Lagrangian transport barriers in three-dimensional unsteady flows

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Joint work with G. Haller

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Banff workshop
Uncovering Transport Barriers in Geophysical Flows

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- v can solve a PDE (e.g. Navier-Stokes) or be obtained from physical measurements
- Relevant structures are time-varying and only exist for finite time (e.g. fronts, oceanic eddies)

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- Sarcasm aside, examples include
  1. Forcasting for natural disasters (Olascoaga, Haller, Mezic, Peacock etc.),
  2. Agulhas eddies and climate change (Haller, Beron-Vera, Froyland, Beal, etc.)
  3. Plasma fusion (del-Castillo-Negrete, Morrison, B., etc.)
  4. Zonal jets (del-Castillo-Negrete, Rypina, Olascoaga, Beron-Vera, Haller, Froyland, Farazmand, B., etc.)
  5. Biological systems (Green, Rowley, Ouellette, Komoutsakous, Dabiri, Shadden, Ross, etc.)
  6. Theoretical descriptions (Haller, Froyland, Mezic, Mancho, Budisic, Allshouse, Thiffeault, Pratt, Kirwan, B., etc.)
  7. Last, but not least, "etc."

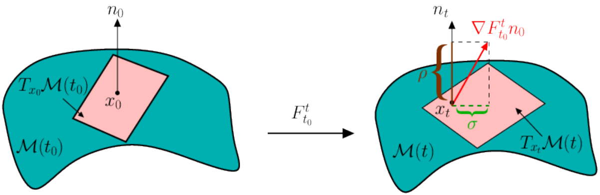
Lagrangian Coherent Structures (LCSs) are barriers to tranaport [1]

- Hyperbolic LCSs locally minimize/maximize normal repulsion ρ

$$\rho_{t_0}^t(x_0, n_0) = \langle n_t, \nabla F_{t_0}^t(x_0) n_0 \rangle$$

- Shear LCSs locally maximize tangential shear σ

$$\sigma_{t_0}^t(x_0, n_0) = |\nabla F_{t_0}^t(x_0) n_0 - \langle n_t, \nabla F_{t_0}^t(x_0) n_0 \rangle n_t|$$



- Transport barriers are hyperbolic or shear LCSs

[1] D.B. and G. Haller, Hyperbolic and Elliptic Transport Barriers in Three-Dimensional Unsteady Flows, *submitted*

ETH Zürich

Characterization of Hyperbolic and Shear LCS as Orthogonal Surfaces

- Theorem characterizing hyperbolic and shear LCSs:
- Let $C_{t_0}^t = \left(\nabla F_{t_0}^t\right)^* \nabla F_{t_0}^t$ be the Cauchy-Green strain tensor, ξ_i, λ_i be the eigenvectors and eigenvalues

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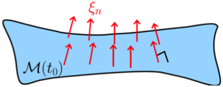
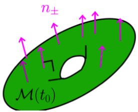
- If $\mathcal{M}(t)$ is a shear LCS then $\mathcal{M}(t_0) \perp n_+$ or $\mathcal{M}(t_0) \perp n_-$

$$n_{\pm} = \sqrt{\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+\sqrt{\lambda_n}}}\xi_1 \pm \sqrt{\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_1}+\sqrt{\lambda_n}}}\xi_3$$





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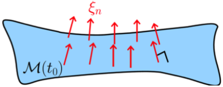

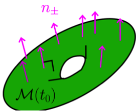

- ξ_3, ξ_1, $n_\pm$ are the optimal directions of repulsion, attraction, and shear.

Zero Helicity Condition for Orthogonal Surfaces

- If $\mathcal{M}(t_0) \perp \pi$ for a vector field π, then the helicity of π

$$H_\pi = \langle \nabla \times \pi, \pi \rangle$$

vanishes on $\mathcal{M}(t_0)$. (General geometric, mathematical constraint for orthogonal surfaces)

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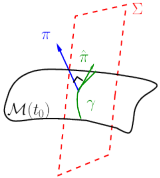
- The intersection γ is tangent to the the reduced field $\hat{\pi} = \pi \times n$, where $n \perp \Sigma$. $\hat{\pi}$ is a vector field on Σ

- Geometry of the reduced fields
  Think of π as ξ_1, ξ_3 or $n_\pm$



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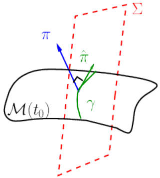


- Punchline: A cut γ of a strain/shear surface is a curve of zero helicity and an integral curve of $\hat{\xi}_1, \hat{\xi}_3$ or $\hat{n}_\pm$

Test case: Steady ABC Flow

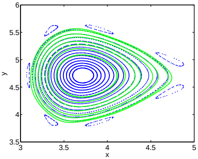
- As a proof of concept, we first consider the steady ABC flow (steady solution of 3D Euler's equation)

$$\dot{x} = A\sin z + C\cos y$$
$$\dot{y} = B\sin x + A\cos z$$
$$\dot{z} = C\sin y + B\cos x$$

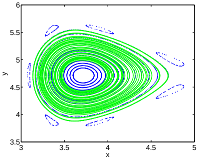
- Poincare plot on $\{z = 0\}$ visually shows KAM-like vortex structures

Elliptic barriers (closed shear LCSs) in the Steady ABC Flow

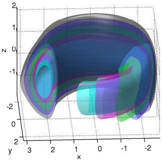
- Closed reduced shearlines (green) are trajectories of the reduced field $\hat{n}_{\pm}$ on $\{z = 0\}$



$t_0 + T = 40$



$t_0 + T = 150$



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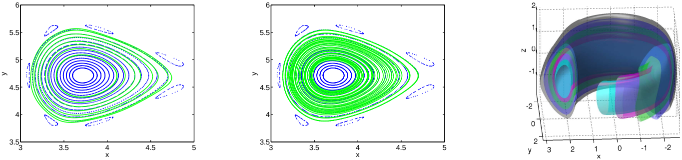

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- Trajectories are integrated for a fixed time for the full 3D flow (i.e. we do not do a 2D analysis of the Poincare map)

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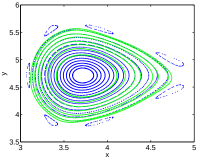

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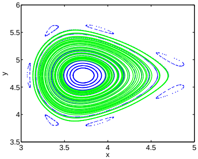

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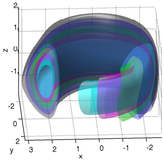
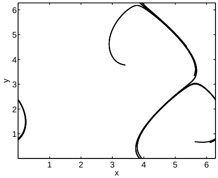


- Trajectories are integrated for a fixed time for the full 3D flow (i.e. we do not do a 2D analysis of the Poincare map)
- Significance: Reconstructed 3D KAM tori without using notions of invariance, steadiness, conjugacy to rotation, Birkhoff Egrodic Theorem, etc.

Repelling LCSs for Steady Case; $t_0 + T = 3$

- Reduced strainlines are integral curves of $\hat{\xi}_3$
- Compute reduced strainlines of zero helicity on $\{z = 0\}$



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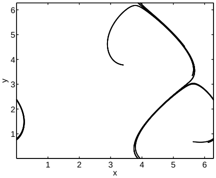
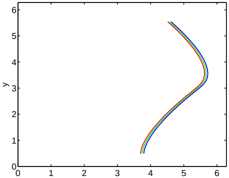
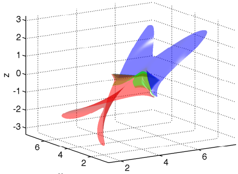


- See that they separate finite-time dynamics of upward and downward motions



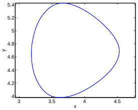


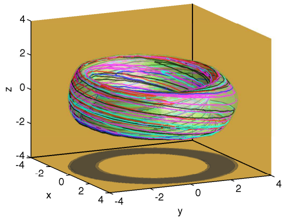
Shear Barriers for Periodic ABC Flow

- Temporally periodic ABC flow

$$\dot{x} = (A + 0.1 \sin t) \sin z + C \cos y$$
$$\dot{y} = B \sin x + (A + 0.1 \sin t) \cos z$$
$$\dot{z} = C \sin y + B \cos x$$

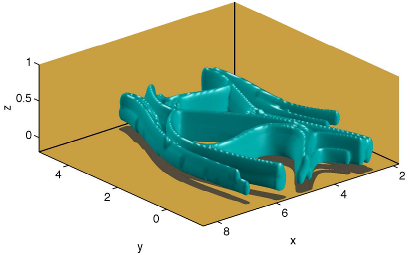
- KAM torus obtained from iterating a single closed reduced shearline under the temporal Poincare map $F^{2\pi}$





Repelling LCSs

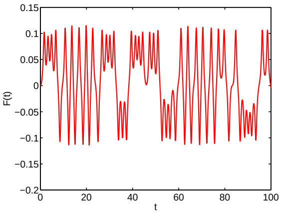
- Find reduced strainlines of zero helicity on multiple z slices
- Parallel computation, one core for each z slice
- Result for 3D barrier for integration time $t_0 + T = 4.0$



Temporally Chaotic Signal Added to ABC Flow

- We consider the aperiodically forced ABC flow

$$\begin{aligned} \dot{x} &= (A + F(t))\sin z + C\cos y \\ \dot{y} &= B\sin x + (A + F(t))\cos z \\ \dot{z} &= C\sin y + B\cos x \end{aligned}$$



Coherent Lagrangian Vortices in the Chaotically Forced ABC Flow: $t_0 = 0, t_0 + T = 100$.

- In this setting, there are no invariant sets of F^T for any time T

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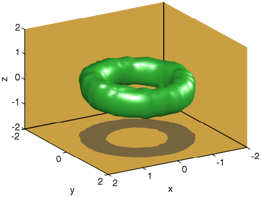
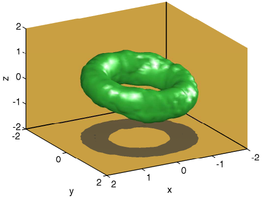
- In this setting, there are no invariant sets of F^T for any time T
- Used a family Π_σ of 150 planes to cut the torus.

Elliptic LCS at $t_0 = 0$



Advected elliptic LCS at $t = 100$



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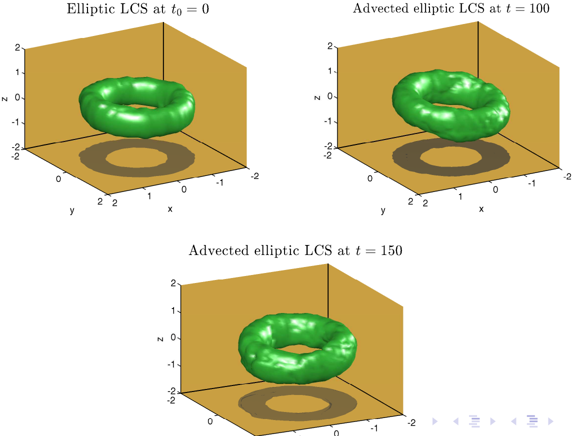

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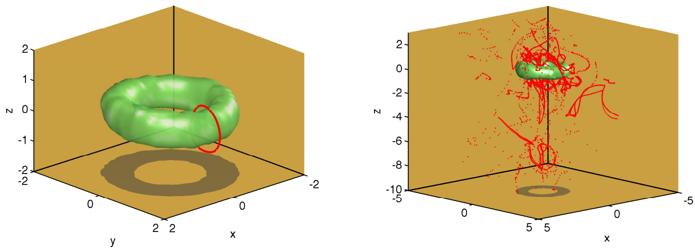

- Coherent Lagrangian vortices maintain their shape over the integration time, and are boundaries of vortices in unsteady flows

Concluding Remarks

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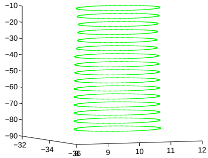
Concluding Remarks

- Presented a theory of shear and hyperbolic transport barriers for 3D unsteady flows
- Based on a rigorous mathematical/physical description (i.e. no heuristics, e.g. from steady flows) that was shown to capture vortices in steady flows.
- Ongoing work includes using the theory to detect elliptic barriers in 3D velocity data



- Thank you for your attention!