Transport barriers and coherent structures in mean-field Hamiltonian systems

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INTRODUCTION

- Reduced models play a key role in the study of coherent structures and transport.
- The basic idea of these models is to capture some of the essential aspect of the dynamics in a mathematically tractable setting.
- These models provide a useful laboratory to test dynamical systems methods and diagnostics.
- Early studies of chaotic advection were based on pretty simple, yes quite linsightful kinematic models. For example, periodically perturbed, one-degrees-of-freedom Hamiltonian dynamical systems.
- The limitations of simple kinematic models are well-known, e.g., flows in nature and in laboratory experiments are typically not time periodic.
- The incorporation of ad hoc, time dependences is to some degree straightforward (although understanding the consequences of this is highly non-trivial !).

INTRODUCTION

- However, arbitrary, mathematically "sensible", ad hoc spatio-temporal dependences might be physically questionable (e.g., violation of potential vorticity conservation).
- At the heart of this issue is the construction of dynamically consistent transport models. That is, models that respect to some controlled level of approximation the underlying physics.
- Our goal is to develop dynamically consistent models of intermediate complexity between the exact, but "difficult" to study primitive equation models, and the highly approximated but "easy" to understand kinematic models.
- An early example of this approach is the linearly dynamically consistent "Bickley jet model" [del-Castillo-Negrete & Morrison, 1993].
- The goal of this talk is to preset a class of weakly nonlinear dynamically consistent mean-field models.

SELF-CONSISTENT TRANSPORT

- Dynamical consistency is closely related to self-consistent transport.

  - Passive transport: transport of a scalar (or vector) field that does not modify the prescribed advection velocity field

Advection-diffusion equation

$$\partial_t C + \nabla \cdot (\mathbf{V} C) = D \nabla^2 C + S$$

$\mathbf{V}$ independent of C

  - Self-consistent transport: transport of an scalar (or vector) field that actively modifies the prescribed advection velocity field

Self-consistent coupling

$$\mathbf{V} = \Omega[C]$$

- This self-consistent coupling is what makes the advection-diffusion model above dynamically consistent.

SELF-CONSISTENT TRANSPORT

Example: Two-dimensional Hydrodynamics

Vorticity equation:

$$(\partial_t + V\cdot\nabla)\,\zeta = \nu\,\nabla^2\,\zeta$$

Self-consistent vorticity-velocity coupling

$$\zeta = \hat{z}\cdot\nabla\times V$$

Streamfunction formulation:

$$\nabla\cdot V = 0 \quad\Rightarrow\quad V = \hat{z}\times\nabla\psi, \qquad V\cdot\nabla\,\zeta = \{\psi,\zeta\}$$

Self-consistent transport problem

$$\partial_t\,\zeta + \{\psi,\zeta\} = \nu\,\nabla^2\,\zeta$$

$$\psi(x,y,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\; G(x,y;x',y')\,\zeta(x',y',t)$$

SELF-CONSISTENT TRANSPORT

Example: One-dimensional electron dynamics

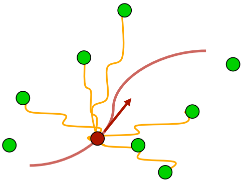
Single-particle electron distribution function:

$$f(x,v,t)$$

Advection-diffusion equation in phase space

$$\partial_t f + u\,\partial_x f + \partial_x \phi\,\partial_u f = \nu\,\partial_u^2 f$$

Vlasov equation



Self-consistent coupling

$$\nabla^2 \phi = \int f\,du - \rho_i$$

Poisson equation

$$H = \frac{u^2}{2} - \phi(x,t)$$

Self-consistent transport problem

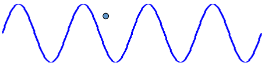
$$\partial_t f + \{H, f\} = \nu \nabla^2 f$$

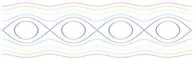
$$\phi(x,t) = \int dx'\,G(x;x') \int du'\,f(x',u',t)$$

SELF-CONSISTENT CHAOTIC TRANSPORT

Integrable motion in a one-wave field

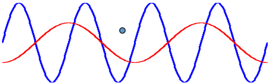
$$\phi = \cos(k_1 x - \omega_1 t)$$

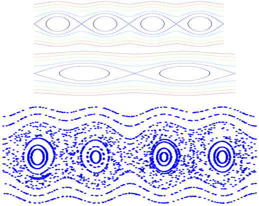




Chaotic motion in a two-waves field

$$\phi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$





How does this well-understood picture change when we take into account self-consistency?

R.T. Pierrehumbert, Geophys. Astrophys. Fluid. Dyn., 58, 285 (1991).
D. del-Castillo-Negrete and P.J. Morrison: Phys. Fluids A, 5, 948 (1993).
J.L. Tennyson, J.D. Meiss, and P.J. Morrison: Phys. D, 71,1, (1994). D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).
J.M. Finn, and D. del-Castillo-Negrete: CHAOS, 11, 4, (2001).

A HIERARCHY OF DYNAMICALLY CONSISTENT MODELS

$$\partial_t C + \mathbf{V} \cdot \nabla C = D \nabla^2 C$$

$$\mathbf{V} = \hat{\mathbf{e}}_z \times \nabla \psi$$

$$\psi(x,z,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \, G(x,z;x',z') \, C(x',z',t)$$

2-D +1 (x,z,t)
Euler type
Self-consistency

$$\mathbf{V} = z \, \hat{\mathbf{e}}_x - \partial_x \Phi \, \hat{\mathbf{e}}_z$$

$$\Phi(x,t) = \int dx' \, K(x;x') \int dz' \, C(x',z',t)$$

1-D +1 (z,t)
Vlasov type
Self-consistency

$$\Phi(x,t) = a(t) \, e^{ix} + a^*(t) \, e^{-ix}$$

$$\frac{da}{dt} = i U a + i \int dx \int dz \, e^{-ix} \, C(x,z,t)$$

0-D +1 (t)
Single wave model
Self-consistency

- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

VLASOV-POISSON REDUCTION OF 2*D* EULER EQUATION

One-dimensional electron dynamics and two-dimensional vortex dynamics

2-D vorticity dynamics ("harder" problem)

$$\partial_t \zeta + \{\psi, \zeta\} = \nu \nabla^2 \zeta \qquad \psi(x,y,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, G(x,y;x',y') \, \zeta(x',y',t)$$

1-D electron dynamics ("simpler" problem)

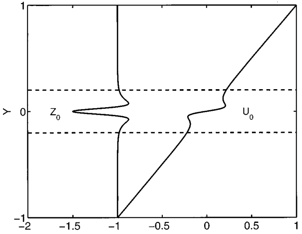
$$\partial_t f + \{H, f\} = \nu \nabla^2 f \qquad \phi(x,t) = \int dx' \, G(x;x') \int du' \, f(x',u',t)$$

- We want to construct a tractable reduced model of self-consistent transport that (under some conditions) applies to vortex dynamics and electron transport

- As a first step: Can we approximate the 2-D vortex dynamics Green's function by a 1-D Green's function (like the one in the electron dynamics)?

VLASOV-POISSON REDUCTION OF 2*D* EULER EQUATION

The vorticity "defect" equation



The vorticity defect approximation assume
a localized small vorticity perturbation
in a strong constant vorticity background

Under these assumptions, a matched
asymptotic expansion leads to the
reduced streamfunction

$$\psi(x,y,t) \quad \rightarrow \quad \psi = -\frac{y^2}{2} + B(x,t)$$

vorticity "defect" dynamics ("simpler" problem)

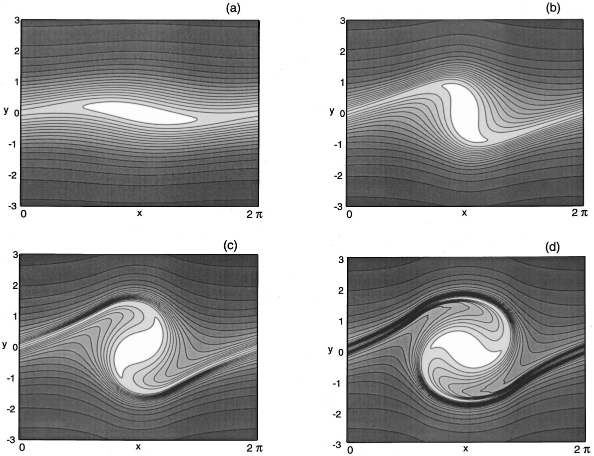
$$\partial_t \zeta + \{\psi, \zeta\} = \nu \nabla^2 \zeta \qquad B(x,t) = \int dx' \, \Gamma(x;x') \int dy' \, \zeta(x',y',t)$$

Mathematically similar to the 1-D electron-dynamics!

N.J. Balmforth, D. del-Castillo-Negrete, and W.R. Young: *J. Fluid Mech.*, 333, 197-230, (1997).

SHEAR FLOW INSTABILITY AND COHERENT STRUCTURE FORMATION

Numerical simulation of the reduced, vorticity defect equation



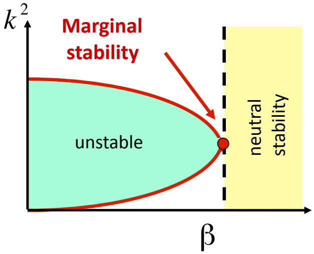
- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

SINGLE WAVE MODEL

$$\partial_t C + \mathbf{V}\cdot \nabla C = D\nabla^2 C$$

$$\mathbf{V} = z\,\hat{\mathbf{e}}_x - \partial_x \Phi\,\hat{\mathbf{e}}_z$$

$$\Phi(x,t) = \int dx'\, K(x;x') \int dz'\, C(x',z',t)$$



Marginally stable equilibrium $C_0 = C_0(z,\ \beta)$

The weakly nonlinear dynamics of perturbations of a marginally stable equilibrium is governed by the single-wave model for which

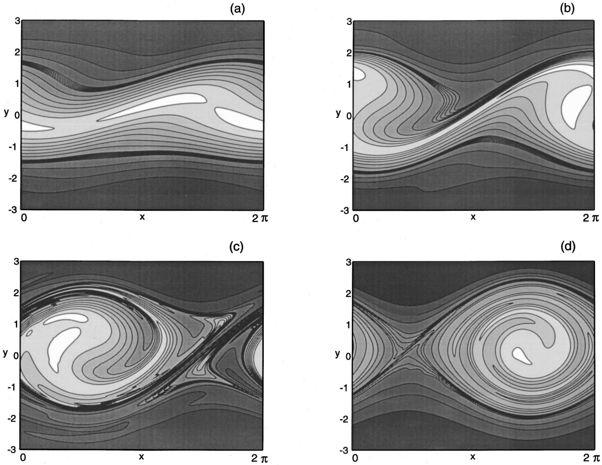
$$\Phi(x,t) = a(t)\,e^{ix} + a^*(t)\,e^{-ix}$$

$$\frac{da}{dt} = iU\,a + i\int dx \int dz\, e^{-ix}\, C(x,z,t)$$

- T.M. O'Neill, Winfrey, and Malmberg, Phys. Fluids., 14, 1204 (1971).
- D. del-Castillo-Negrete: Phys. of Plasmas, 5, (11), 3886-3900, (1998).

SHEAR FLOW INSTABILITY AND COHERENT STRUCTURE FORMATION

Numerical simulation of the reduced, single wave model



- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

SINGLE WAVE MODEL

•The SWM is a very general model describing the weakly nonlinear dynamics of a large family of fluids and plasma systems near marginal stability

•This is probably the simplest model for studying self-consistent chaotic transport

Continuum representation

$$\partial_t C + y\,\partial_x C + \partial_x \Phi\, \partial_y C = 0$$

$$\Phi(x,t) = a(t)\, e^{ix} + a^*(t)\, e^{-ix}$$

$$\frac{da}{dt} = iU\,a + i\int dx \int dy\, e^{-ix}\, C(x,y,t)$$

Discrete point vortex representation

$$C = 2\pi \sum_{j=1}^{N} \Gamma_j\, \delta\left[x - x_j(t)\right] \delta\left[y - y_j(t)\right]$$

particles:

$$\frac{d x_j}{dt} = y_j \qquad j = 1,2,\ldots\ldots N$$

$$\frac{dy_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

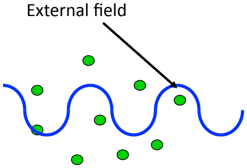
Mean field:

$$\frac{d}{dt}\left(\rho\, \mathrm{e}^{-i\theta}\right) + i\,U\rho\, \mathrm{e}^{-i\theta} = i \sum_k \Gamma_k\, e^{-ix_k}$$

- T.M. O'Neill, Winfrey, and Malmberg, Phys. Fluids., 14, 1204 (1971).
- D. del-Castillo-Negrete: Phys. of Plasmas, 5, (11), 3886-3900, (1998).

MEAN-FIELD HAMILTONIAN DYNAMICS

The single wave model is a *N*-degrees of freedom Hamiltonian mean field model.

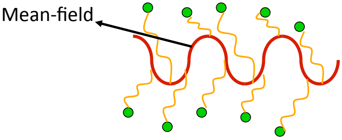


No coupling between particles.
Chaos due to explicit time dependence added to the amplitude

$$\frac{d\,x_j}{dt} = y_j$$

$$\frac{d\,y_j}{dt} = -\left[1+\varepsilon\cos\omega t\right]\sin x_j$$

Single-Wave Model



Chaos due to self-consistent time dependence in ρ and θ

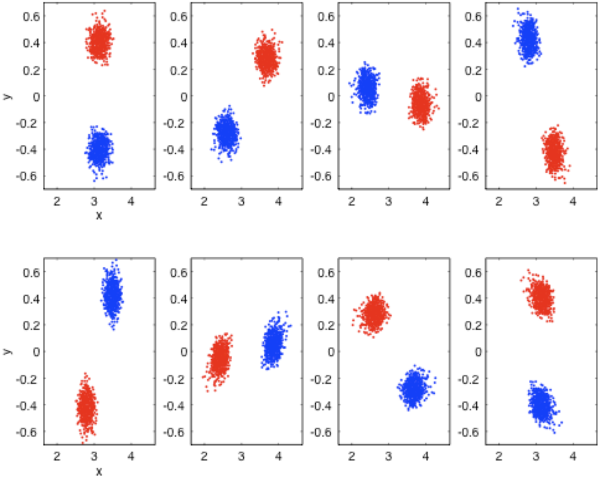
$$\frac{d\,x_j}{dt} = y_j$$

$$\frac{dy_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

$$\frac{d}{dt}\left(\rho\,e^{-i\theta}\right) + i\,U\rho\,e^{-i\theta} = i\sum_k \Gamma_k\,e^{-ix_k}$$

DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

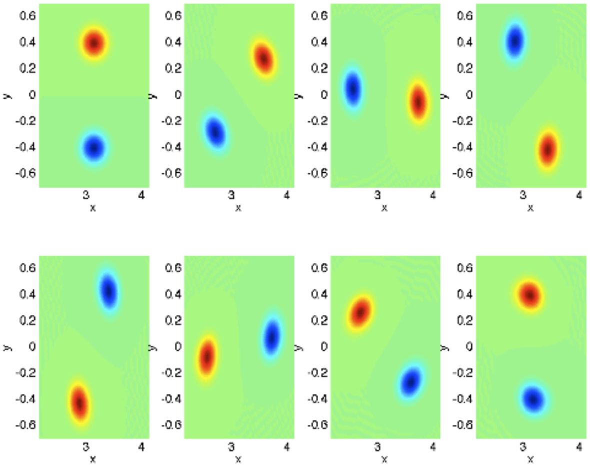
Numerical simulation of the finite-N single wave model



- D. del-Castillo-Negrete, M.C. Firpo: CHAOS, 12, 496-507, (2002).

DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

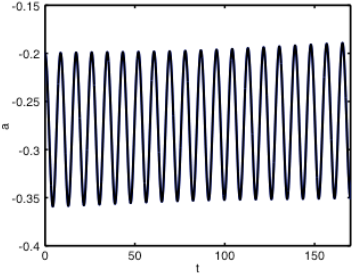
Numerical simulation of the continuum single wave model

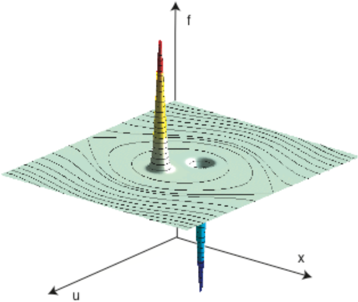


- D. del-Castillo-Negrete, M.C. Firpo: CHAOS, 12, 496-507, (2002).

DIPOLE COHERENT STRUCTURE: MEAN FIELD EVOLUTION

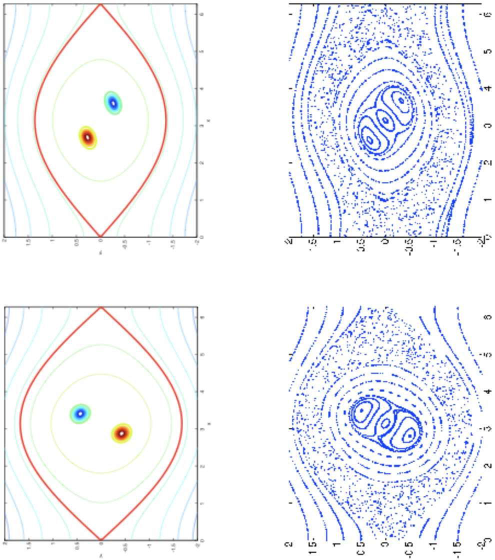
Time periodic mean field
Coherent rotating dipole





$$\frac{da}{dt} = \frac{i}{2\pi} \int dx \int du \, e^{-ix} \, C(x,u,t)$$

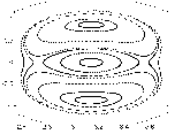
DIPOLE ROTATION AND AND SELF-CONSISTENT CHAOS



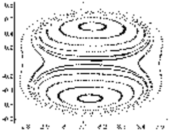
DIPOLE ROTATION AND AND SELF-CONSISTENT CHAOS

Parametric instability in Poincare section
for test particle in dipole field

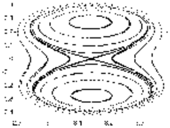
$P > P_c$



$P = P_c$



$P < P_c$



$$P_c = \left(\frac{5\Gamma^2}{6} \right)^{2/3}$$

Particle mean field resonance

$$a(t) = \frac{\omega^2}{2}\left[1 - 2\,\varepsilon \cos \Omega t + \mathrm{K}\,\right]$$

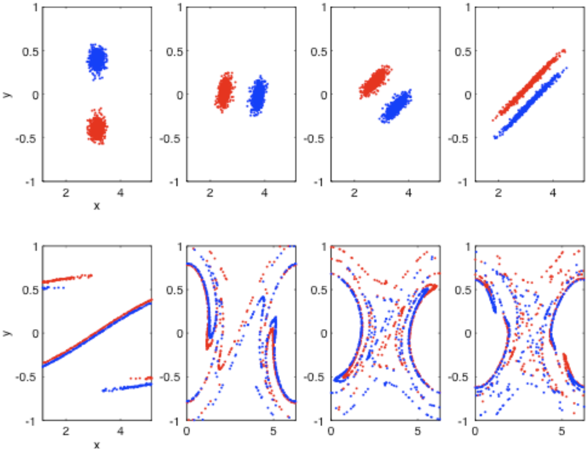
$$x(t) = \omega^3\left[\,\varepsilon \sin\, \Omega t + \mathrm{K}\,\right]$$

Stability of the origin

$$q''(t) = 2\,a(t)\,q$$

HYPERBOLIC-ELLIPTIC BIFURCATION IN MEAN FIELD HAMILTONIAN DYNAMICS

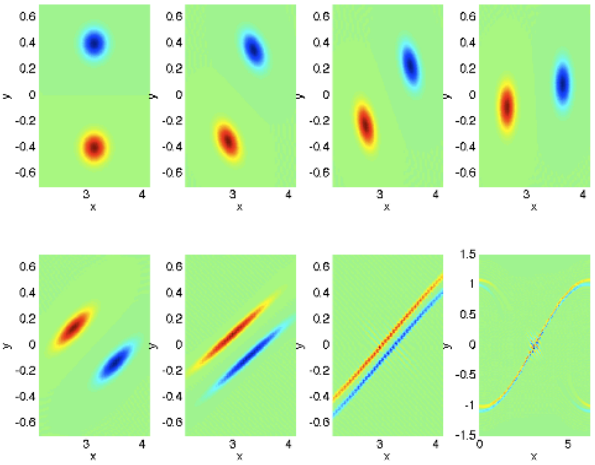
Numerical simulation of the finite-N single wave model



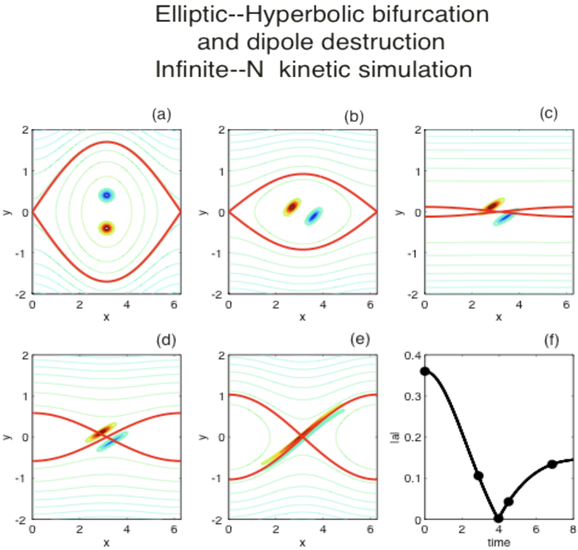
• D. del-Castillo-Negrete, M.C. Firpo: CHAOS, 12, 496-507, (2002).

HYPERBOLIC-ELLIPTIC BIFURCATION IN MEAN FIELD HAMILTONIAN DYNAMICS

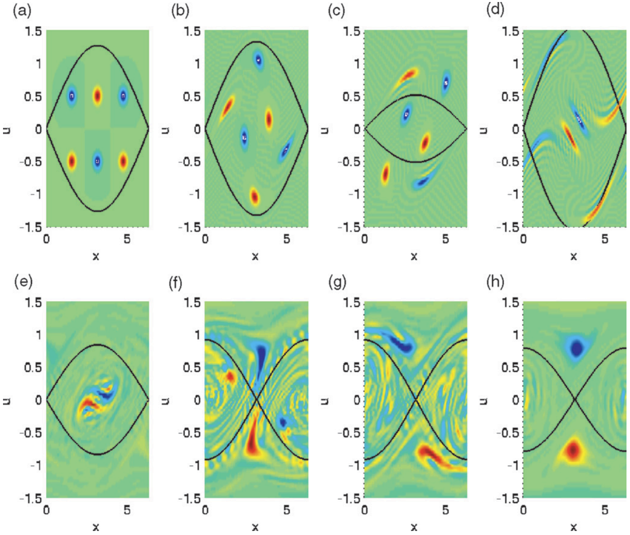
Numerical simulation of the continuum single wave model



HYPERBOLIC-ELLIPTIC BIFURCATION IN MEAN FIELD HAMILTONIAN DYNAMICS



RELAXATION TOWARDS ROTATING DIPOLE STATE



• D. del-Castillo-Negrete, Plasma Physics and Controlled Fusion **47**, 1-11 (2005).

STANDARD MEAN FIELD MAP

$$\frac{d\,x_j}{dt} = y_j \qquad j = 1,2,\ldots\ldots N$$

$$\frac{dy_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

particles

$$x_k^{n+1} = x_k^n + y_k^{n+1}$$

$$y_k^{n+1} = y_k^n - \kappa^{n+1}\sin\left(x_k^n - \theta^n\right)$$

$$\frac{d}{dt}\left(\rho\, e^{-i\theta}\right) + i\,U\rho\, e^{-i\theta} = i\sum_k \Gamma_k\, e^{-ix_k}$$

mean field

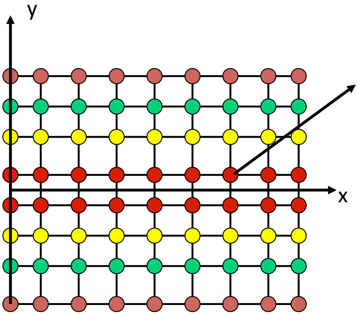
$$\kappa^{n+1} = \sqrt{\left(\kappa^n\right)^2 + \left(\eta^n\right)^2} + \eta^n$$

$$\theta^{n+1} = \theta^n + \frac{1}{\kappa^{n+1}}\frac{\partial\eta^n}{\partial\theta^n}$$

$$\eta^n = \sum_{j=1}^{N}\gamma_j\sin\left(x_j^n - \theta^n\right)$$

- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

STANDARD MEAN FIELD MAP



$\left(x_k^0\ ,y_k^0\right)$ Particles initial positions

$$k = 1,2,.......2\times10^5$$

Particle's "charge"

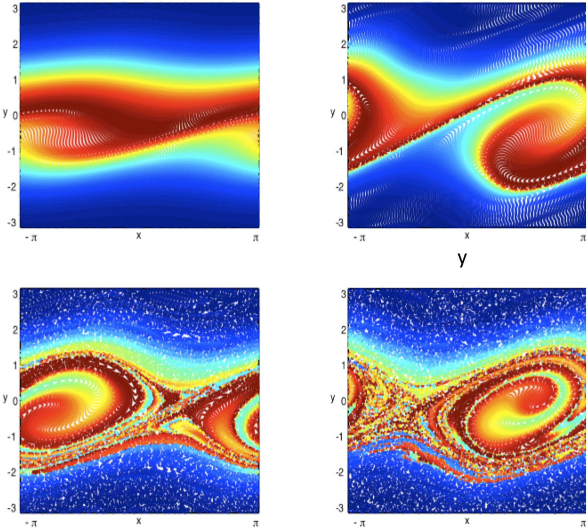
$$\gamma_k = \frac{\tau^3}{\pi}\exp\left(\frac{-y_k^2}{2}\right)$$

Mean field initial condition

$$\theta^0 = 0 \qquad \kappa^0 = 0.001$$

STANDARD MEAN FIELD MAP

Shear flow instability and coherent structure formation



• D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

NONTWIST MEAN FIELD MAP

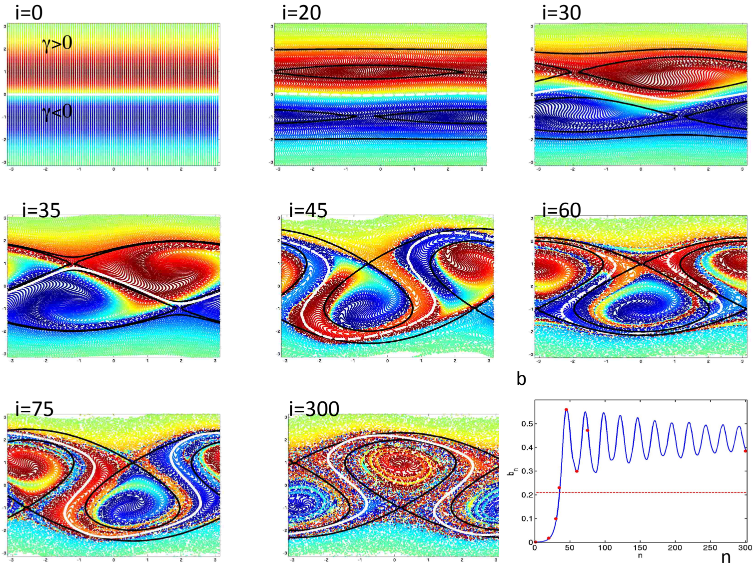
$k = 1, 2, \ldots N$

$$
\begin{aligned}
x_k^{n+1} &= x_k^n + a\left[1 - \left(\frac{\tau}{\Gamma_k} p_k^{n+1}\right)^2\right] , \\
p_k^{n+1} &= p_k^n - 2\tau\Gamma_k\sqrt{J^{n+1}}\sin\left(x_k^n - \theta^n\right), \\
\theta^{n+1} &= \theta^n - U\tau - \frac{\tau}{\sqrt{J^{n+1}}}\sum_{k=1}^{N}\Gamma_k\cos\left(x_k^n - \theta^n\right), \\
J^{n+1} &= J^n + 2\tau\sqrt{J^{n+1}}\sum_{k=1}^{N}\Gamma_k\sin\left(x_k^n - \theta^n\right),
\end{aligned}
\tag{1}
$$

• L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22 013137 (2012).

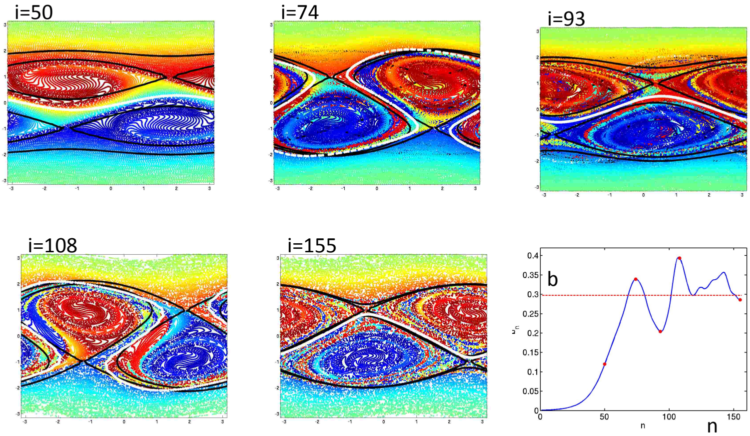
NONTWIST MEAN FIELD MAP

Shear flow instability and coherent structure formation



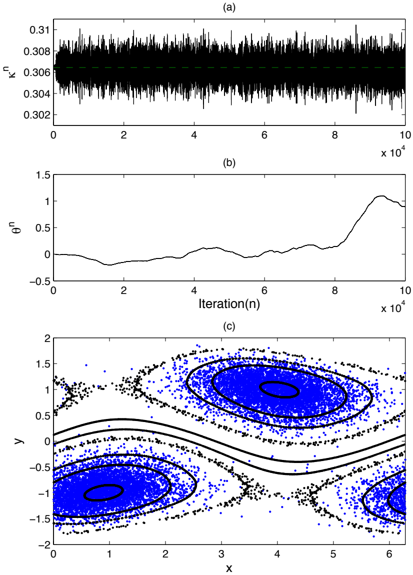
NONTWIST MEAN FIELD MAP

Shear flow instability and coherent structure formation



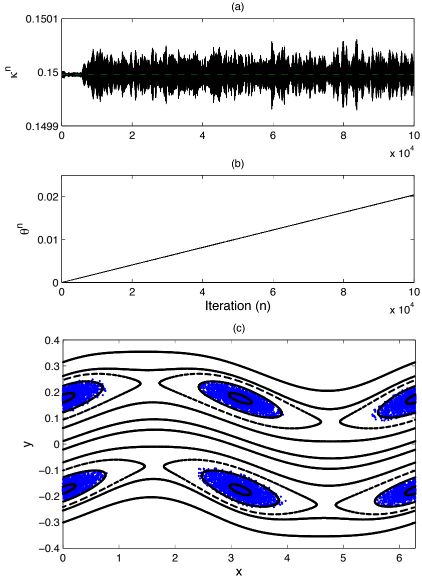
NONTWIST MEAN FIELD MAP

Period-one coherent structures



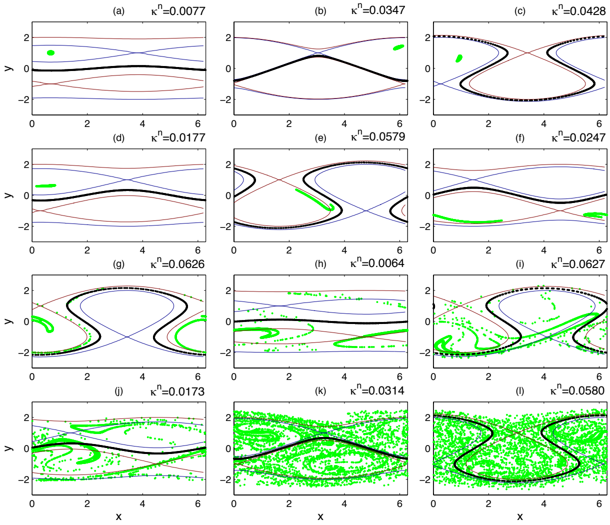
NONTWIST MEAN FIELD MAP

Period-two coherent structures



NONTWIST MEAN FIELD MAP

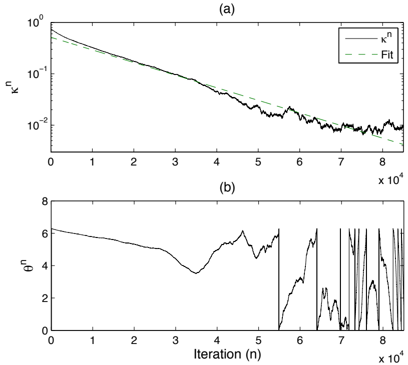
Self-consistent separatrix reconnection in the mean-field map

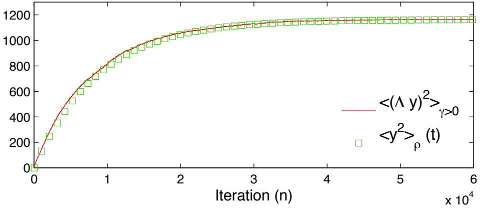


- L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22

NONTWIST MEAN FIELD MAP

Self-consistent suppression of diffusion

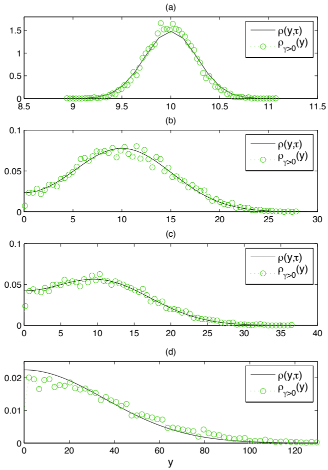




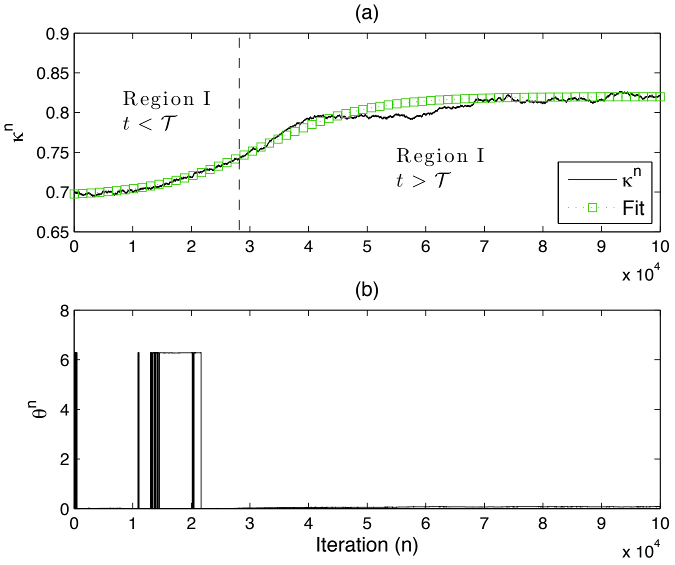
NONTWIST MEAN FIELD MAP

Self-consistent suppression of diffusion

$$\frac{\partial \rho}{\partial t} = D_{QL}(t)\,\frac{\partial^2 \rho}{\partial y^2}\,, \qquad D_{QL}(t) = \frac{\kappa_0^2}{4}\,e^{-2t\nu}\,.$$



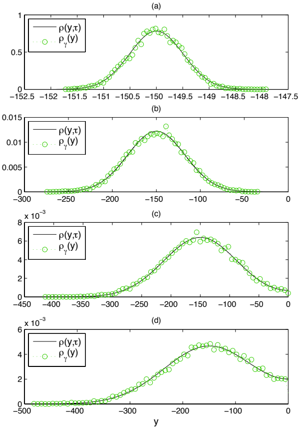
Self-consistent transition to global chaos

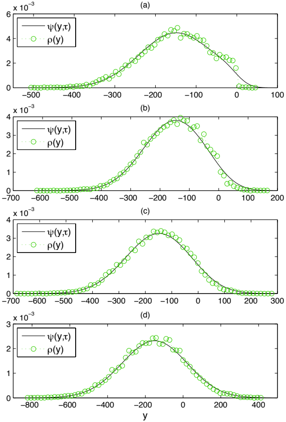


NONTWIST MEAN FIELD MAP

Self-consistent transport across shearless barrier

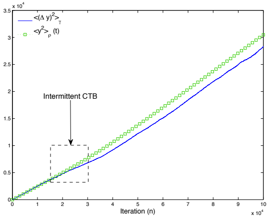
$$\frac{\partial \rho}{\partial t} = D_{QL}(t)\,\frac{\partial^2 \rho}{\partial y^2}\,, \qquad D_{QL}(t) = \frac{1}{4}\left[K + \alpha \tanh\left(\frac{t-\mu}{\beta}\right)\right]^2$$

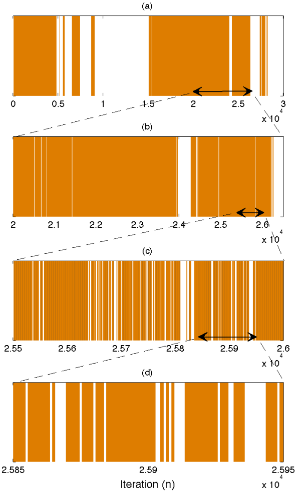




NONTWIST MEAN FIELD MAP

Self-consistent intermittent transport near criticality





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