A frame energy for tori immersed in $\mathbb{R}^m$: sharp Willmore-conjecture type lower bound, regularity of critical points and applications

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Indeed: Best frame$\rightarrow$ global conformal structure of the underlying abstract surface + local conformal coordinates.

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- We define the frame energy

$$\mathcal{F}(\vec{\Phi}, \vec{e}) := \frac{1}{4} \int_{\mathbb{T}^2} |d\vec{e}|^2 dvol_g$$

where $dvol_g$ is the volume form associated to $g := \vec{\Phi}^*(g_{\mathbb{R}^m})$ and $|d\vec{e}|^2 := \sum_{i,j,k=1}^{2} g^{ij} \partial_{x^i} \vec{e}_k \cdot \partial_{x^j} \vec{e}_k$.

Relation of the frame energy $\mathcal{F}$ with the Willmore functional W

By projecting on the tangent and the normal space $d\vec{e}_i$ one gets

$$\mathcal{F}(\vec{\Phi}, \vec{e}) = \mathcal{F}_T(\vec{\Phi}, \vec{e}) + W(\vec{\Phi})$$

where

$$\mathcal{F}_T(\vec{\Phi}, \vec{e}) = \frac{1}{2} \int_{\mathbb{T}^2} |\vec{e}_1 \cdot d\vec{e}_2|^2 dvol_g \quad \text{Tangential frame energy}$$

and

$$W(\vec{\Phi}) := \int_{\mathbb{T}^2} H^2 dvol_g = \frac{1}{4} \int_{\mathbb{T}^2} |\mathbb{I}|^2 dvol_g \quad \text{Willmore functional}$$

($\mathbb{I}$ is the second fundamental form of $\vec{\Phi}$ and $H = \frac{1}{2} g^{ij} \mathbb{I}_{ij}$ is the mean curvature).

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$\Rightarrow$ $\mathcal{F}$ can be seen as a more coercive Willmore energy where the extra term $\mathcal{F}_T$ prevents

- degeneration under Moebius transformations of $\mathbb{R}^m$
- degeneration of conformal classes of the underlying abstract surface

(both the last two difficulties are present, and are non trivial issues, for the Willmore functional)

$\Rightarrow$ good chances to perform minimization of $\mathcal{F}$.

Calculus of variations of $\mathcal{F}$: weak setting

- Weak immersions: fix a reference metric g_0 on $\mathbb{T}^2$, we say that $\vec{\Phi} \in \mathcal{E}(\mathbb{T}^2, \mathbb{R}^3)$ iff
  i) $\vec{\Phi} \in W^{1,\infty}(\mathbb{T}^2, \mathbb{R}^3)$ and called $g_{\vec{\Phi}} := \vec{\Phi}^* g_{\mathbb{R}^3}$ there exists $C_{\vec{\Phi}} > 1$ s.t.
  $$C_{\vec{\Phi}}^{-1} g_{\vec{\Phi}} \leq g_0 \leq C_{\vec{\Phi}} g_{\vec{\Phi}} \quad a.e.$$

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- Weak framed immersions$=\{(\vec{\Phi}, \vec{e}) : \vec{\Phi}$ and $\vec{e}$ as above$\}$ form a Banach manifold.

Calculus of variations of $\mathcal{F}$: Frechét differentability and the PDE

Proposition

$\mathcal{F}$ is Frechét differentiable on the space of weak framed immersions and $(\vec{\Phi}, \vec{e})$ is a critical point of $\mathcal{F}$ iff

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\begin{aligned}
0 \;=\; & div\Big[\frac{1}{2}\Big(\nabla \vec{H} - 3\nabla H\,\vec{n} + \nabla^{\perp}\vec{n} \times \vec{H}\Big) - \vec{\mathbb{I}}\llcorner_g(\vec{e}_2 \cdot \nabla^{\perp}\vec{e}_1) \\
& \quad - \vec{e}_2 \cdot \nabla^{\perp}\vec{e}_1\,(\vec{e}_2 \cdot \nabla\vec{e}_1, \nabla\vec{\Phi})_g + \frac{1}{2}|\vec{e}_2 \cdot \nabla\vec{e}_1|_g^2 \nabla^{\perp}\vec{\Phi}\Big].
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\end{aligned}
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Remark: The equation is 4^{th} order non linear elliptic and critical (criticality is a common feature of geometric PDEs: Willmore,Harmonic maps, CMC surfaces, Yang Mills, Yamabe, etc.)

$\Rightarrow$ challenging to prove the regularity of critical points of the frame energy.

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Theorem

Let $\vec{\Phi}$ be a weak immersion of the disc D^2 into $\mathbb{R}^3$ and let $\vec{e} = (\vec{e}_1, \vec{e}_2)$ be a moving frame on $\vec{\Phi}$ such that $(\vec{\Phi}, \vec{e})$ is a critical point of the frame energy $\mathcal{F}$. Then, up to a bilipschitz reparametrization we have locally that $\vec{\Phi}$ is conformal and $\vec{e}$ is the coordinate moving frame associated to $\vec{\Phi}$, i.e.
$(\vec{e}_1, \vec{e}_2) = \left(\frac{\partial_{x_1}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi}|}, \frac{\partial_{x_2}\vec{\Phi}}{|\partial_{x_2}\vec{\Phi}|} \right)$. *Moreover, there exist $\rho \in (0,1)$ such that $\vec{\Phi}|_{B_\rho(0)}$ is a C^∞ immersion.*

Application to regular homotopy classes: basic defintions

IDEA: Morse Theory of the frame energy $\mathcal{F}$ on the space of framed tori $\to$ explore the topology of the space of framed tori.

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DEFINITION: Let Σ^2 be a closed surface, then two immersions $f, g : \Sigma^2 \hookrightarrow \mathbb{R}^m$ are regularly homotopic if there exists $H : \Sigma^2 \times [0,1] \rightarrow \mathbb{R}^m$ smooth s.t.
(i) $H_0(\cdot) = H(\cdot, 0) = f$, $H_1(\cdot) = H(., 1) = g$
(ii) H_t is an immersion of Σ^2 in $\mathbb{R}^m$ for every $t \in [0,1]$.

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REMARK:
- intermediate notion between homotopy (just topological) and isotopy (for every $t \in [0,1]$ H_t is an embedding)
-to be more precise we will consider regular homotopic immersions UP TO DIFFEOMORPHISMS IN THE DOMAIN

Application to regular homotopy classes: our problem

Some history

- 1958 Smale: $\forall f, g : S^2 \hookrightarrow \mathbb{R}^3$ are regularly homotopic ($\to$ sphere eversion); in $\mathbb{R}^4$ there are instead countably many regular homotopy classes of spheres.

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Minimization of $\mathcal{F}$ in regular homotopy classes

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Theorem

Fix σ a regular homotopy class of immersions of the 2-torus $\mathbb{T}^2$ into $\mathbb{R}^3$. Then there exists a smooth conformal immersion $\vec{\Phi} : \mathbb{T}^2 \hookrightarrow \mathbb{R}^3$, with $\vec{\Phi} \in \sigma$, such that, called $\vec{e} := (\vec{e}_1, \vec{e}_2) := \left(\frac{\partial_{x_1}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi}|}, \frac{\partial_{x_2}\vec{\Phi}}{|\partial_{x_2}\vec{\Phi}|} \right)$ the coordinate moving frame, the couple $(\vec{\Phi}, \vec{e})$ minimizes the frame energy $\mathcal{F}$ among all weak immersions of $\mathbb{T}^2$ into $\mathbb{R}^3$ lying in σ and all $W^{1,2}$ moving frames on $\vec{\Phi}(\mathbb{T}^2)$:

$$\mathcal{F}(\vec{\Phi}, \vec{e}) = \min \left\{ \mathcal{F}(\tilde{\vec{\Phi}}, \tilde{\vec{e}}) : \tilde{\vec{\Phi}} \in \mathcal{E}(\mathbb{T}^2, \mathbb{R}^3), \tilde{\vec{\Phi}} \in \sigma, \tilde{\vec{e}} \in W^{1,2}(\mathbb{T}^2) \right\}.$$

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b) The minimization of the Willmore functional in regular homotopy classes is more difficult (maybe even not possible) because of
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here both are excluded.
The first by the previous Proposition, the second by a Wente-type estimate of λ in terms of $\mathcal{F}$.

Free minimization

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Theorem Let $\vec{\Phi} : \mathbb{T}^2 \hookrightarrow \mathbb{R}^m$ be a smooth immersion of the 2-dimensional torus into the Euclidean $3 \leq m$-dimensional space and let $\vec{e} = (\vec{e}_1, \vec{e}_2)$ be any moving frame along $\vec{\Phi}$. Then

$$\mathcal{F}(\vec{\Phi}, \vec{e}) := \frac{1}{4} \int_{\mathbb{T}^2} |d\vec{e}|^2 \; dvol_g \geq 2\pi^2 \quad .$$

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Question: rigidity?
YES! If equality holds then it must be $m \geq 4$, $\vec{\Phi}(\mathbb{T}^2) \subset \mathbb{R}^m$ must be, up to isometries and dilations in $\mathbb{R}^m$, the Clifford torus

$$T_{Cl} := S^1 \times S^1 \subset \mathbb{R}^4 \subset \mathbb{R}^m \quad ,$$

and $\vec{e}$ must be, up to a constant rotation on $T(\vec{\Phi}(\mathbb{T}^2))$, the moving frame given by $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi})$, where of course (θ, φ) are natural flat the coordinates on $S^1 \times S^1$.

Comments on the lower bound

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- In codimension one the lower bound is of course implied by the proof of Willmore conjecture by Marques-Neves, but our lower bound holds in arbitrary codimension.
- Surprisingly, our lower bound works better in higher codimension: it is sharp and rigid in codimension at least 2, but in codimension one it is not realized.
- Topping (2000), using integral geometry, proved an analogous lower bound for an analogous energy for immersions of rectangular tori into S^3.

Sketch of the proof of the lower bound

Lemma Let $(\vec{\Phi}, \vec{e})$ be a framed immersion of $\mathbb{T}^2$ into $\mathbb{R}^m$, $m \geq 3$, and denote $\tau \in M$ the conformal class induced by $\vec{\Phi}$. Then

$$\mathcal{F}(\vec{\Phi}, \vec{e}) \geq \pi^2 \left(\tau_2 + \frac{1}{\tau_2}\right) \frac{\sin^2\theta}{\sin^2\theta + \cos^4\theta}.$$

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-Now let $f(\tau)$ denote the right hand side and define $\Omega := \left\{(\tau_1, \tau_2) : \left(\tau_1 - \frac{1}{2}\right)^2 + (\tau_2 - 1)^2 \leq \frac{1}{4}\right\} \cap M^+$.

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$\Omega := \left\{(\tau_1, \tau_2) : \left(\tau_1 - \frac{1}{2}\right)^2 + (\tau_2 - 1)^2 \leq \frac{1}{4}\right\} \cap M^+$.
-Then, by direct computation, $f|_{\partial\Omega} \geq 2\pi^2$ and is monotone strictly increasing in τ_2 for $\tau_2 \geq 1 \to$ lower bound true for $\tau \notin \Omega$.
-But if $\tau \in \Omega$ then the Willmore conjecture holds by the work of Li-Yau and Montiel-Ros. So we conclude.

Open problems

- Who is the global minimizer of $\mathcal{F}$ in $\mathbb{R}^3$? The Clifford torus?
- Who is the knotted minimizer of $\mathcal{F}$ in $\mathbb{R}^3$? The diagonal double cover of the Clifford torus (proposed by Kusner in 1983)?
- Minimization of $\mathcal{F}$ in regular homotopy classes in $\mathbb{R}^4$

!!THANK YOU FOR THE ATTENTION!!