue of Lusztig induction. Ideally, this process would allow

- the generalization of DeBacker and Reeder's methods beyond the depth 0 case,
- better understanding of the functoriality of the local Langlands correspondence,
- new descriptions of L-packets.

Clifton and I are currently pursuing these questions.

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Thank you.