Three-Dimensional Chaotic Advection in an Idealized Ocean Eddy

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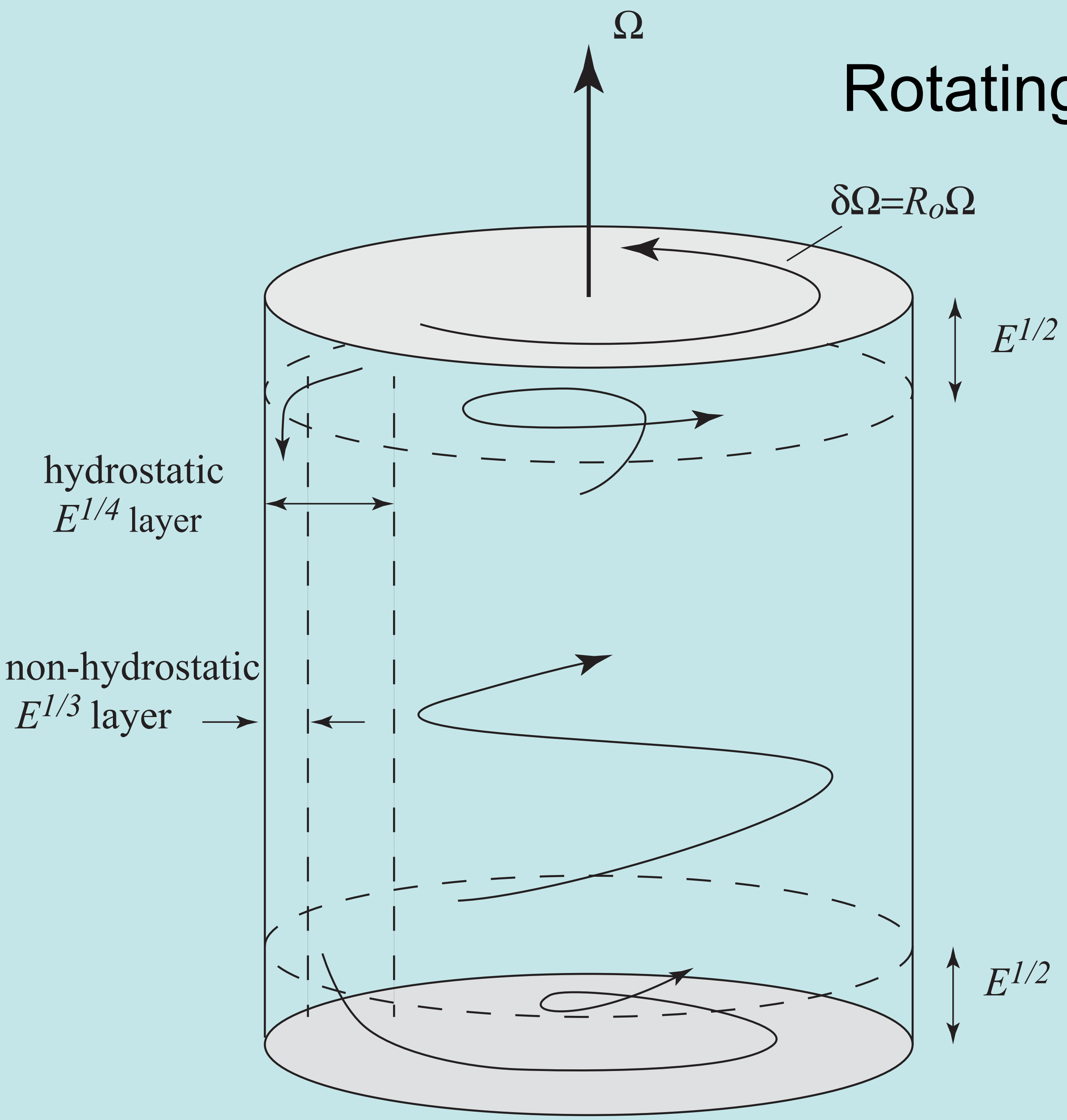
J. Fluid Mech. submitted

How did we choose our model?

- something our friends can relate to
- a platform for discussing dynamics (F=ma). (so we seek dynamical consistency)
- fully 3D: $\frac{\partial w}{\partial z}$ is important in $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \boxed{\frac{\partial w}{\partial z}} = 0$

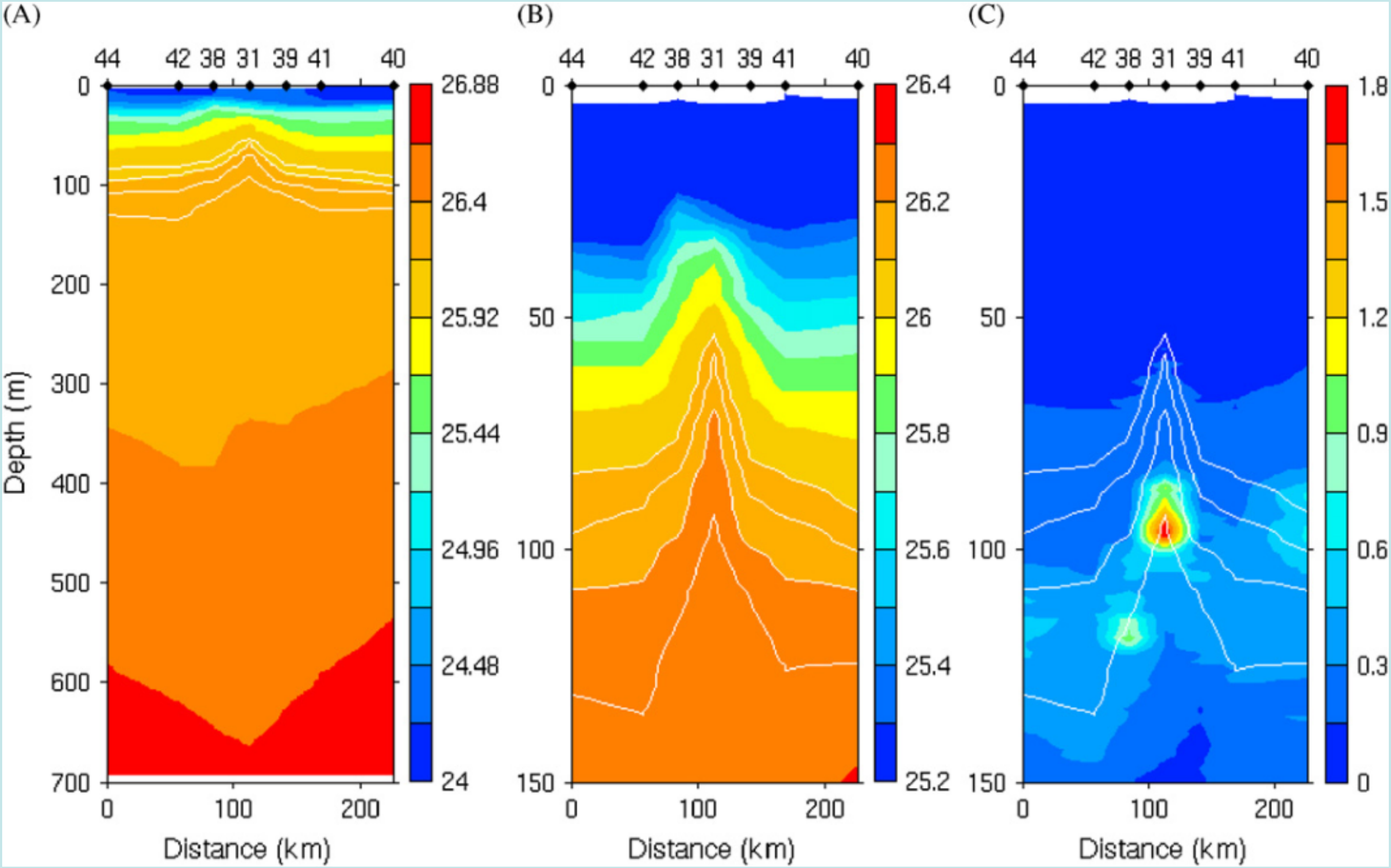
Rotating Can Flow



Linear Thoery for strong Rotation: Greenspan's 'Theory of Rotating Fluids'

Non-rotating versions:
Fountain, et al. 2000
Lackey and Sotiropoulos 2006

Ocean Eddy with Overturning Circulation



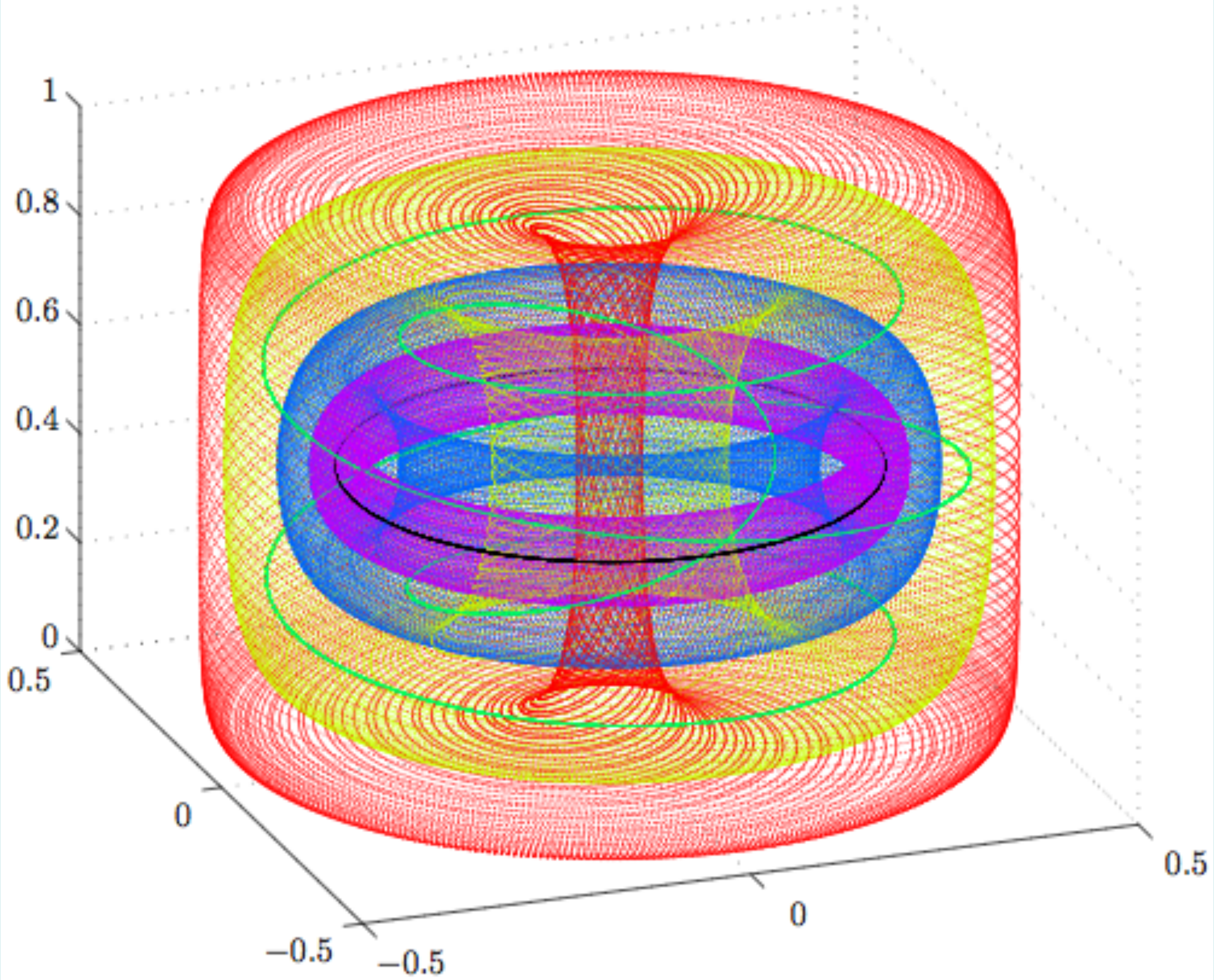
From Ledwell, McGillicuddy and Anderson DSR-II (2008)

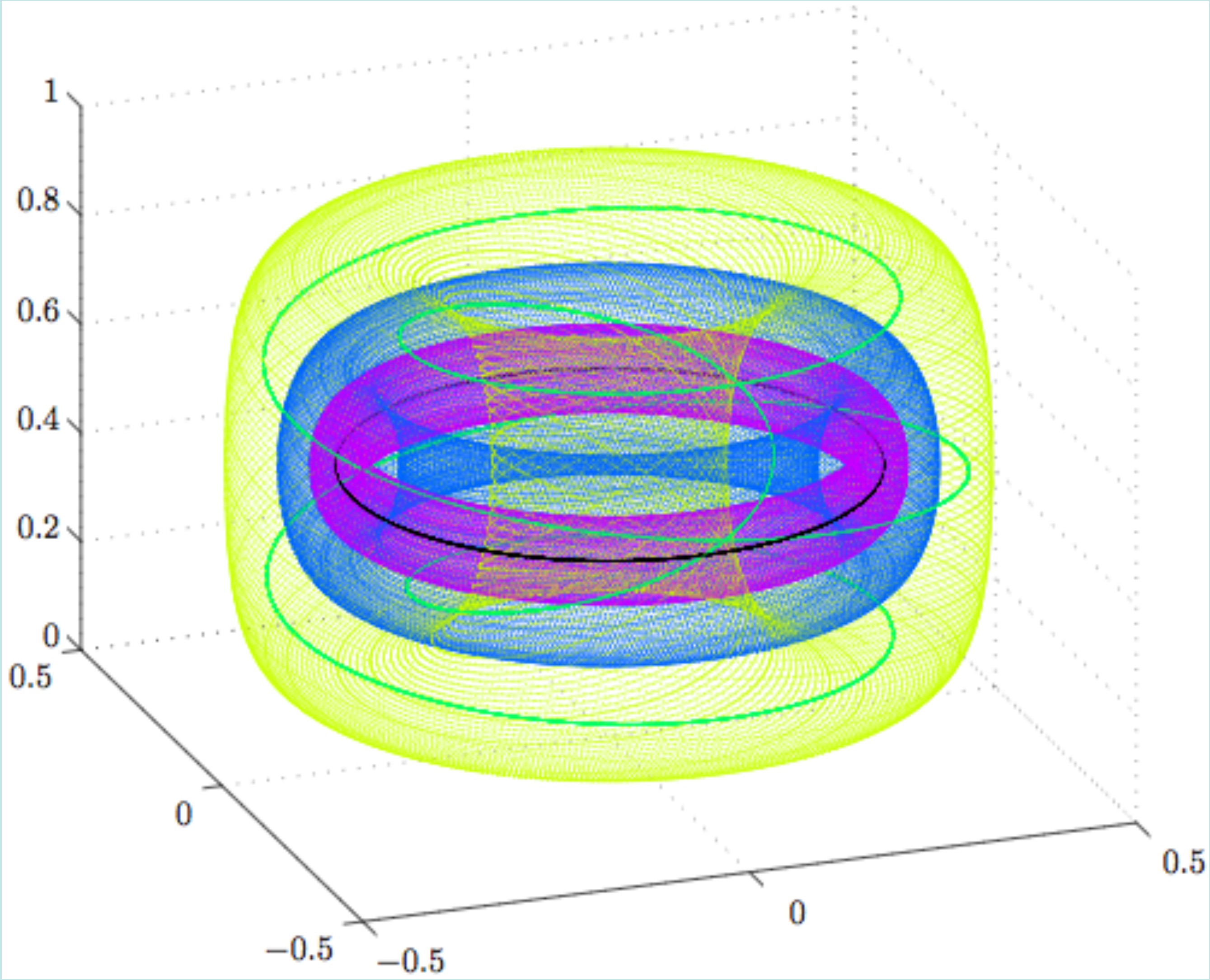
Velocity Fields

1) Navier-Stokes integration.

2) Kinematic (3D velocity non-divergent but no dynamics)

3) Linear asymptotic solution for small Rossby number.


1
0.8
0.6
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5



1
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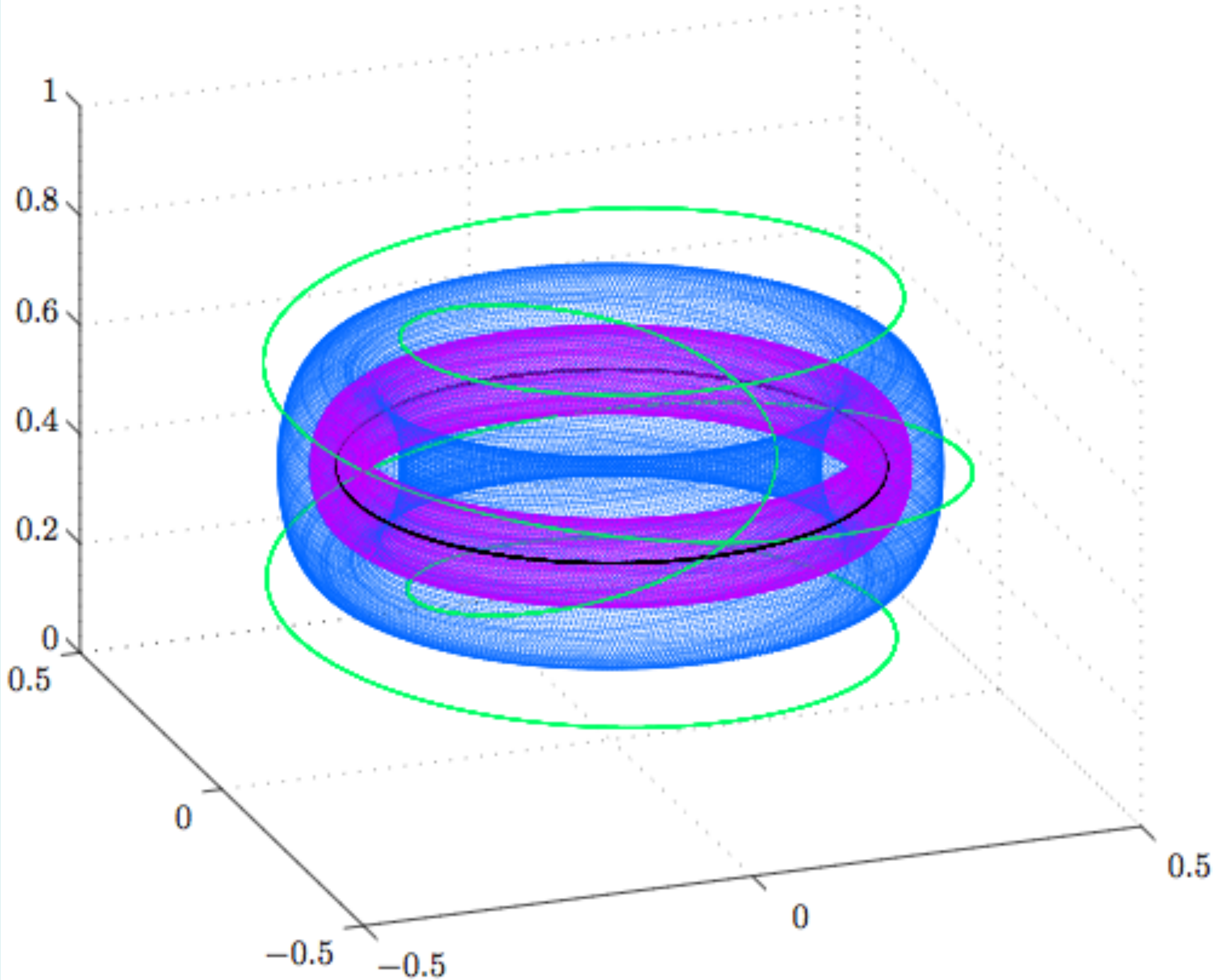

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0.2
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−0.5
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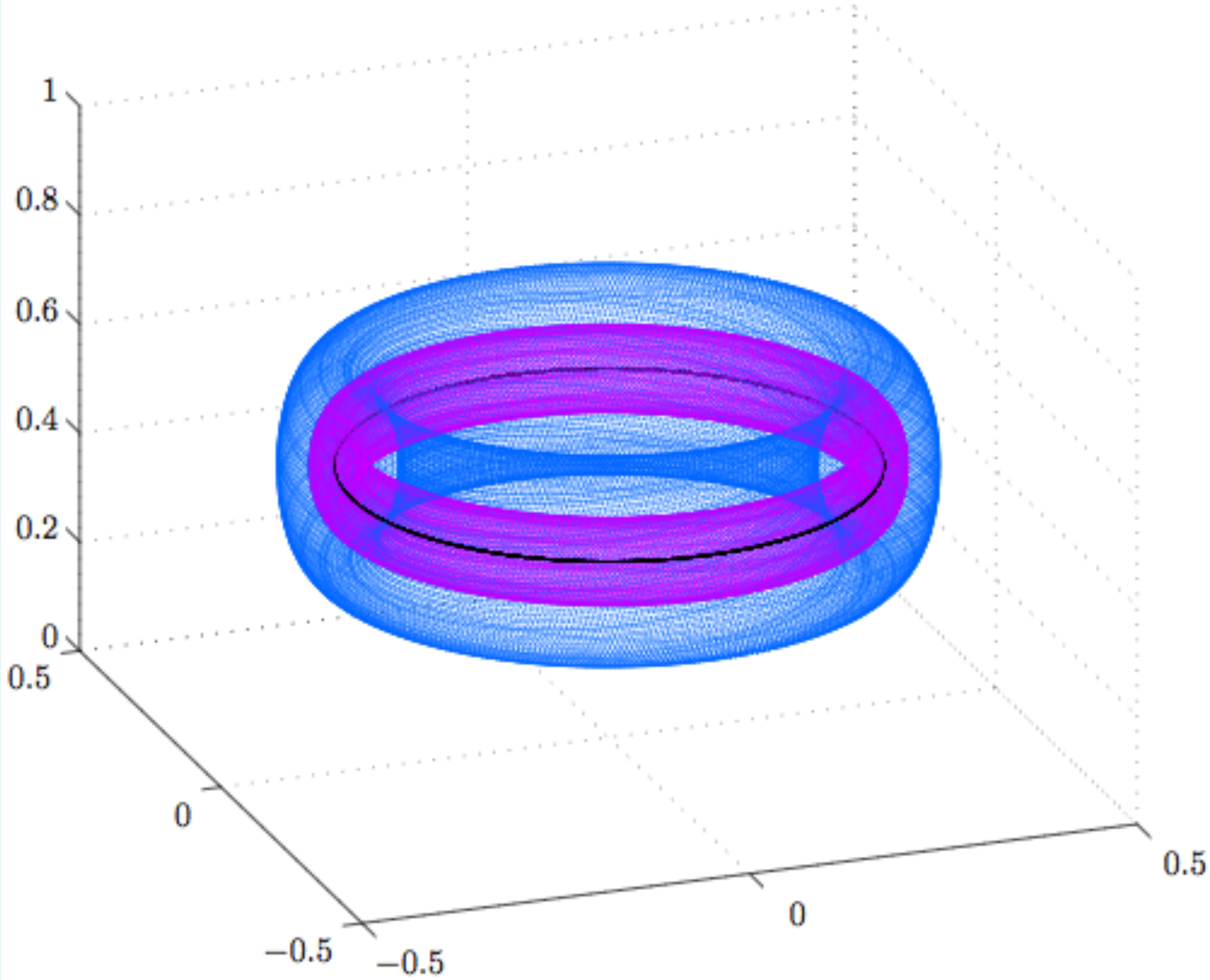

1
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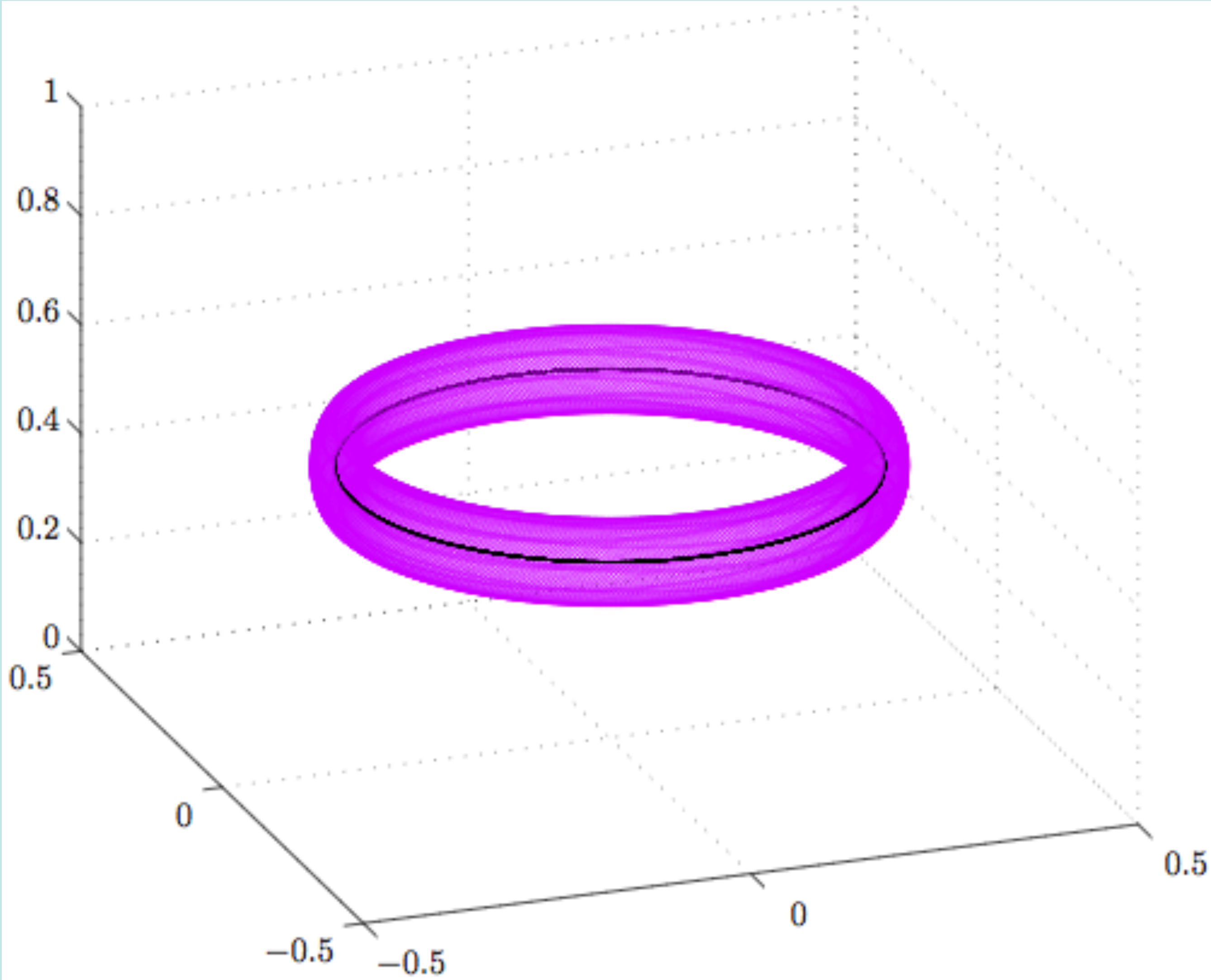
0.6
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5


1
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0.5
0
−0.5
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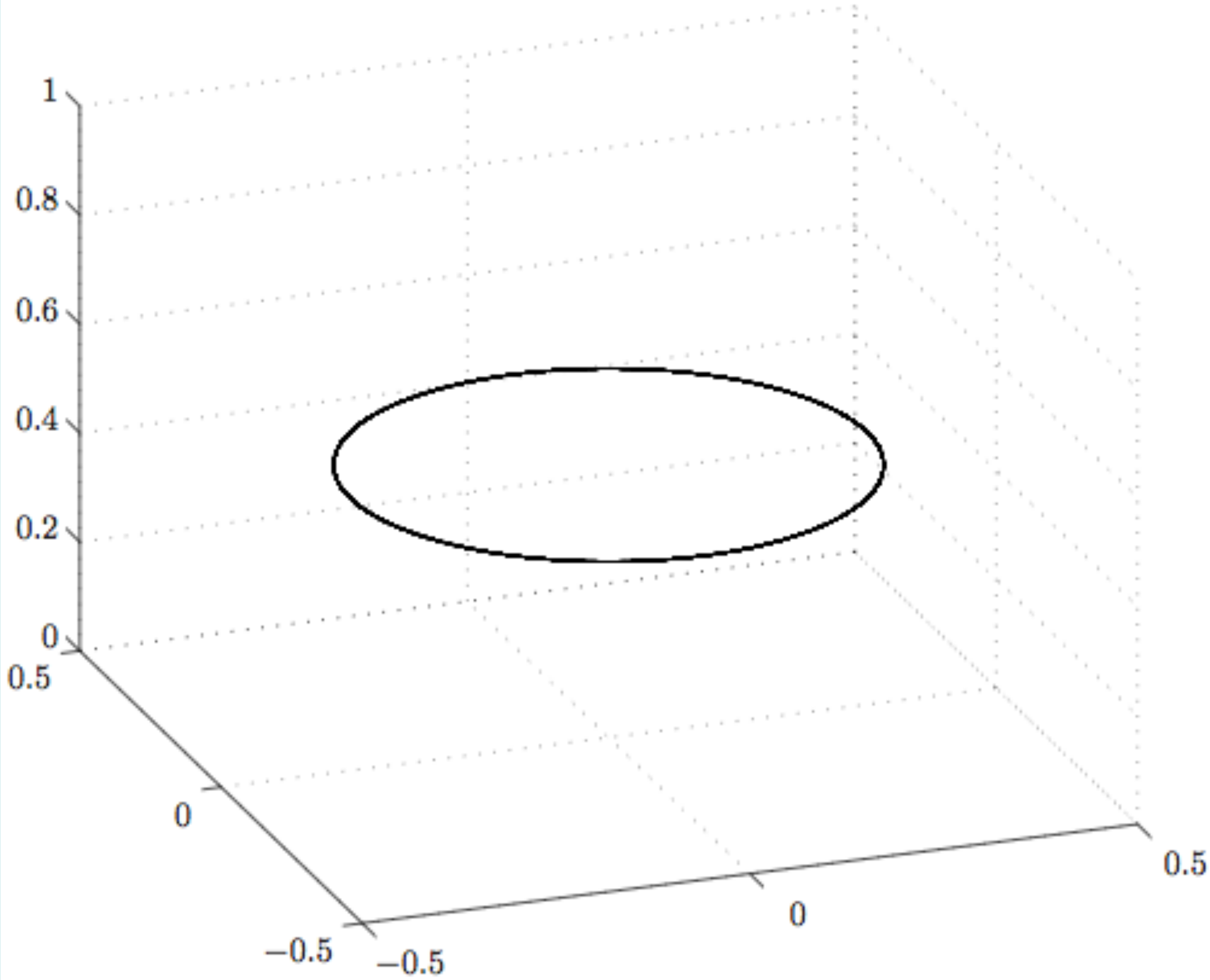
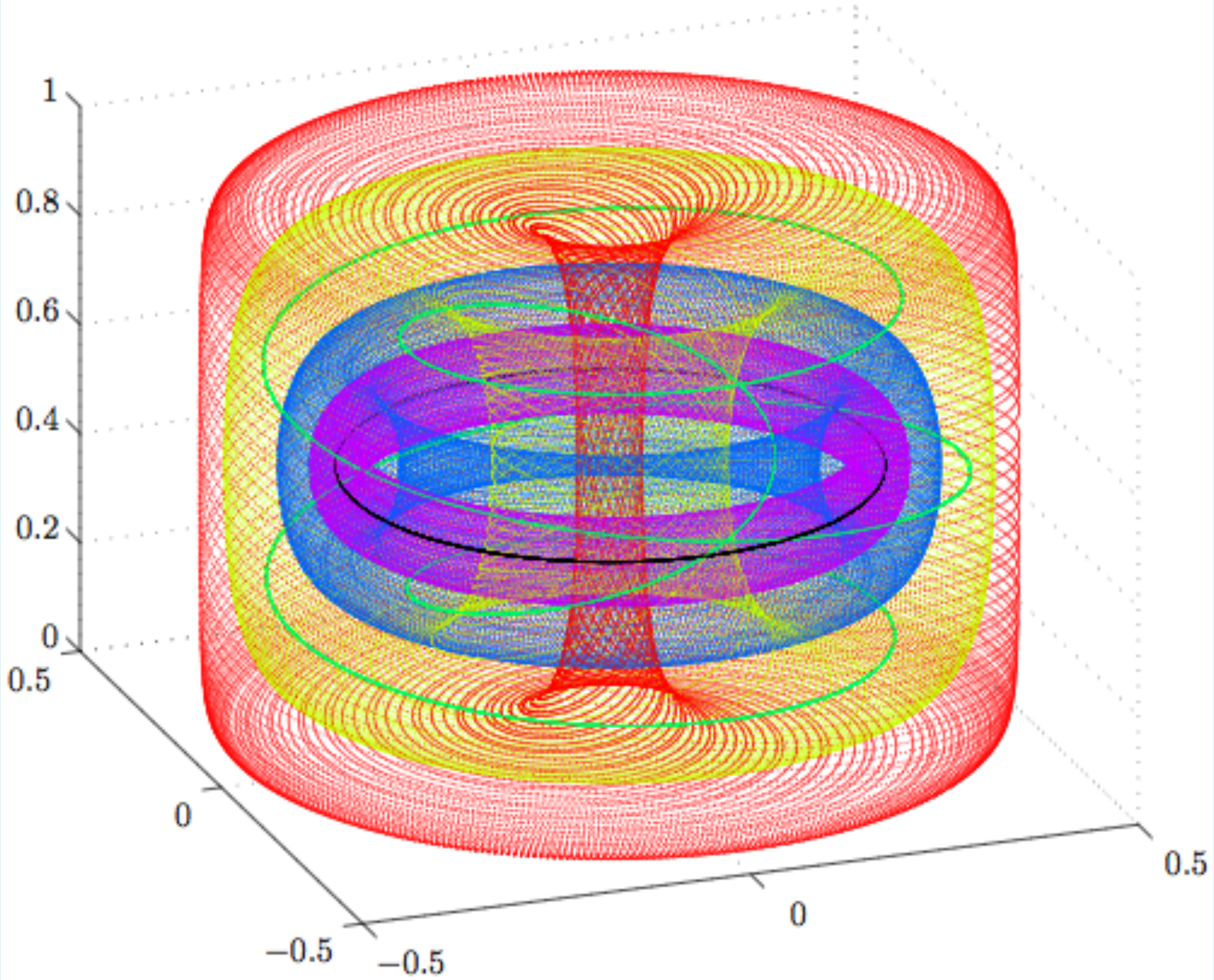
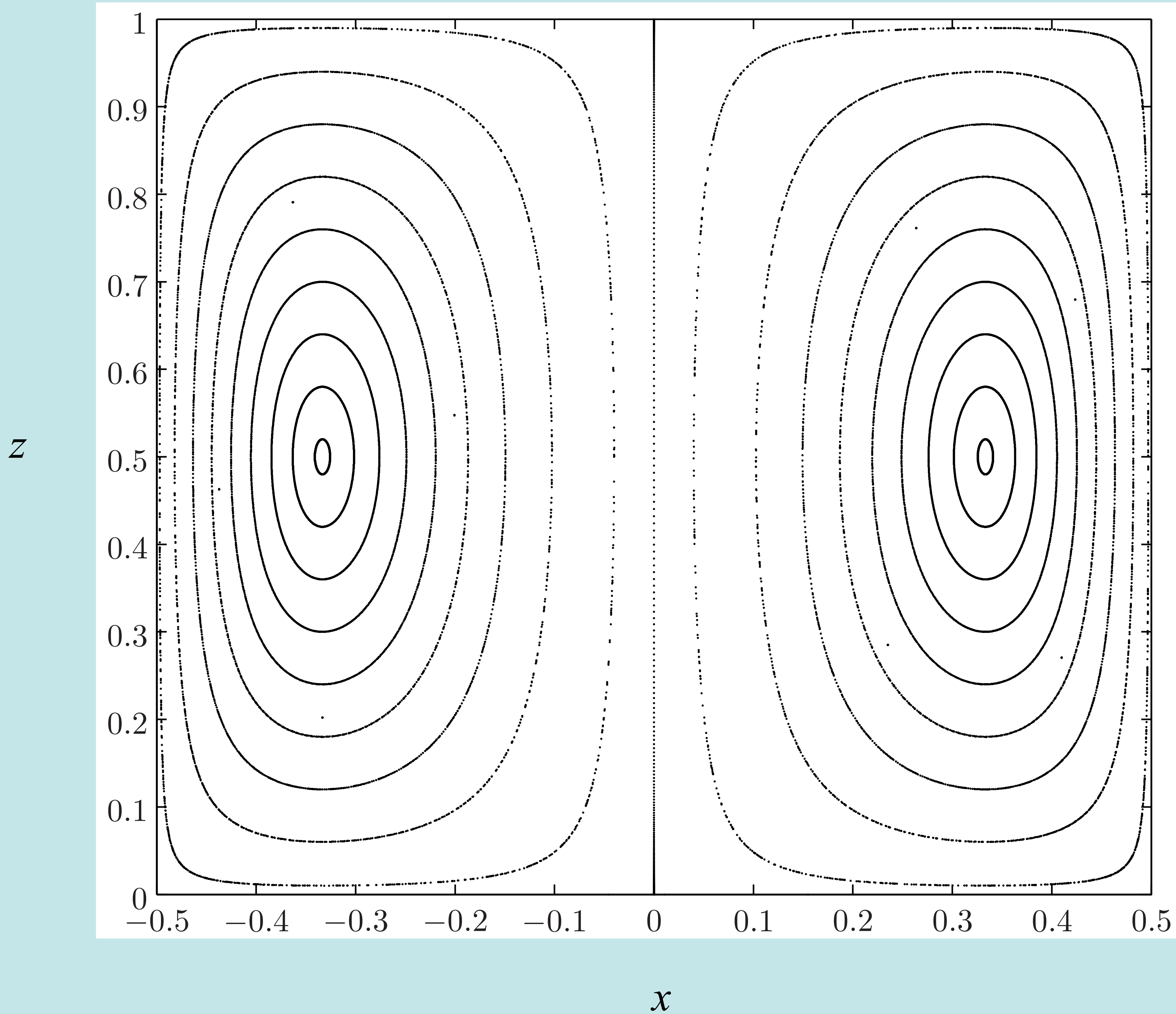

1
0.8
0.6
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5

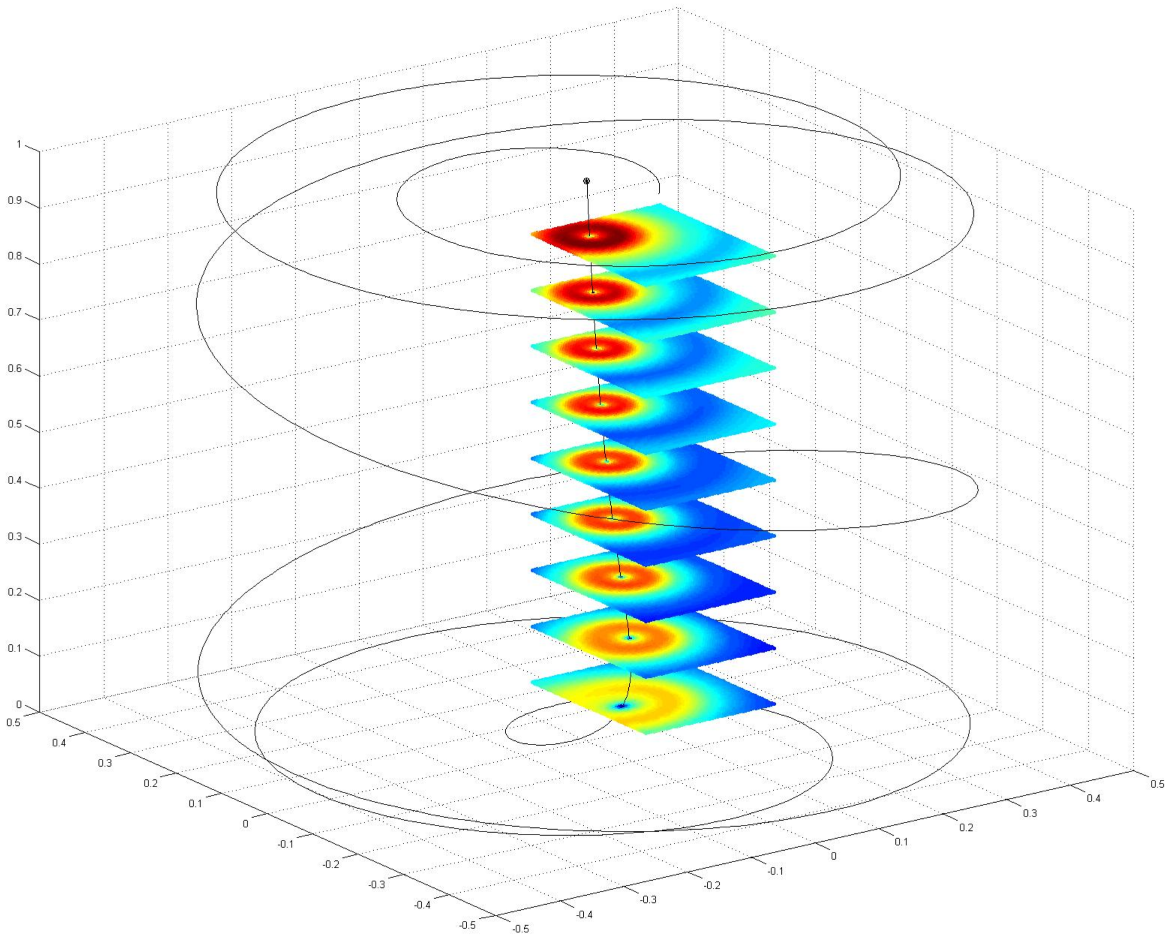


1
0.8
0.6
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5

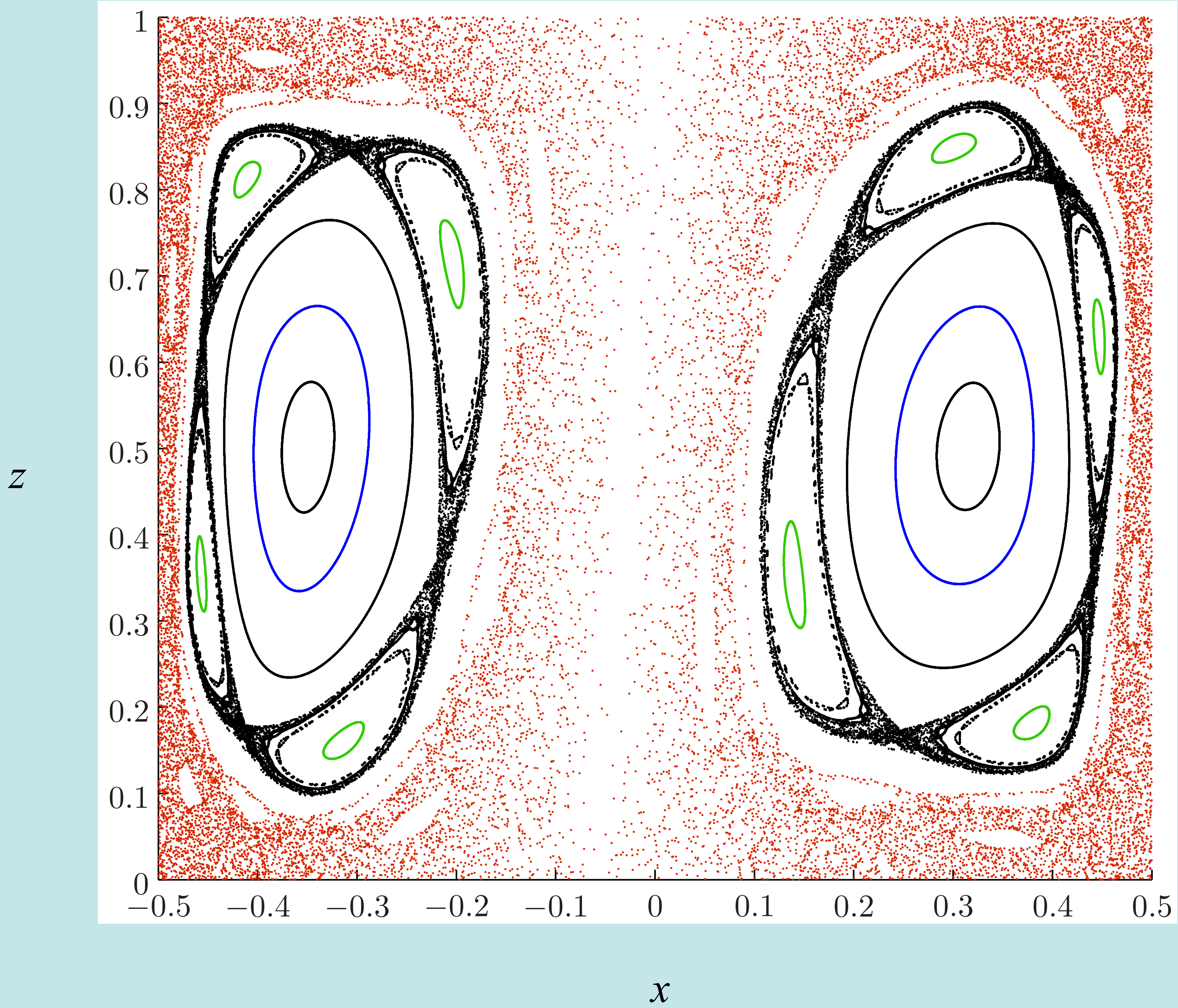


z
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
x

Stable Manifold

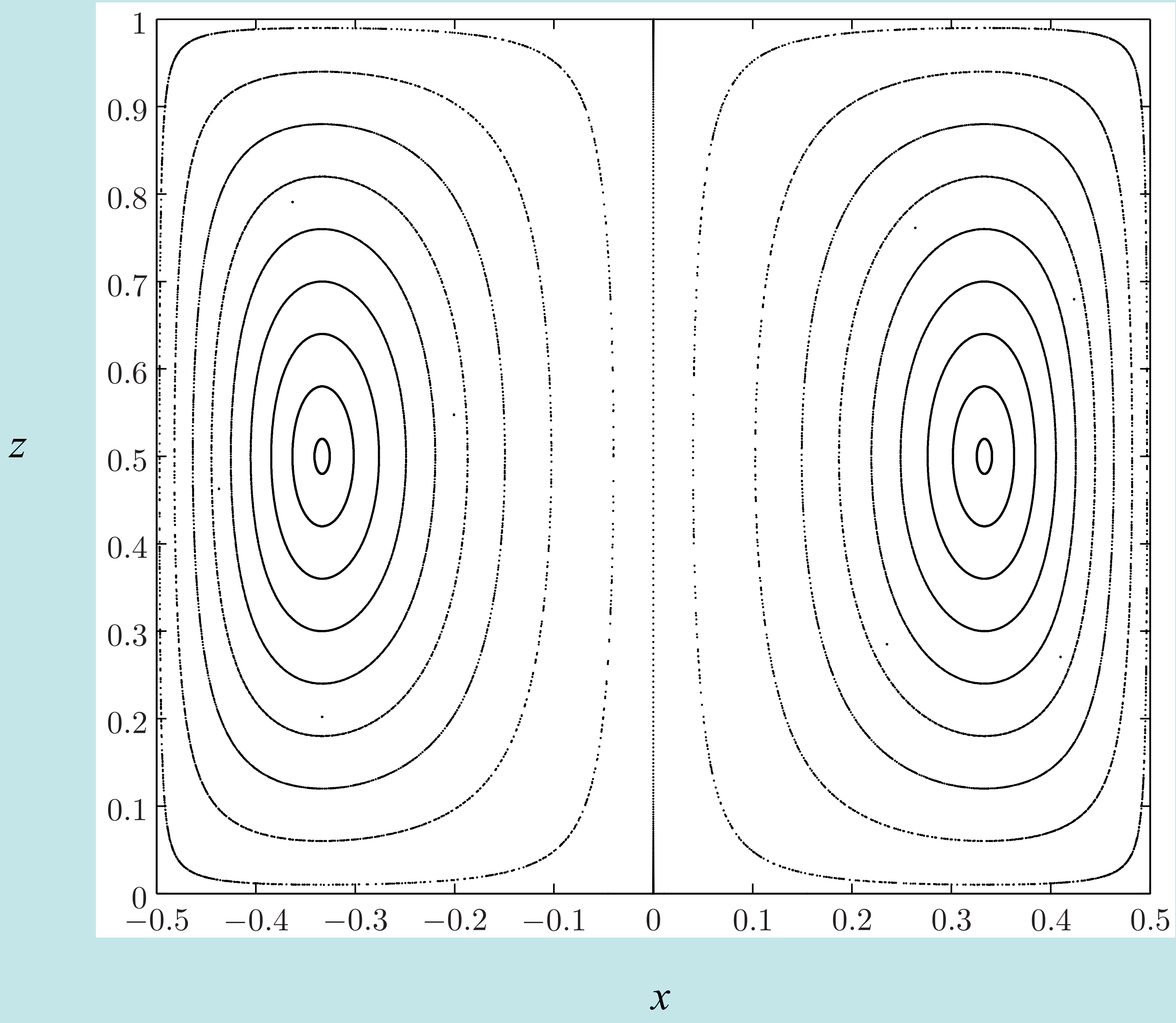


Computed from Complexity Measures (Rypina, Scott, Pratt, Brown: NPG 2011)



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
z
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
x

Weak KAM theorem: Mezic and Wiggins (1994)

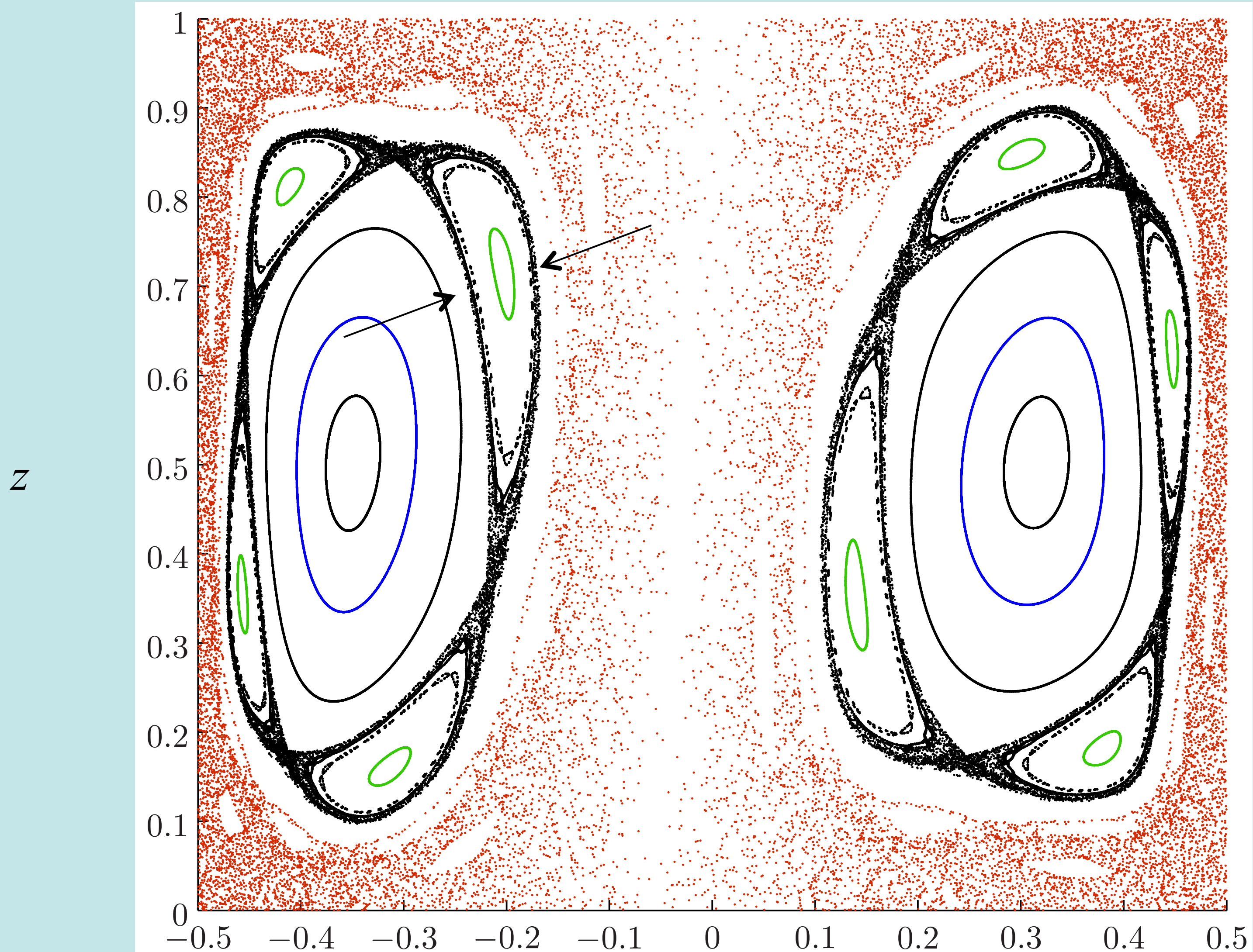


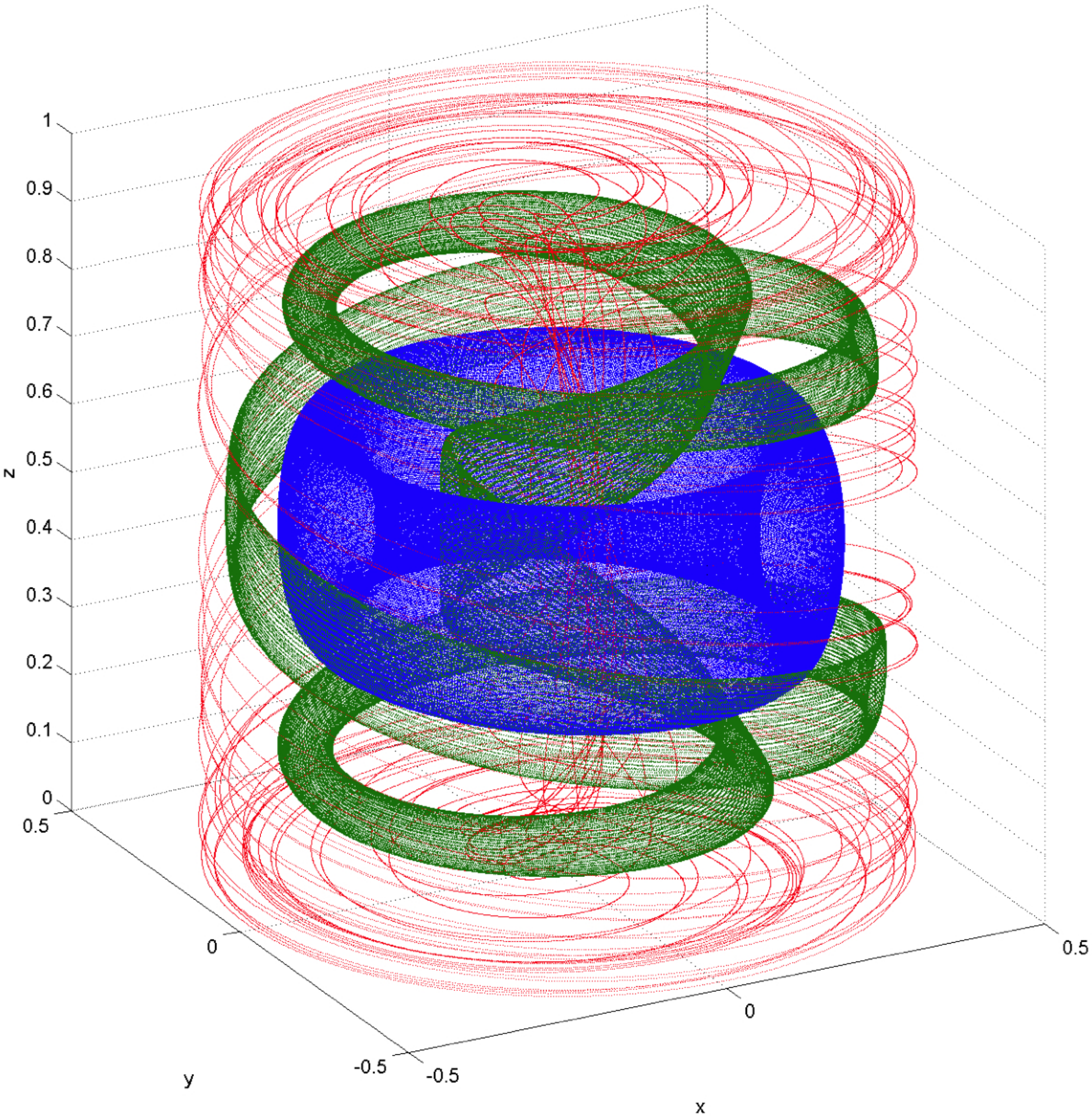
resonance width

$$\Delta I = \sqrt{\frac{\varepsilon F^{o}_{nm}}{\left| n\Omega_{\phi}\left[\frac{d}{dI}\ln\left(\frac{T_{\theta}}{T_{\phi}}\right)\right]\right|_{I=I_o}}}$$

← projection of forcing on trajectory

← measure of how close neighboring tori are to resonance



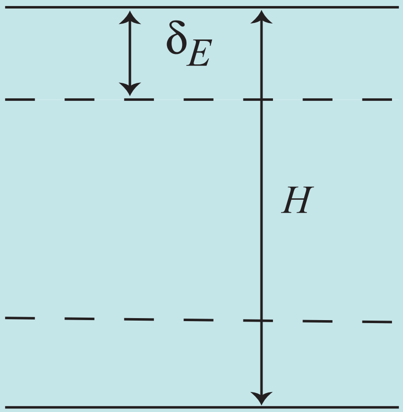


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
z
0.5
0
-0.5
y
-0.5
0
0.5
x

Parameters (Numerical Model)

Ekman Number $E = \frac{\nu}{\Omega H^2} = \left(\frac{\delta_E}{H}\right)^2$

$$1/2000 < E < 1$$

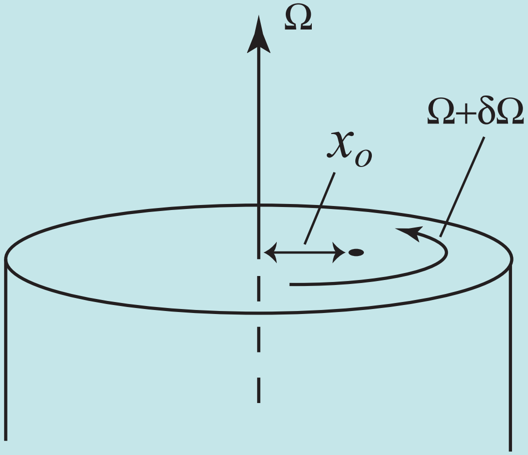


Rossby Number $R_o = \delta\Omega / \Omega$

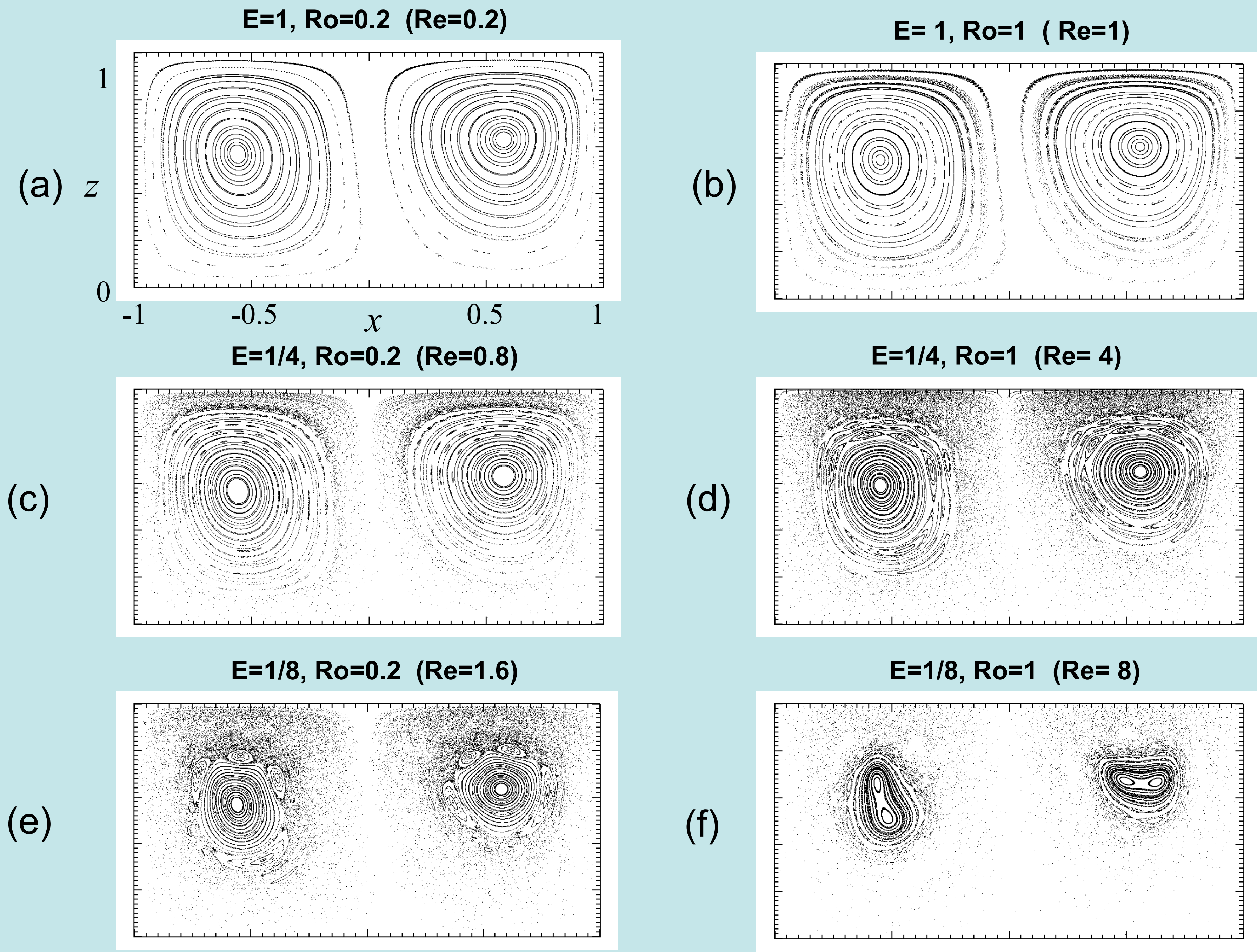
$$0.2 \le R_o \le 1$$

Perturbation Amplitude x_o

Aspect Ratio H/R=1

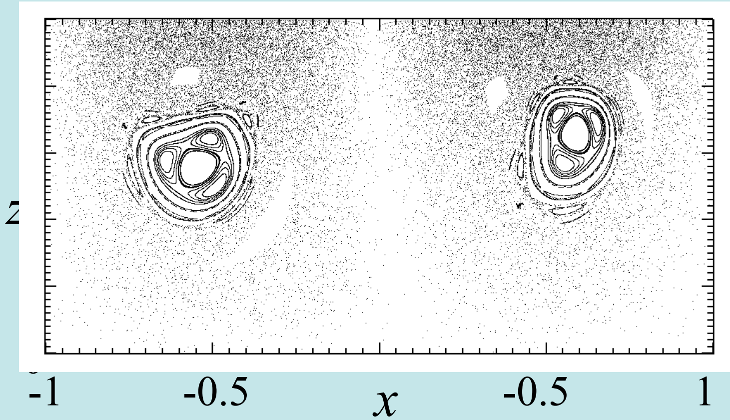


Note: $R_e = R_o / (EH^2 / R^2)$


E=1, Ro=0.2 (Re=0.2)
E= 1, Ro=1 (Re=1)
(a)
z
1
0
-1
-0.5
x
0.5
1
(b)
E=1/4, Ro=0.2 (Re=0.8)
E=1/4, Ro=1 (Re= 4)
(c)
(d)
E=1/8, Ro=0.2 (Re=1.6)
E=1/8, Ro=1 (Re= 8)
(e)
(f)

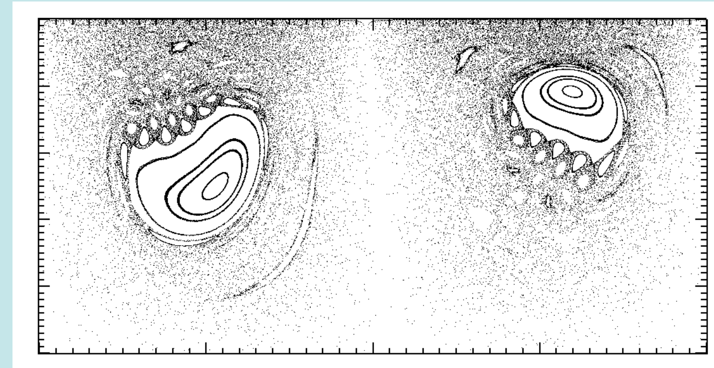
E=1/20, Ro=0.2 (Re=4)

(g)



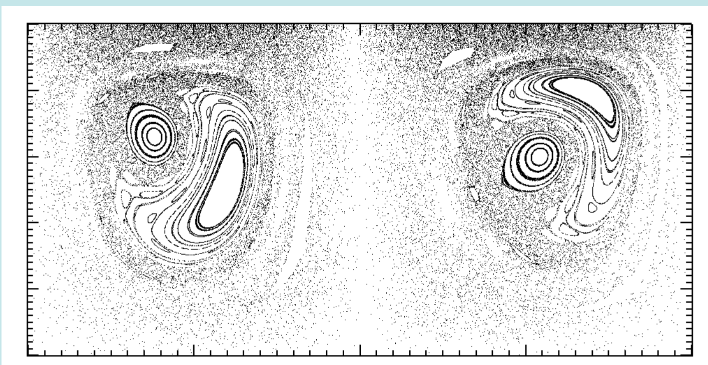
E=1/20, Ro=1 (Re=20)

(h)



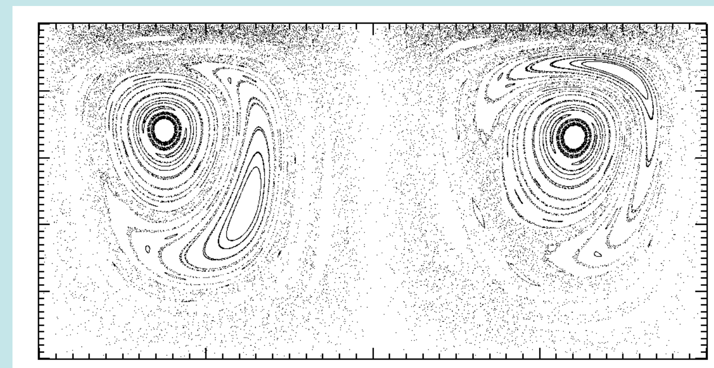
E=1/50, Ro=0.2 (Re=10)

(i)



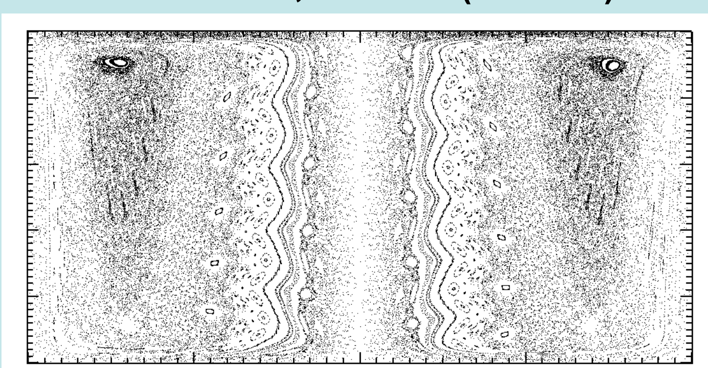
E=1/50, Ro=1 (Re=50)

(j)



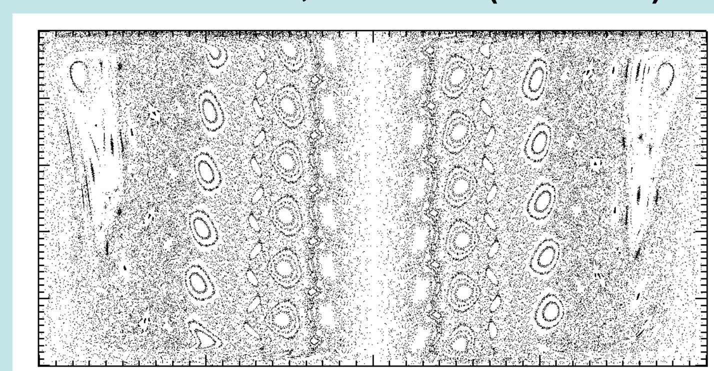
E=1/2000, Ro=0.2 (Re=400)

(k)



E=1/2000, Ro=1.0 (Re=2000)

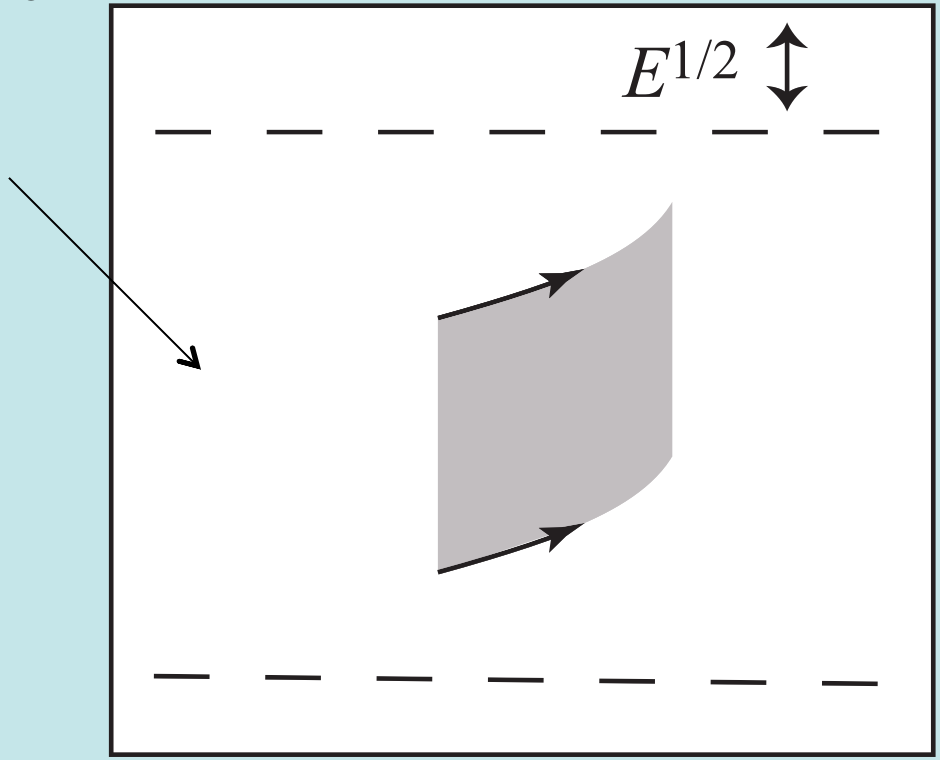
(l)



Taylor-Proudman Theorem

If $R_o \ll 1$ and $E \ll 1$ and $\dfrac{\partial}{\partial t} = 0$,

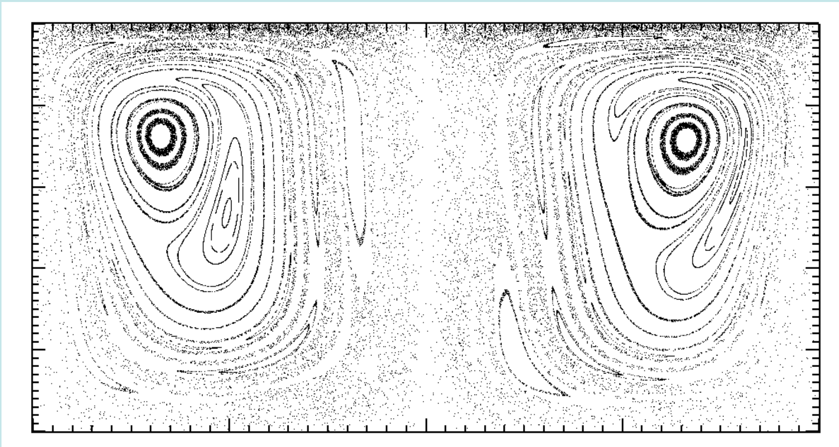
then $\dfrac{\partial \mathbf{u}}{\partial z} = 0$ in interior.



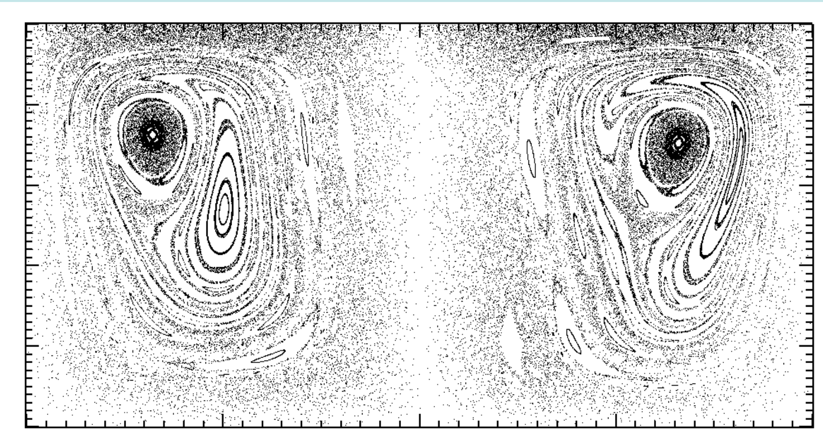
Interior trajectories live on vertically aligned sheets.

Hypothesis: stirring rate will decrease like *$E^{1/2}$ as E decreases.*

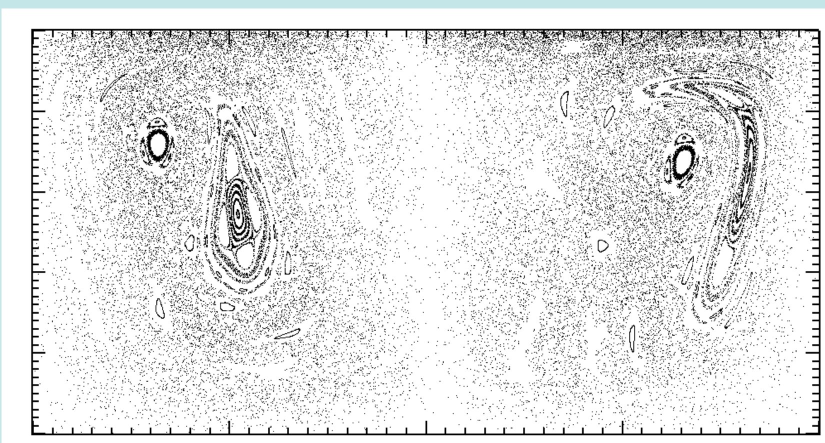
x0=-0.02



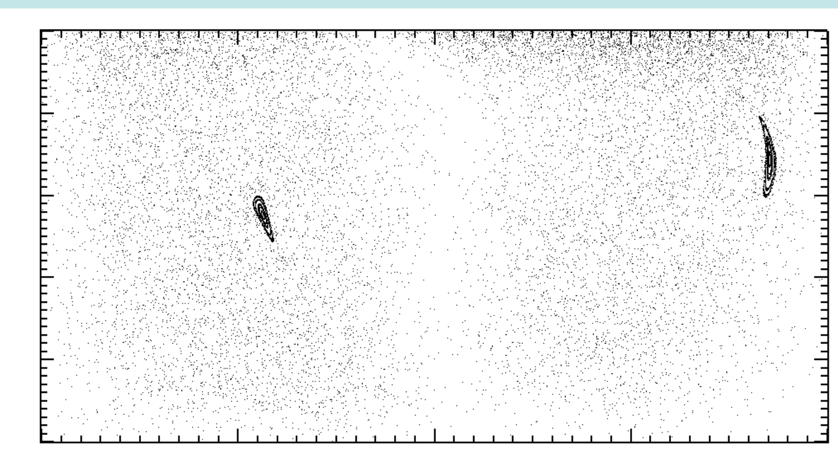
x0=-0.04



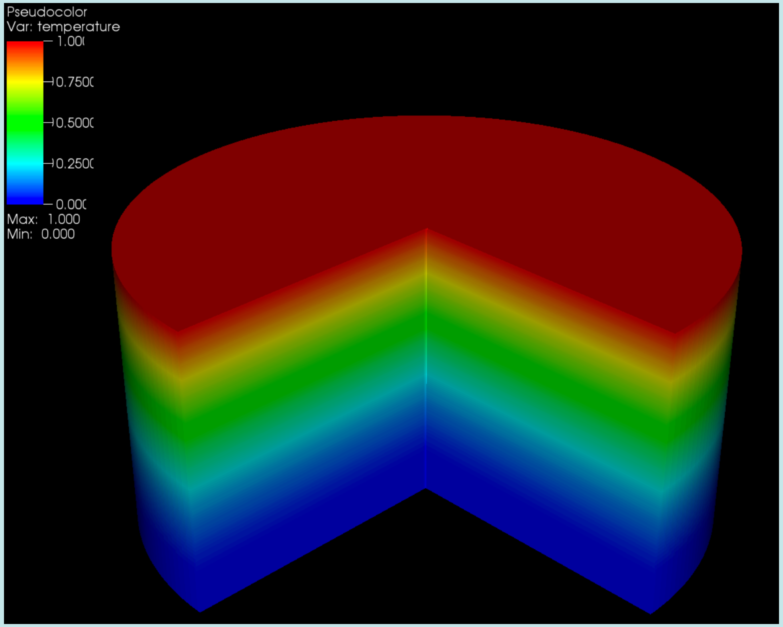
x0=-0.08



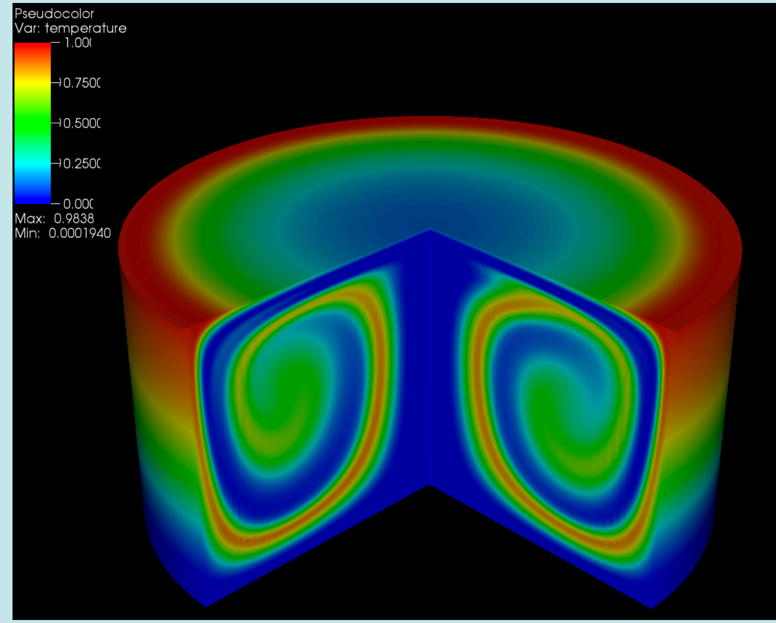
x0=-0.16



E=1/100, Ro=1

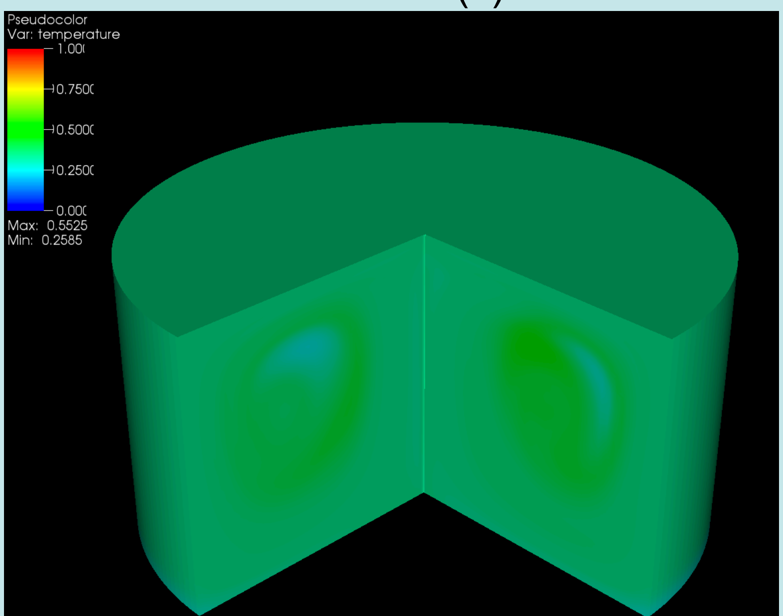


(a): t=0

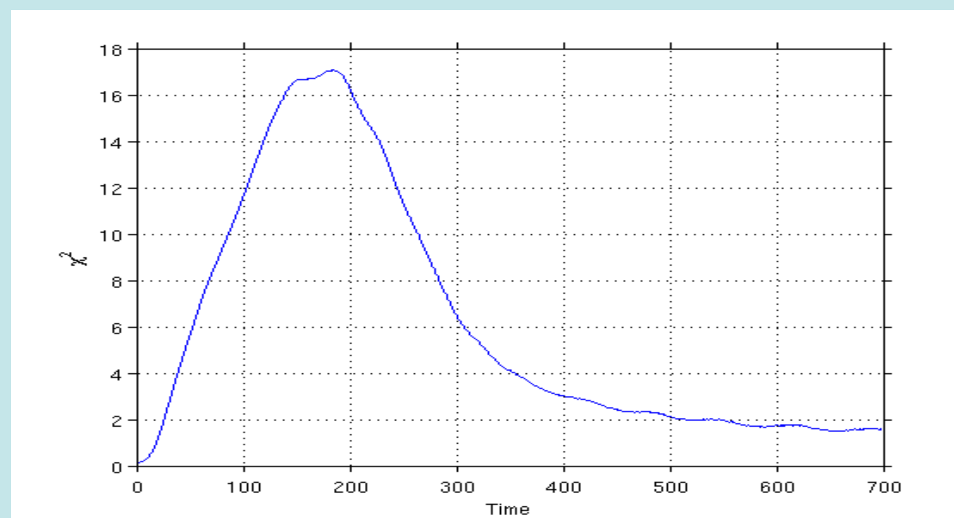


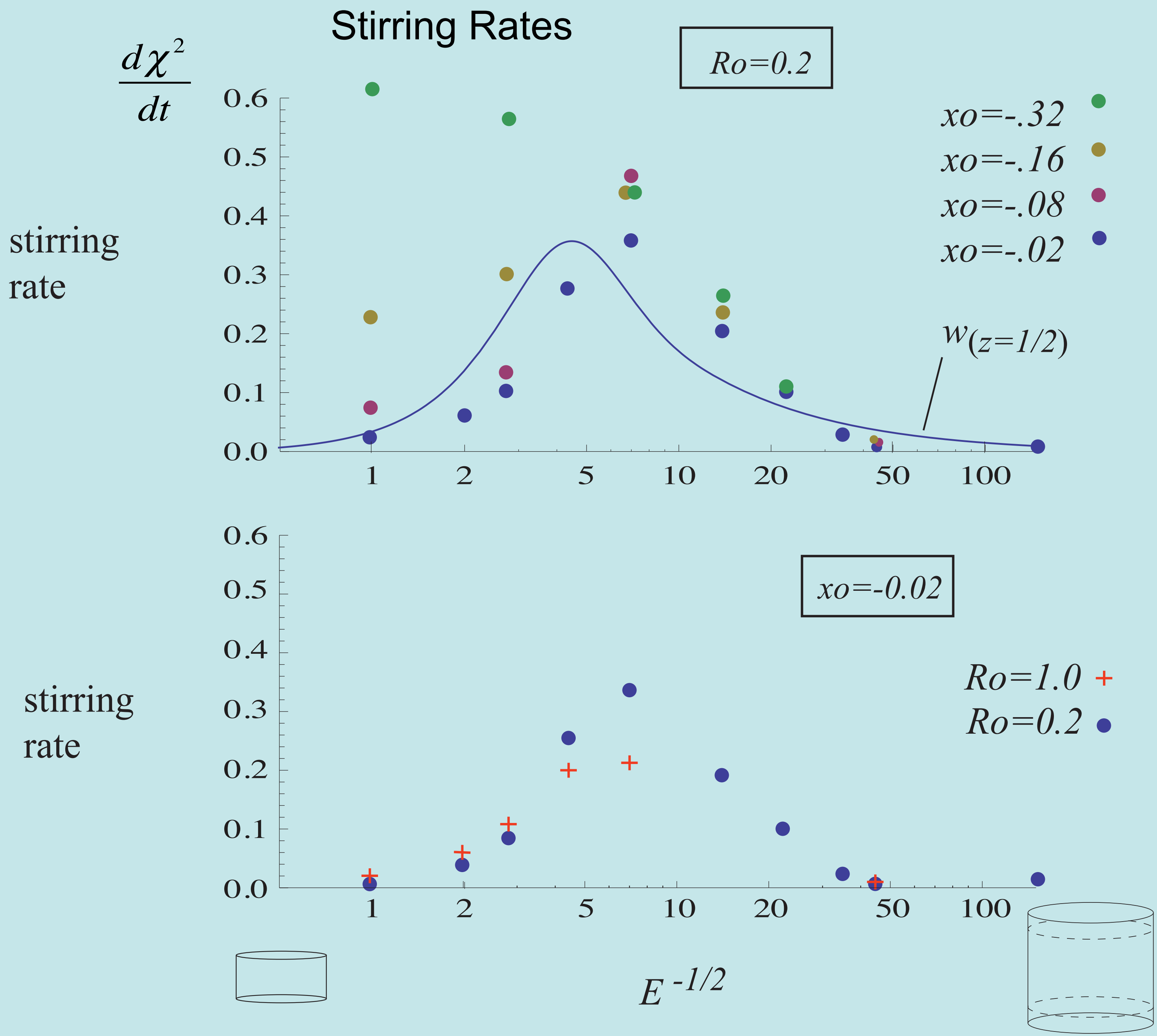
(b): t=23

(c): t=309



$$\chi^2(t) = \int_V |\nabla C|^2 dv \Big/ \int_V |C|^2 dv$$





Stirring Rates
$\frac{d\chi^2}{dt}$
Ro=0.2
stirring rate
xo=-.32
xo=-.16
xo=-.08
xo=-.02
w(z=1/2)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
1
2
5
10
20
50
100
xo=-0.02
stirring rate
Ro=1.0
Ro=0.2
$E^{-1/2}$

Challenges for this Group

1) We want to define and locate barriers in 3D flows with more general time dependence. Many of the methods discussed at this conference have the potential for doing so <u>in models</u>. But observations in 3D are not even remotely extensive enough to apply them.

2) How would one design a dye release experiment in order to visualize these structures?

3) Do the effects of background turbulence overwhelm chaotic advection?

4) How do we get a handle on stirring when the perturbation if finite. (No KAM; no resonance width formula.)

Boston Museum of Science.
(Spring, 2013)

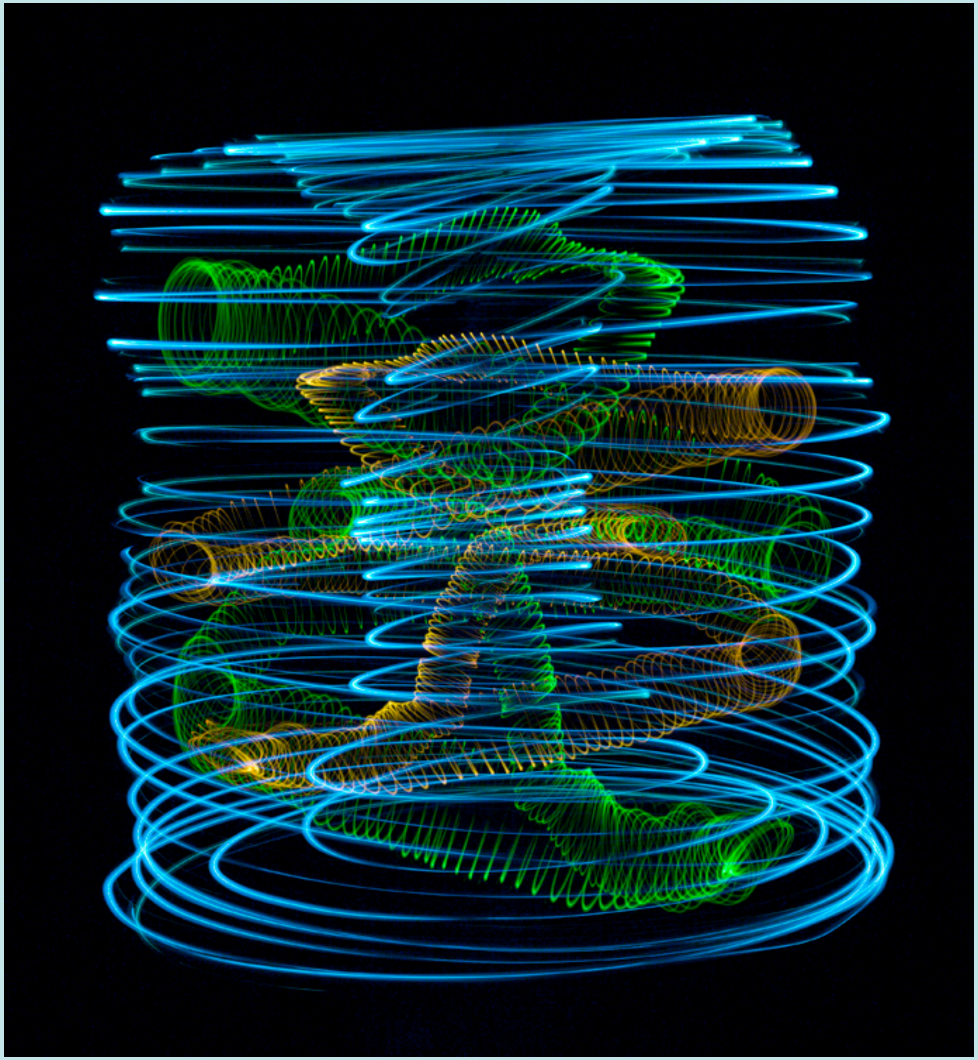
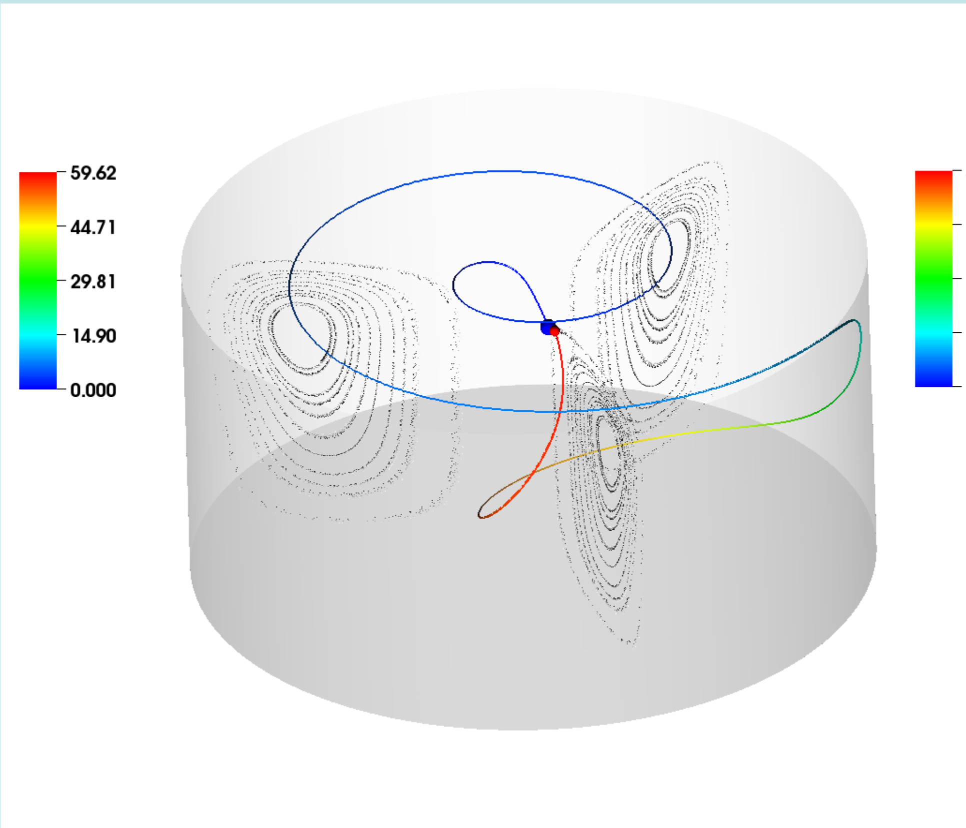


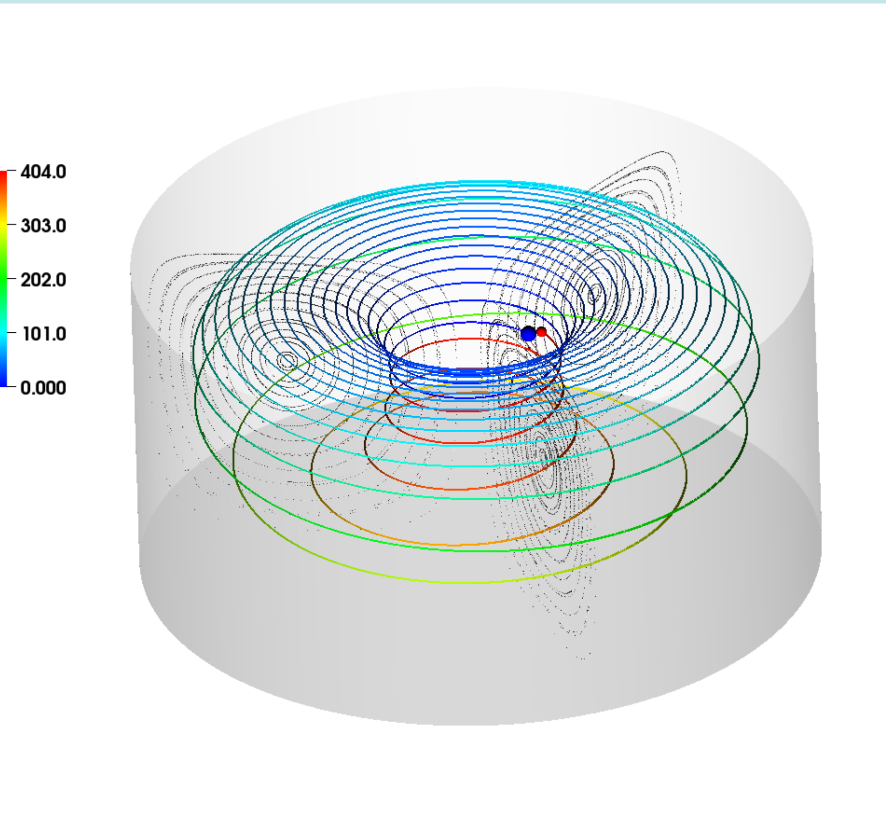
Photo by L. Pratt and A. Azure

Ro=1, *E*=1/100

Ro=1, *E*=1



(a)

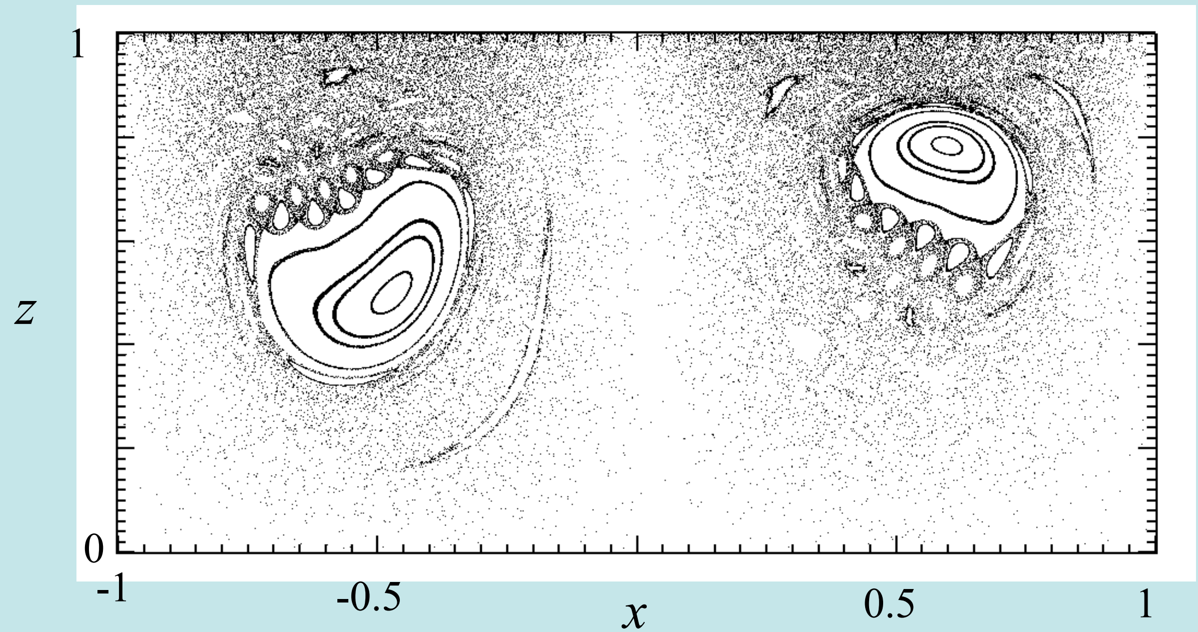


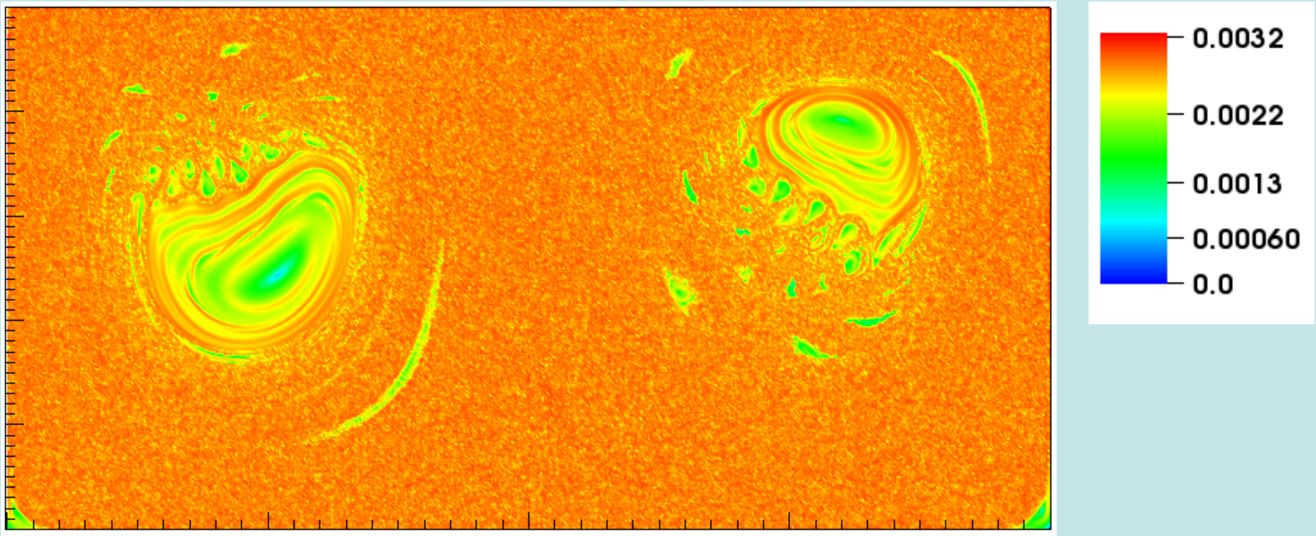
(b)

Summary

1) Stirring in a canonical model of a 3D flow with swirl and overturning can be highly nonhomogeneous due to the presence of complex barriers that separate mixed (chaotic) regions.
2) The stirring rate increases then decreases as E decreases below unity.
3) The addition of periodic time dependence and double resonance yields new structures.
4) Most promising application is to sub-mesoscale eddies at the ocean surface. w=.02m/s H=30m: $T_{overturn}$=hrs to days.
5) For larger features (mesoscale eddies, hurricanes), overturning time > life time of eddy.

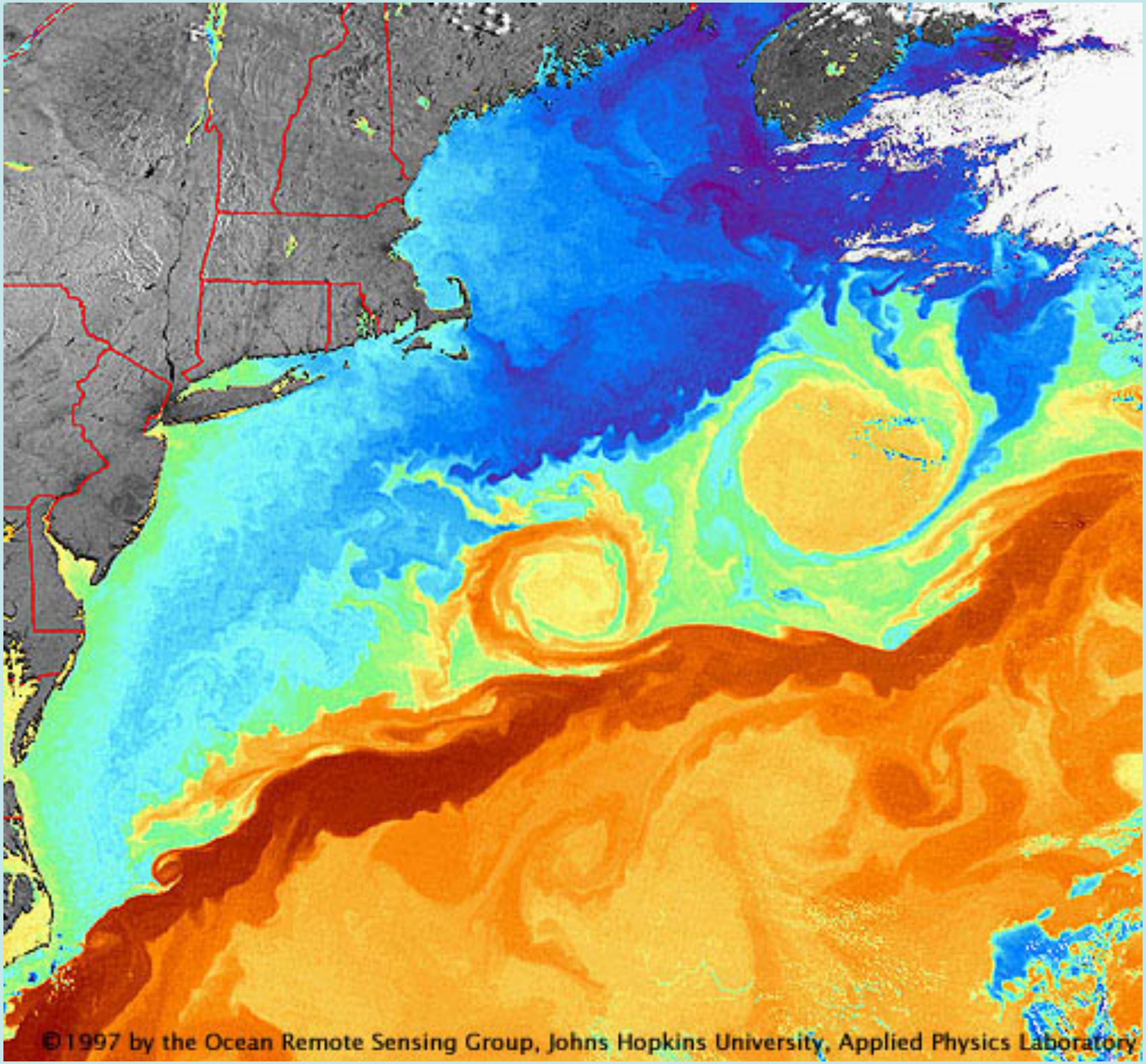
Re=20, Ro=1, x0=-0.02





Finite Time Lyapunov Exponents


Within, the ways of water

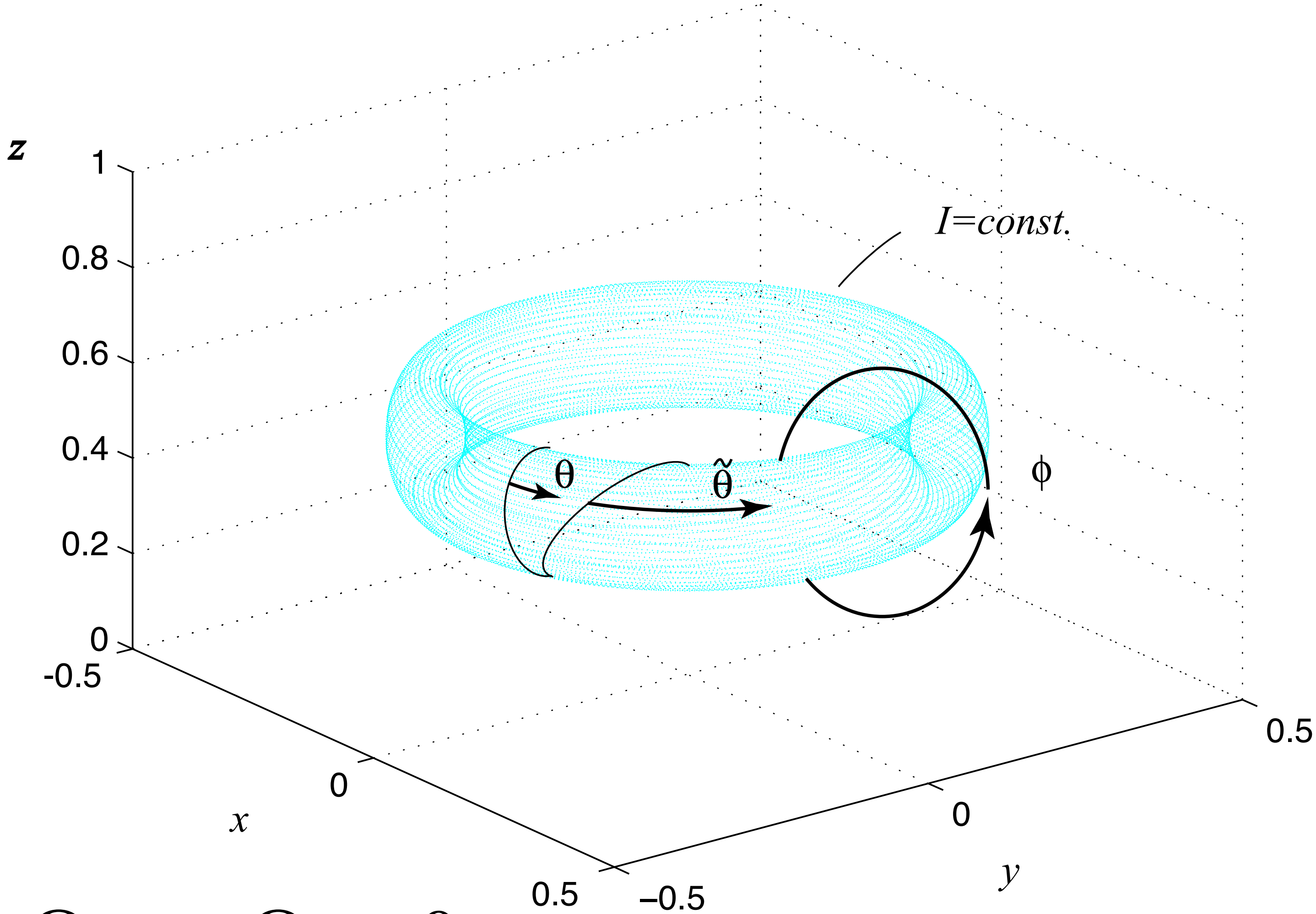

©1997 by the Ocean Remote Sensing Group, Johns Hopkins University, Applied Physics Laboratory

Action-Angle-Angle System (Mezic and Wiggins 1994)

$$\frac{d\tilde{\theta}}{dt} = \Omega_\theta(I)$$

$$\frac{d\phi}{dt} = \Omega_\phi(I)$$

$$\frac{dI}{dt} = 0$$



$$T_\phi = \frac{2\pi}{\Omega_\phi}$$

$$T_\theta = \frac{2\pi}{\Omega_\theta}$$

resonance: $n\Omega_\phi + m\Omega_\theta = 0$