The Topology of the Limits of Sequences of Embedded Minimal Disks (joint with G. Tinaglia)

J. Bernstein

Johns Hopkins University

December 16
Geometric Variational Problems

Outline

Background and Motivation

Basic Definitions

Fix a Riemannian three-manifold Ω – for instance an open subset of $\mathbb{R}^3$.

Recall:

- A set $\Sigma \subset \Omega$ is an *embedded (smooth) surface* if, locally, it can be smoothly straightened – i.e., $\forall p \in \Sigma$, there is an open neighborhood $U_p \subset \Omega$ of p and a C^∞ diffeomorphism $\phi_p : (U_p, p) \to (B_1, 0)$ so that $\phi_p(\Sigma \cap U_p) = \{x_3 = 0\} \cap B_1$;
- $\Sigma \subset \Omega$ is a *properly embedded surface* if it is an embedded surface and is a relatively closed in Ω – i.e. $\bar{\Sigma} = \Sigma$;
- An embedded surface is *minimal* if $\mathbf{H} \equiv 0$.

If Σ is properly embedded, then it is minimal if and only if its area is stationary with respect to compactly supported (in Ω) variations.

Basic Definitions

Fix a Riemannian three-manifold Ω – for instance an open subset of $\mathbb{R}^3$.

Recall:

- A set $\Sigma \subset \Omega$ is an *embedded (smooth) surface* if, locally, it can be smoothly straightened – i.e., $\forall p \in \Sigma$, there is an open neighborhood $U_p \subset \Omega$ of p and a C^∞ diffeomorphism $\phi_p : (U_p, p) \to (B_1, 0)$ so that $\phi_p(\Sigma \cap U_p) = \{x_3 = 0\} \cap B_1$;
- $\Sigma \subset \Omega$ is a *properly embedded surface* if it is an embedded surface and is a relatively closed in Ω – i.e. $\bar{\Sigma} = \Sigma$;
- An embedded surface is *minimal* if $\mathbf{H} \equiv 0$.

If Σ is properly embedded, then it is minimal if and only if its area is stationary with respect to compactly supported (in Ω) variations.

Basic Definitions

Fix a Riemannian three-manifold Ω – for instance an open subset of $\mathbb{R}^3$.

Recall:

- A set $\Sigma \subset \Omega$ is an *embedded (smooth) surface* if, locally, it can be smoothly straightened – i.e., $\forall p \in \Sigma$, there is an open neighborhood $U_p \subset \Omega$ of p and a C^∞ diffeomorphism $\phi_p : (U_p, p) \to (B_1, 0)$ so that $\phi_p(\Sigma \cap U_p) = \{x_3 = 0\} \cap B_1$;
- $\Sigma \subset \Omega$ is a *properly embedded surface* if it is an embedded surface and is a relatively closed in Ω – i.e. $\bar{\Sigma} = \Sigma$;
- An embedded surface is *minimal* if $\mathbf{H} \equiv 0$.

If Σ is properly embedded, then it is minimal if and only if its area is stationary with respect to compactly supported (in Ω) variations.

Basic Definitions

Fix a Riemannian three-manifold Ω – for instance an open subset of $\mathbb{R}^3$.

Recall:

- A set $\Sigma \subset \Omega$ is an *embedded (smooth) surface* if, locally, it can be smoothly straightened – i.e., $\forall p \in \Sigma$, there is an open neighborhood $U_p \subset \Omega$ of p and a C^∞ diffeomorphism $\phi_p : (U_p, p) \to (B_1, 0)$ so that $\phi_p(\Sigma \cap U_p) = \{x_3 = 0\} \cap B_1$;
- $\Sigma \subset \Omega$ is a *properly embedded surface* if it is an embedded surface and is a relatively closed in Ω – i.e. $\bar{\Sigma} = \Sigma$;
- An embedded surface is *minimal* if $\mathbf{H} \equiv 0$.

If Σ is properly embedded, then it is minimal if and only if its area is stationary with respect to compactly supported (in Ω) variations.

Basic Definitions

Fix a Riemannian three-manifold Ω – for instance an open subset of $\mathbb{R}^3$.

Recall:

- A set $\Sigma \subset \Omega$ is an *embedded (smooth) surface* if, locally, it can be smoothly straightened – i.e., $\forall p \in \Sigma$, there is an open neighborhood $U_p \subset \Omega$ of p and a C^∞ diffeomorphism $\phi_p : (U_p, p) \to (B_1, 0)$ so that $\phi_p(\Sigma \cap U_p) = \{x_3 = 0\} \cap B_1$;
- $\Sigma \subset \Omega$ is a *properly embedded surface* if it is an embedded surface and is a relatively closed in Ω – i.e. $\bar{\Sigma} = \Sigma$;
- An embedded surface is *minimal* if $\mathbf{H} \equiv 0$.

If Σ is properly embedded, then it is minimal if and only if its area is stationary with respect to compactly supported (in Ω) variations.

Sequences of Minimal Surfaces

We begin by posing some (vague) motivational questions:

Question

- *What classes of smooth minimal surfaces have good (pre-)compactness properties?*

That is, for a sequence of surfaces $\mathcal{S} = \{\Sigma_i\}$ in some class $\mathcal{M}$:

- *When do sequences always smoothly subconverge?*
- *When is there a well-behaved set away from which the sequence smoothly subconverges?*
- *How much bigger is $\bar{\mathcal{M}}$, than $\mathcal{M}$?*

Ideally, answers to these questions yield information about both the structure of $\mathcal{M}$ and of its elements.

Sequences of Minimal Surfaces

We begin by posing some (vague) motivational questions:

Question

- *What classes of smooth minimal surfaces have good (pre-)compactness properties?*

That is, for a sequence of surfaces $\mathcal{S} = \{\Sigma_i\}$ in some class $\mathcal{M}$:

- *When do sequences always smoothly subconverge?*
- *When is there a well-behaved set away from which the sequence smoothly subconverges?*
- *How much bigger is $\bar{\mathcal{M}}$, than $\mathcal{M}$?*

Ideally, answers to these questions yield information about both the structure of $\mathcal{M}$ and of its elements.

Sequences of Minimal Surfaces

We begin by posing some (vague) motivational questions:

Question

- *What classes of smooth minimal surfaces have good (pre-)compactness properties?*

That is, for a sequence of surfaces $\mathcal{S} = \{\Sigma_i\}$ in some class $\mathcal{M}$:

- *When do sequences always smoothly subconverge?*
- *When is there a well-behaved set away from which the sequence smoothly subconverges?*
- *How much bigger is $\bar{\mathcal{M}}$, than $\mathcal{M}$?*

Ideally, answers to these questions yield information about both the structure of $\mathcal{M}$ and of its elements.

Sequences of Minimal Surfaces

We begin by posing some (vague) motivational questions:

Question

- *What classes of smooth minimal surfaces have good (pre-)compactness properties?*

That is, for a sequence of surfaces $\mathcal{S} = \{\Sigma_i\}$ in some class $\mathcal{M}$:

- *When do sequences always smoothly subconverge?*
- *When is there a well-behaved set away from which the sequence smoothly subconverges?*
- *How much bigger is $\bar{\mathcal{M}}$, than $\mathcal{M}$?*

Ideally, answers to these questions yield information about both the structure of $\mathcal{M}$ and of its elements.

Sequences of Minimal Surfaces

We begin by posing some (vague) motivational questions:

Question

- *What classes of smooth minimal surfaces have good (pre-)compactness properties?*

That is, for a sequence of surfaces $\mathcal{S} = \{\Sigma_i\}$ *in some class* $\mathcal{M}$*:*

- *When do sequences always smoothly subconverge?*
- *When is there a well-behaved set away from which the sequence smoothly subconverges?*
- *How much bigger is* $\bar{\mathcal{M}}$*, than* $\mathcal{M}$*?*

Ideally, answers to these questions yield information about both the structure of $\mathcal{M}$ and of its elements.

Classes of Minimal Surfaces

We introduce some classes of minimal surfaces in a fixed Riemannian three-manifold Ω:

- Let $\mathcal{E}_\Omega$ be set of properly embedded minimal surfaces in Ω;
- Let $\mathcal{E}_\Omega(e, g) \subset \mathcal{E}_\Omega$ be oriented surfaces which are topologically genus-g surfaces with e punctures;
- Let $\mathcal{A}_\Omega(N) \subset \mathcal{E}_\Omega$ be set of surfaces so that for any $\Sigma \in \mathcal{A}_\Omega(N)$ and any $B_\rho(p) \subset \Omega$,

$$\frac{Area(B_\rho(p) \cap \Sigma)}{\pi \rho^2} \leq N.$$

- Let $\mathcal{TC}_\Omega(N) \subset \mathcal{E}_\Omega$ be surfaces Σ so that

$$\int_\Sigma |A_\Sigma|^2 \leq 8\pi N;$$

Classes of Minimal Surfaces

We introduce some classes of minimal surfaces in a fixed Riemannian three-manifold Ω:

- Let $\mathcal{E}_\Omega$ be set of properly embedded minimal surfaces in Ω;
- Let $\mathcal{E}_\Omega(e, g) \subset \mathcal{E}_\Omega$ be oriented surfaces which are topologically genus-g surfaces with e punctures;
- Let $\mathcal{A}_\Omega(N) \subset \mathcal{E}_\Omega$ be set of surfaces so that for any $\Sigma \in \mathcal{A}_\Omega(N)$ and any $B_\rho(p) \subset \Omega$,

$$\frac{Area(B_\rho(p) \cap \Sigma)}{\pi \rho^2} \leq N.$$

- Let $\mathcal{TC}_\Omega(N) \subset \mathcal{E}_\Omega$ be surfaces Σ so that

$$\int_\Sigma |A_\Sigma|^2 \leq 8\pi N;$$

Classes of Minimal Surfaces

We introduce some classes of minimal surfaces in a fixed Riemannian three-manifold Ω:

- Let $\mathcal{E}_\Omega$ be set of properly embedded minimal surfaces in Ω;
- Let $\mathcal{E}_\Omega(e, g) \subset \mathcal{E}_\Omega$ be oriented surfaces which are topologically genus-g surfaces with e punctures;
- Let $\mathcal{A}_\Omega(N) \subset \mathcal{E}_\Omega$ be set of surfaces so that for any $\Sigma \in \mathcal{A}_\Omega(N)$ and any $B_\rho(p) \subset \Omega$,

$$\frac{Area(B_\rho(p) \cap \Sigma)}{\pi \rho^2} \leq N.$$

- Let $\mathcal{TC}_\Omega(N) \subset \mathcal{E}_\Omega$ be surfaces Σ so that

$$\int_\Sigma |A_\Sigma|^2 \leq 8\pi N;$$

Classes of Minimal Surfaces

We introduce some classes of minimal surfaces in a fixed Riemannian three-manifold Ω:

- Let $\mathcal{E}_\Omega$ be set of properly embedded minimal surfaces in Ω;
- Let $\mathcal{E}_\Omega(e,g) \subset \mathcal{E}_\Omega$ be oriented surfaces which are topologically genus-g surfaces with e punctures;
- Let $\mathcal{A}_\Omega(N) \subset \mathcal{E}_\Omega$ be set of surfaces so that for any $\Sigma \in \mathcal{A}_\Omega(N)$ and any $B_\rho(p) \subset \Omega$,

$$\frac{Area(B_\rho(p) \cap \Sigma)}{\pi \rho^2} \leq N.$$

- Let $\mathcal{TC}_\Omega(N) \subset \mathcal{E}_\Omega$ be surfaces Σ so that

$$\int_\Sigma |A_\Sigma|^2 \leq 8\pi N;$$

Geometric Arzela-Ascoli Theorem

Theorem

Fix an $N > 0$, a Riemannian three-manifold Ω and an increasing sequence of open subsets

$$\Omega_1 \subset \Omega_2 \subset \cdots$$

so that $\Omega = \bigcup_i \Omega_i$. If $\Sigma_i \in \mathcal{E}_{\Omega_i} \cap \mathcal{A}_{\Omega_i}(N)$ satisfy

$$\sup_{K \cap \Sigma_i} |A_{\Sigma_i}| \leq C(K)$$

for every $K \subset\subset \Omega$, then the Σ_i subconverge smoothly on compact subsets of Ω (with multiplicity) to $\Sigma \in \mathcal{E}_\Omega \cap \mathcal{A}_\Omega(N)$.

Sketch of Proof

- For any smooth surface Σ and point $p \in \Sigma$ can express Σ near p as a graph (suitable understood) over $T_p\Sigma$;
- The L^∞ norm of A_Σ gives a fixed scale for such a graph and gives quantiative $C^{1,1}$ bounds;
- Schauder theory gives uniform C^∞ estimates (on a smaller scale) – usual Arzela-Ascoli gives good convergence of these graphs;
- These estimates together with area bounds imply that there are at most finitely many such “sheets” of Σ near p;
- Embeddedness is preserved by the strict maximum principle.

Sketch of Proof

- For any smooth surface Σ and point $p \in \Sigma$ can express Σ near p as a graph (suitable understood) over $T_p\Sigma$;
- The L^∞ norm of A_Σ gives a fixed scale for such a graph and gives quantiative $C^{1,1}$ bounds;
- Schauder theory gives uniform C^∞ estimates (on a smaller scale) – usual Arzela-Ascoli gives good convergence of these graphs;
- These estimates together with area bounds imply that there are at most finitely many such “sheets” of Σ near p;
- Embeddedness is preserved by the strict maximum principle.

Sketch of Proof

- For any smooth surface Σ and point $p \in \Sigma$ can express Σ near p as a graph (suitable understood) over $T_p\Sigma$;
- The L^∞ norm of A_Σ gives a fixed scale for such a graph and gives quantiative $C^{1,1}$ bounds;
- Schauder theory gives uniform C^∞ estimates (on a smaller scale) – usual Arzela-Ascoli gives good convergence of these graphs;
- These estimates together with area bounds imply that there are at most finitely many such “sheets” of Σ near p;
- Embeddedness is preserved by the strict maximum principle.

Sketch of Proof

- For any smooth surface Σ and point $p \in \Sigma$ can express Σ near p as a graph (suitable understood) over $T_p\Sigma$;
- The L^∞ norm of A_Σ gives a fixed scale for such a graph and gives quantiative $C^{1,1}$ bounds;
- Schauder theory gives uniform C^∞ estimates (on a smaller scale) – usual Arzela-Ascoli gives good convergence of these graphs;
- These estimates together with area bounds imply that there are at most finitely many such "sheets" of Σ near p;
- Embeddedness is preserved by the strict maximum principle.

Sketch of Proof

- For any smooth surface Σ and point $p \in \Sigma$ can express Σ near p as a graph (suitable understood) over $T_p\Sigma$;
- The L^∞ norm of A_Σ gives a fixed scale for such a graph and gives quantiative $C^{1,1}$ bounds;
- Schauder theory gives uniform C^∞ estimates (on a smaller scale) – usual Arzela-Ascoli gives good convergence of these graphs;
- These estimates together with area bounds imply that there are at most finitely many such "sheets" of Σ near p;
- Embeddedness is preserved by the strict maximum principle.

Generalizations

Some natural directions to generalize this theorem.

- Drop the curvature bound – leads to measure theoretic considerations;
- Drop the area bound – leads to limits which are no longer surfaces but rather unions of smooth surfaces – i.e. dimension can jump up;
- Ulitmately, we will drop both area and curvature bounds – leads to decomposing ambient space into singular part where curvature of sequence blows up and regular part where one has convergence (in second sense).

Generalizations

Some natural directions to generalize this theorem.

- Drop the curvature bound – leads to measure theoretic considerations;
- Drop the area bound – leads to limits which are no longer surfaces but rather unions of smooth surfaces – i.e. dimension can jump up;
- Ulitmately, we will drop both area and curvature bounds – leads to decomposing ambient space into singular part where curvature of sequence blows up and regular part where one has convergence (in second sense).

Generalizations

Some natural directions to generalize this theorem.

- Drop the curvature bound – leads to measure theoretic considerations;
- Drop the area bound – leads to limits which are no longer surfaces but rather unions of smooth surfaces – i.e. dimension can jump up;
- Ulitmately, we will drop both area and curvature bounds – leads to decomposing ambient space into singular part where curvature of sequence blows up and regular part where one has convergence (in second sense).

Generalizations

Some natural directions to generalize this theorem.

- Drop the curvature bound – leads to measure theoretic considerations;
- Drop the area bound – leads to limits which are no longer surfaces but rather unions of smooth surfaces – i.e. dimension can jump up;
- Ulitmately, we will drop both area and curvature bounds – leads to decomposing ambient space into singular part where curvature of sequence blows up and regular part where one has convergence (in second sense).

Minimal Laminations

Minimal Laminations

Definition

Fix a Riemannian three-manifold Ω. A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism $\psi_p : (U_p, p) \to (B_1, 0)$ so $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.

As observed by Solomon, Lipschitz regularity of “straightening maps”, ψ_p, is best possible.

Minimal Laminations

Definition

Fix a Riemannian three-manifold Ω. A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism $\psi_p : (U_p, p) \to (B_1, 0)$ so $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.

As observed by Solomon, Lipschitz regularity of "straightening maps", ψ_p, is best possible.

Minimal Laminations

Definition

Fix a Riemannian three-manifold Ω. A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism $\psi_p : (U_p, p) \to (B_1, 0)$ so $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.

As observed by Solomon, Lipschitz regularity of “straightening maps”, ψ_p, is best possible.

Minimal Laminations

Definition

Fix a Riemannian three-manifold Ω. A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism $\psi_p : (U_p, p) \to (B_1, 0)$ so $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.

As observed by Solomon, Lipschitz regularity of "straightening maps", ψ_p, is best possible.

Minimal Laminations

Definition

Fix a Riemannian three-manifold Ω. A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism $\psi_p : (U_p, p) \to (B_1, 0)$ so $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.

As observed by Solomon, Lipschitz regularity of “straightening maps”, ψ_p, is best possible.

Regular and Singular Points

Definition

For a Riemannian three-manifold Ω, an increasing sequence of open subsets $\Omega_1 \subset \Omega_2 \subset \cdots$ with $\Omega = \cup_i \Omega_i$ and a sequence $\mathcal{S} = \{\Sigma_i\}$ with $\Sigma_i \in \mathcal{E}_{\Omega_i}$ define the *singular set* of $\mathcal{S}$ to be

$$sing(\mathcal{S}) := \left\{ p \in \Omega : \exists p_i \to p \text{ s.t. } |A_{\Sigma_i}|(p_i) \to \infty \right\}.$$

this is a (relatively) closed subset of Ω. Likewise,

$$reg(\mathcal{S}) := \left\{ p \in \Omega : \exists \rho > 0 \text{ s.t. } \sup_i \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}|(p_i) < \infty \right\}.$$

this is an open subset of Ω.

NB: Can pass to a subsequence so $\Omega = sing(\mathcal{S}) \cup reg(\mathcal{S})$

Regular and Singular Points

Definition

For a Riemannian three-manifold Ω, an increasing sequence of open subsets $\Omega_1 \subset \Omega_2 \subset \cdots$ with $\Omega = \cup_i \Omega_i$ and a sequence $\mathcal{S} = \{\Sigma_i\}$ with $\Sigma_i \in \mathcal{E}_{\Omega_i}$ define the *singular set* of $\mathcal{S}$ to be

$$sing(\mathcal{S}) := \left\{ p \in \Omega : \exists p_i \to p \text{ s.t. } |A_{\Sigma_i}|(p_i) \to \infty \right\}.$$

this is a (relatively) closed subset of Ω. Likewise,

$$reg(\mathcal{S}) := \left\{ p \in \Omega : \exists \rho > 0 \text{ s.t. } \sup_i \sup_{B_\rho(p) \cap \Sigma_i} |A_{\Sigma_i}|(p_i) < \infty \right\}.$$

this is an open subset of Ω.

NB: Can pass to a subsequence so $\Omega = sing(\mathcal{S}) \cup reg(\mathcal{S})$

Generalized Arzela-Ascoli Theorem

Theorem

Fix a Riemannian three-manifold Ω, an increasing sequence of open subsets

$$\Omega_1 \subset \Omega_2 \subset \cdots$$

with $\Omega = \bigcup_i \Omega_i$ and a sequence $\mathcal{S} = \{\Sigma_i\}$ where $\Sigma_i \in \mathcal{E}_{\Omega_i}$. There is a subsequence, $\mathcal{S}'$ of $\mathcal{S}$ and a relatively closed subset K of Ω so that

- *$sing(\mathcal{S}') = K$;*
- *The $\Sigma_i \backslash K$ converge (in a suitably sense) to a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$*

We call such a quadruple $(\Omega, \mathcal{S}', K, \mathcal{L})$ a *minimal surface sequence*.

Generalized Arzela-Ascoli Theorem

Theorem

Fix a Riemannian three-manifold Ω, an increasing sequence of open subsets

$$\Omega_1 \subset \Omega_2 \subset \cdots$$

with $\Omega = \bigcup_i \Omega_i$ and a sequence $\mathcal{S} = \{\Sigma_i\}$ where $\Sigma_i \in \mathcal{E}_{\Omega_i}$. There is a subsequence, $\mathcal{S}'$ of $\mathcal{S}$ and a relatively closed subset K of Ω so that

- *$sing(\mathcal{S}') = K$;*
- *The $\Sigma_i \backslash K$ converge (in a suitably sense) to a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$*

We call such a quadruple $(\Omega, \mathcal{S}', K, \mathcal{L})$ a *minimal surface sequence*.

Generalized Arzela-Ascoli Theorem

Theorem

Fix a Riemannian three-manifold Ω*, an increasing sequence of open subsets*

$$\Omega_1 \subset \Omega_2 \subset \cdots$$

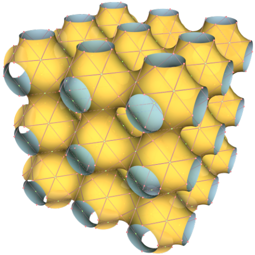
with $\Omega = \bigcup_i \Omega_i$ *and a sequence* $\mathcal{S} = \{\Sigma_i\}$ *where* $\Sigma_i \in \mathcal{E}_{\Omega_i}$*. There is a subsequence,* $\mathcal{S}'$ *of* $\mathcal{S}$ *and a relatively closed subset* K *of* Ω *so that*

- $sing(\mathcal{S}') = K$*;*
- *The* $\Sigma_i \backslash K$ *converge (in a suitably sense) to a proper minimal lamination* $\mathcal{L}$ *of* $\Omega \backslash K$

We call such a quadruple $(\Omega, \mathcal{S}', K, \mathcal{L})$ a *minimal surface sequence*.

Example: Rescaling of a Triply Periodic Surface

Let Σ be triply periodic minimal surface – e.g.

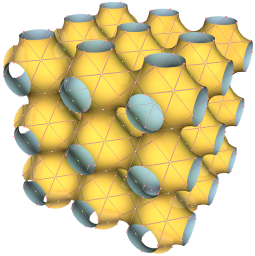


Setting $\Sigma_i = \frac{1}{i}\Sigma$, then

- $K = \mathbb{R}^3$;
- $\mathcal{L} = \emptyset$.

Example: Rescaling of a Triply Periodic Surface

Let Σ be triply periodic minimal surface – e.g.

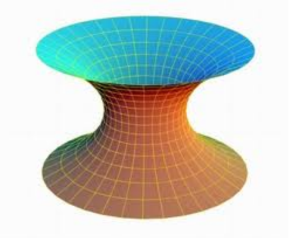


Setting $\Sigma_i = \frac{1}{i}\Sigma$, then

- $K = \mathbb{R}^3$;
- $\mathcal{L} = \emptyset$.

Example: Rescaling of a Catenoid

Let C be a vertical catenoid – i.e.,



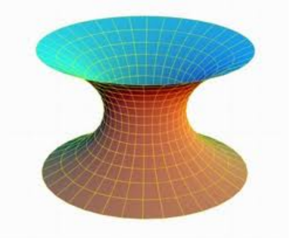
Setting $\Sigma_i = \frac{1}{i} C$, then

- $K = \vec{0}$;
- $\mathcal{L}$ has a single leaf $\{x_3 = 0\} \setminus \vec{0}$.

NB: The leaf extends smoothly to a surface in $\mathcal{E}_{\mathbb{R}^3}$. Hence also, the lamination $\mathcal{L}$ extends to a proper lamination of $\mathbb{R}^3$.

Example: Rescaling of a Catenoid

Let C be a vertical catenoid – i.e.,



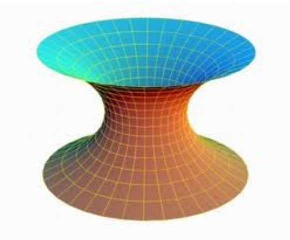
Setting $\Sigma_i = \frac{1}{i}C$, then

- $K = \vec{0}$;
- $\mathcal{L}$ has a single leaf $\{x_3 = 0\}\backslash\vec{0}$.

NB: The leaf extends smoothly to a surface in $\mathcal{E}_{\mathbb{R}^3}$. Hence also, the lamination $\mathcal{L}$ extends to a proper lamination of $\mathbb{R}^3$.

Example: Rescaling of a Catenoid

Let C be a vertical catenoid – i.e.,



Setting $\Sigma_i = \frac{1}{i}C$, then

- $K = \vec{0}$;
- $\mathcal{L}$ has a single leaf $\{x_3 = 0\} \setminus \vec{0}$.

NB: The leaf extends smoothly to a surface in $\mathcal{E}_{\mathbb{R}^3}$. Hence also, the lamination $\mathcal{L}$ extends to a proper lamination of $\mathbb{R}^3$.

Compactness Results

More generaly,

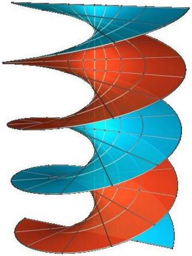
Theorem (Choi-Schoen, 1985; Anderson, 1985; White, 1987)

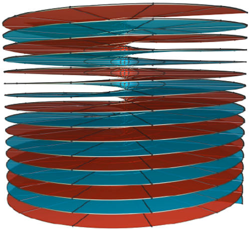
Fix $N \geq 0$, If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence where each $\Sigma_i \in \mathcal{S}$ satisfies $\Sigma_i \in \mathcal{TC}_{\Omega_i}(N)$, then

- *K consists of at most $\lfloor N \rfloor$ points of Ω;*
- *The lamination $\mathcal{L}$ extends to a proper lamination of Ω.*

Example: Rescaling of a Helicoid

Let H be a vertical helicoid – i.e.,





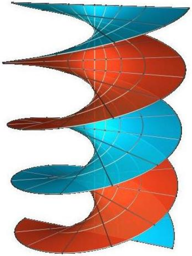
Set $\Sigma_i = \frac{1}{i} H$, so

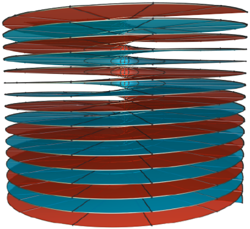
- $K = x_3 -$ axis;
- $\mathcal{L}$ a foliation of $\mathbb{R}^3 \backslash K$ by horizontal (punctued) planes.

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Example: Rescaling of a Helicoid

Let H be a vertical helicoid – i.e.,





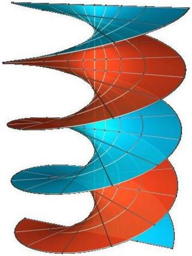
Set $\Sigma_i = \frac{1}{i} H$, so

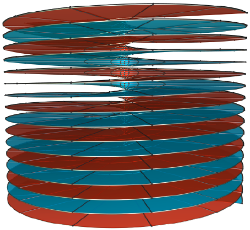
- $K = x_3 -$ axis;
- $\mathcal{L}$ a foliation of $\mathbb{R}^3 \backslash K$ by horizontal (punctued) planes.

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Example: Rescaling of a Helicoid

Let H be a vertical helicoid – i.e.,





Set $\Sigma_i = \frac{1}{i} H$, so

- $K = x_3 -$ axis;
- $\mathcal{L}$ a foliation of $\mathbb{R}^3 \backslash K$ by horizontal (punctued) planes.

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Colding-Minicozzi Theory

Colding-Minicozzi Theory – Global Case

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence with each $\Sigma_i \in \mathcal{E}_{\Omega_i}(1, 0)$, then we have a *minimal disk sequence*.

Theorem (Colding-Minicozzi, 2004)

If $(\mathbb{R}^3, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel (punctured) planes;*
- *K is a line perpendicular to these planes. (due to Meeks)*

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Colding-Minicozzi Theory – Global Case

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence with each $\Sigma_i \in \mathcal{E}_{\Omega_i}(1, 0)$, then we have a *minimal disk sequence*.

Theorem (Colding-Minicozzi, 2004)

If $(\mathbb{R}^3, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel (punctured) planes;*
- *K is a line perpendicular to these planes. (due to Meeks)*

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Colding-Minicozzi Theory – Global Case

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence with each $\Sigma_i \in \mathcal{E}_{\Omega_i}(1,0)$, then we have a *minimal disk sequence*.

Theorem (Colding-Minicozzi, 2004)

If $(\mathbb{R}^3, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel (punctured) planes;*
- *K is a line perpendicular to these planes. (due to Meeks)*

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Colding-Minicozzi Theory – Global Case

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence with each $\Sigma_i \in \mathcal{E}_{\Omega_i}(1, 0)$, then we have a *minimal disk sequence*.

Theorem (Colding-Minicozzi, 2004)

If $(\mathbb{R}^3, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel (punctured) planes;*
- *K is a line perpendicular to these planes. (due to Meeks)*

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

Colding-Minicozzi Theory – Global Case

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal surface sequence with each $\Sigma_i \in \mathcal{E}_{\Omega_i}(1, 0)$, then we have a *minimal disk sequence*.

Theorem (Colding-Minicozzi, 2004)

If $(\mathbb{R}^3, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$\mathcal{L}$ is a foliation of $\mathbb{R}^3 \backslash K$ by parallel (punctured) planes;*
- *K is a line perpendicular to these planes. (due to Meeks)*

NB: The leaves extend smoothly to surfaces in $\mathcal{E}_{\mathbb{R}^3}$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Local Case

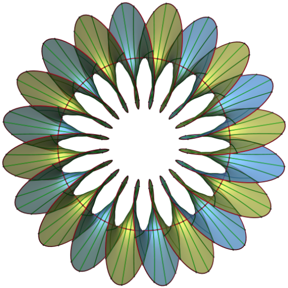
The situation is very different when $\Omega = B_1 \subset \mathbb{R}^3$ (also when Ω is a curved, possibly complete, three-manifold).

For instance the singular set K can be:

- a point (Colding-Minicozzi);
- a finite set of points (Dean);
- a closed line segment (Kahn);
- any closed subset of the x_3-axis (Hoffman-White, Kleene);
- non-straight curves (Meeks-Weber).

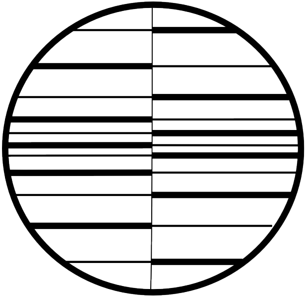
In contrast to the other constructions, Hoffman-White use variational methods which carry over to, for instance, $\Omega = \mathbb{H}^3$.

The Example of Meeks-Weber



Sequence of minimal annuli in a solid torus of revolution whose singular set is the central circle of the solid torus.

The Example of Colding-Minicozzi

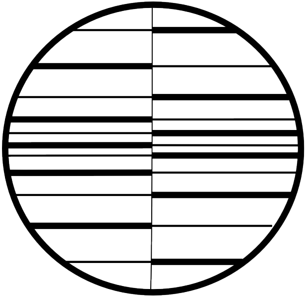


Minimal disk sequence, $(B_1, \mathcal{S}, K, \mathcal{L})$ with

- $K = \vec{0}$
- $\mathcal{L} = L_- \cup L_0 \cup L_+$ where $L_0 = B_1 \cap \{x_3 = 0\} \setminus \vec{0}$ and $L_\pm$ are non-proper embedded disks (in $B_1 \setminus \vec{0}$).

NB: $\bar{L}_0$, the closure in B_1 is an embedded disk. However, $\mathcal{L}$ does not extend to a lamination of B_1

The Example of Colding-Minicozzi



Minimal disk sequence, $(B_1, \mathcal{S}, K, \mathcal{L})$ with

- $K = \vec{0}$
- $\mathcal{L} = L_- \cup L_0 \cup L_+$ where $L_0 = B_1 \cap \{x_3 = 0\} \setminus \vec{0}$ and $L_\pm$ are non-proper embedded disks (in $B_1 \setminus \vec{0}$).

NB: $\bar{L}_0$, the closure in B_1 is an embedded disk. However, $\mathcal{L}$ does not extend to a lamination of B_1

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Colding-Minicozzi Theory – Local Case

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi, 2004)

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence and $K \neq \emptyset$, then

- *$K \subset K'$ a properly embedded Lipschitz curve in Ω;*
- *For any $p \in K$ there exists a leaf L of $\mathcal{L}$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface in Ω near p.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap K \neq \emptyset$, then $\overline{L}$ meets K "transversely".*

- Meeks showed that if $K = K'$ (i.e., K has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that K is contained in a C^1 curve.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \backslash K$;
- a proper disk or annulus in $\Omega \backslash K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus> 0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \backslash K$;
- a proper disk or annulus in $\Omega \backslash K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus > 0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \backslash K$;
- a proper disk or annulus in $\Omega \backslash K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus > 0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \backslash K$;
- a proper disk or annulus in $\Omega \backslash K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus > 0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \backslash K$;
- a proper disk or annulus in $\Omega \backslash K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus>0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Possible Leaves

What can be said about the the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus K$;
- a proper disk or annulus in $\Omega \setminus K$ with $\bar{L}$ a proper disk in Ω;
- a proper annulus in Ω disjoint from K (Hoffman-White).

Question (Hoffman-White)

Can L be a surface of genus> 0 occur? A planar domain with more than two punctures?

Answer (B. -Tinaglia)

Under natural geometric conditions on Ω it cannot.

Topology of Leaves

Main Result

Theorem (B.-Tinaglia)

Let Ω be the interior of an oriented compact three-manifold with boundary $\bar{\Omega}$ so that:

- *$\partial\Omega$ is strictly mean convex;*
- *There are no closed minimal surfaces in $\bar{\Omega}$.*

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence, then

- *Each leaf L of $\mathcal{L}$ is either a disk or an annulus,*
- *If L is a regular leaf of $\mathcal{L}$ (i.e. $\overline{L} \in \mathcal{E}_\Omega$), then $\overline{L}$ is either a disk (possibly meeting K) or an annulus disjoint from K.*

NB: Colding-Minicozzi $\implies$ $K \cap \overline{L}$ is a discrete subset of $\overline{L}$.

Main Result

Theorem (B.-Tinaglia)

Let Ω be the interior of an oriented compact three-manifold with boundary $\bar{\Omega}$ so that:

- *$\partial\Omega$ is strictly mean convex;*
- *There are no closed minimal surfaces in $\bar{\Omega}$.*

If $(\Omega, \mathcal{S}, K, \mathcal{L})$ is a minimal disk sequence, then

- *Each leaf L of $\mathcal{L}$ is either a disk or an annulus,*
- *If L is a regular leaf of $\mathcal{L}$ (i.e. $\overline{L} \in \mathcal{E}_\Omega$), then $\overline{L}$ is either a disk (possibly meeting K) or an annulus disjoint from K.*

NB: Colding-Minicozzi $\implies$ $K \cap \overline{L}$ is a discrete subset of $\overline{L}$.

Idea of Proof

Idea of proof:

- The nature of the convergence implies that the disks Σ_i in the sequence $\mathcal{S}$ act as a sort of "effective" universal cover of L;
- Specifically, one can "lift" closed curves in L to curves in the Σ_i;
- The geometry of the Σ_i – in particular the fact that they are minimally embedded and live in a mean convex set restricts the topology of the L – essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on Ω ensure – by a result of White – that minimal surfaces in Ω satisfy an isoperimetric inequality.

Idea of Proof

Idea of proof:

- The nature of the convergence implies that the disks Σ_i in the sequence $\mathcal{S}$ act as a sort of "effective" universal cover of L;
- Specifically, one can "lift" closed curves in L to curves in the Σ_i;
- The geometry of the Σ_i – in particular the fact that they are minimally embedded and live in a mean convex set restricts the topology of the L – essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on Ω ensure – by a result of White – that minimal surfaces in Ω satisfy an isoperimetric inequality.

Idea of Proof

Idea of proof:

- The nature of the convergence implies that the disks Σ_i in the sequence $\mathcal{S}$ act as a sort of "effective" universal cover of L;
- Specifically, one can "lift" closed curves in L to curves in the Σ_i;
- The geometry of the Σ_i – in particular the fact that they are minimally embedded and live in a mean convex set restricts the topology of the L – essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on Ω ensure – by a result of White – that minimal surfaces in Ω satisfy an isoperimetric inequality.

Idea of Proof

Idea of proof:

- The nature of the convergence implies that the disks Σ_i in the sequence $\mathcal{S}$ act as a sort of "effective" universal cover of L;
- Specifically, one can "lift" closed curves in L to curves in the Σ_i;
- The geometry of the Σ_i – in particular the fact that they are minimally embedded and live in a mean convex set restricts the topology of the L – essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on Ω ensure – by a result of White – that minimal surfaces in Ω satisfy an isoperimetric inequality.

Lifts

Henceforth, we fix a minimal disk sequence $(\Omega, \mathcal{S}, K, \mathcal{L})$ and also fix a leaf L of $\mathcal{L}$.

Definition

If

$$\gamma \colon S^1 \to L$$

is a piece-wise C^1 closed curve, then γ has the *closed-lift property* if there exists a sequence of closed "lifts"

$$\gamma_i \colon S^1 \to \Sigma_i$$

converging to γ. Otherwise, γ has the *open-lift property*.

If γ is embedded so are its lifts.

Lifts

Henceforth, we fix a minimal disk sequence $(\Omega, \mathcal{S}, K, \mathcal{L})$ and also fix a leaf L of $\mathcal{L}$.

Definition

If

$$\gamma\colon S^1 \to L$$

is a piece-wise C^1 closed curve, then γ has the *closed-lift property* if there exists a sequence of closed "lifts"

$$\gamma_i\colon S^1 \to \Sigma_i$$

converging to γ. Otherwise, γ has the *open-lift property*.

If γ is embedded so are its lifts.

Lifts

Henceforth, we fix a minimal disk sequence $(\Omega, \mathcal{S}, K, \mathcal{L})$ and also fix a leaf L of $\mathcal{L}$.

Definition

If

$$\gamma \colon S^1 \to L$$

is a piece-wise C^1 closed curve, then γ has the *closed-lift property* if there exists a sequence of closed "lifts"

$$\gamma_i \colon S^1 \to \Sigma_i$$

converging to γ. Otherwise, γ has the *open-lift property*.

If γ is embedded so are its lifts.

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial \Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial \Omega$.

□

Separating Lemma

Lemma (Separating Lemma)

If $\gamma : S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof.

- Let $\gamma_i : S^1 \to \Sigma_i$ be embedded closed lifts of γ;
- Each γ_i bounds a closed minimal disk $\Delta_i \subset \Sigma_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega \backslash \gamma)$; $\Delta \subset L \backslash \gamma$ is open and closed;
- If γ does not separate L then $\Delta = L \backslash \gamma$;
- $\partial\Omega$ strictly mean convex $\implies$ Δ_i cannot get close to $\partial\Omega$.

□

Commutator Lemma

Lemma (Commutator Lemma)

Let L be two-sided and let

$$\alpha : [0,1] \to L \text{ and } \beta : [0,1] \to L$$

be closed piece-wise C^1 Jordan curves. If α and β have the open lift property and $\alpha \cap \beta = p_0$ where $p_0 = \alpha(0) = \beta(0)$, then

$$\nu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$$

has the closed lift property and the lifts are "embedded."

Proof

Proof.

Let α_i^+ be a lift of α and let α_i^- be a lift of α^{-1} and likewise for β.

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.

"embedded": If a lift of ν is not embedded, then either a lift of $\alpha \circ \beta$ or a lift of $\beta \circ \alpha^{-1}$ is closed and embedded. □

Proof

Proof.

Let α_i^+ be a lift of α and let α_i^- be a lift of α^{-1} and likewise for β.

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.

"embedded": If a lift of ν is not embedded, then either a lift of $\alpha \circ \beta$ or a lift of $\beta \circ \alpha^{-1}$ is closed and embedded. $\square$

Proof

Proof.

Let α_i^+ be a lift of α and let α_i^- be a lift of α^{-1} and likewise for β.

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.

"embedded": If a lift of ν is not embedded, then either a lift of $\alpha \circ \beta$ or a lift of $\beta \circ \alpha^{-1}$ is closed and embedded. $\square$

Proof

Proof.

Let α_i^+ be a lift of α and let α_i^- be a lift of α^{-1} and likewise for β.

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.

"embedded": If a lift of ν is not embedded, then either a lift of $\alpha \circ \beta$ or a lift of $\beta \circ \alpha^{-1}$ is closed and embedded. $\square$

Proof

Proof.

Let α_i^+ be a lift of α and let α_i^- be a lift of α^{-1} and likewise for β.

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.

"embedded": If a lift of ν is not embedded, then either a lift of $\alpha \circ \beta$ or a lift of $\beta \circ \alpha^{-1}$ is closed and embedded. □

No Pants

Proposition (No Pants)

If L is two-sided, then L is either a disk or an annulus.

Proof.

L is oriented. If L is not a disk or annulus, then

- There exist embedded closed curves α and β separating L into 3 components, L_1, L_2 and L_3 so that L_3 satisfies $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus. We allow $L_1 = L_2$.
- Let σ be an embedded arc in L_3 with endpoints in α and β. Note σ does not separate L_3.
- With $\gamma = \sigma \circ \alpha \circ \sigma^{-1}$, Commutator Lemma $\Longrightarrow$ $\gamma \circ \beta \circ \gamma^{-1} \circ \beta^{-1}$ has the closed lift property.

□

No Pants

Proposition (No Pants)

If L is two-sided, then L is either a disk or an annulus.

Proof.

L is oriented. If L is not a disk or annulus, then

- There exist embedded closed curves α and β separating L into 3 components, L_1, L_2 and L_3 so that L_3 satisfies $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus. We allow $L_1 = L_2$.
- Let σ be an embedded arc in L_3 with endpoints in α and β. Note σ does not separate L_3.
- With $\gamma = \sigma \circ \alpha \circ \sigma^{-1}$, Commutator Lemma $\implies$ $\gamma \circ \beta \circ \gamma^{-1} \circ \beta^{-1}$ has the closed lift property.

□

No Pants

Proposition (No Pants)

If L is two-sided, then L is either a disk or an annulus.

Proof.

L is oriented. If L is not a disk or annulus, then

- There exist embedded closed curves α and β separating L into 3 components, L_1, L_2 and L_3 so that L_3 satisfies $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus. We allow $L_1 = L_2$.
- Let σ be an embedded arc in L_3 with endpoints in α and β. Note σ does not separate L_3.
- With $\gamma = \sigma \circ \alpha \circ \sigma^{-1}$, Commutator Lemma $\implies$ $\gamma \circ \beta \circ \gamma^{-1} \circ \beta^{-1}$ has the closed lift property.

□

No Pants

Proposition (No Pants)

If L is two-sided, then L is either a disk or an annulus.

Proof.

L is oriented. If L is not a disk or annulus, then

- There exist embedded closed curves α and β separating L into 3 components, L_1, L_2 and L_3 so that L_3 satisfies $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus. We allow $L_1 = L_2$.
- Let σ be an embedded arc in L_3 with endpoints in α and β. Note σ does not separate L_3.
- With $\gamma = \sigma \circ \alpha \circ \sigma^{-1}$, Commutator Lemma $\implies$ $\gamma \circ \beta \circ \gamma^{-1} \circ \beta^{-1}$ has the closed lift property.

□

No Pants (cont.)

Proof.

- A sequence of embedded minimal disks Δ_i must converge to an open and closed subset of $L\backslash(\gamma \circ \beta)$.
- In particular, the limit must contain either L_1, L_2 or L_3.
- Forces interior point of Δ_i to come arbitrarily close to $\partial\Omega$ violating strict maximum principle.

□

No Pants (cont.)

Proof.

- A sequence of embedded minimal disks Δ_i must converge to an open and closed subset of $L\backslash(\gamma \circ \beta)$.
- In particular, the limit must contain either L_1, L_2 or L_3.
- Forces interior point of Δ_i to come arbitrarily close to $\partial\Omega$ violating strict maximum principle.

□

No Pants (cont.)

Proof.

- A sequence of embedded minimal disks Δ_i must converge to an open and closed subset of $L \backslash (\gamma \circ \beta)$.
- In particular, the limit must contain either L_1, L_2 or L_3.
- Forces interior point of Δ_i to come arbitrarily close to $\partial\Omega$ violating strict maximum principle.

□

Two-sidedness of leaves

Proposition (Two-sidedness)

A leaf L is two-sided.

Proof.

- If L one-sided, then there is a closed non-separating curve along which L does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;
- Following lift around in a Σ_i violates either properness or embeddedness.

□

Two-sidedness of leaves

Proposition (Two-sidedness)

A leaf L is two-sided.

Proof.

- If L one-sided, then there is a closed non-separating curve along which L does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;
- Following lift around in a Σ_i violates either properness or embeddedness.

□

Two-sidedness of leaves

Proposition (Two-sidedness)

A leaf L is two-sided.

Proof.

- If L one-sided, then there is a closed non-separating curve along which L does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;
- Following lift around in a Σ_i violates either properness or embeddedness.

□

Understanding Geometric Condition

Question

To what extent can the assumptions on Ω be relaxed?

To understand this suppose that Σ is an embedded (but not necessarily properly embedded) minimal disk in Ω with the property that

- There is a closed set K of Ω disjoint from Σ;
- The curvatures of Σ blow-up at K.
- The closure, $\bar{\Sigma}$, of Σ in $\Omega \backslash K$ is a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$

We denote such an object by $(\Omega, \Sigma, K, \mathcal{L})$ which we call a *minimal disk closure*.

Understanding Geometric Condition

Question

To what extent can the assumptions on Ω be relaxed?

To understand this suppose that Σ is an embedded (but not necessarily properly embedded) minimal disk in Ω with the property that

- There is a closed set K of Ω disjoint from Σ;
- The curvatures of Σ blow-up at K.
- The closure, $\bar{\Sigma}$, of Σ in $\Omega \backslash K$ is a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$

We denote such an object by $(\Omega, \Sigma, K, \mathcal{L})$ which we call a *minimal disk closure*.

Understanding Geometric Condition

Question

To what extent can the assumptions on Ω be relaxed?

To understand this suppose that Σ is an embedded (but not necessarily properly embedded) minimal disk in Ω with the property that

- There is a closed set K of Ω disjoint from Σ;
- The curvatures of Σ blow-up at K.
- The closure, $\bar{\Sigma}$, of Σ in $\Omega \backslash K$ is a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$

We denote such an object by $(\Omega, \Sigma, K, \mathcal{L})$ which we call a *minimal disk closure*.

Understanding Geometric Condition

Question

To what extent can the assumptions on Ω be relaxed?

To understand this suppose that Σ is an embedded (but not necessarily properly embedded) minimal disk in Ω with the property that

- There is a closed set K of Ω disjoint from Σ;
- The curvatures of Σ blow-up at K.
- The closure, $\bar{\Sigma}$, of Σ in $\Omega\backslash K$ is a proper minimal lamination $\mathcal{L}$ of $\Omega\backslash K$

We denote such an object by $(\Omega, \Sigma, K, \mathcal{L})$ which we call a *minimal disk closure*.

Understanding Geometric Condition

Question

To what extent can the assumptions on Ω be relaxed?

To understand this suppose that Σ is an embedded (but not necessarily properly embedded) minimal disk in Ω with the property that

- There is a closed set K of Ω disjoint from Σ;
- The curvatures of Σ blow-up at K.
- The closure, $\bar{\Sigma}$, of Σ in $\Omega \backslash K$ is a proper minimal lamination $\mathcal{L}$ of $\Omega \backslash K$

We denote such an object by $(\Omega, \Sigma, K, \mathcal{L})$ which we call a *minimal disk closure*.

Topology of Minimal Disk Closures

It turns out that leaves of minimal disk closures behave almost identical to those of the limit leaves of a minimal disk sequence.

Theorem (B.-Tinaglia)

Let Ω be the interior of an oriented compact three-manifold with boundary $\bar{\Omega}$ so that:

- *$\partial\Omega$ is strictly mean convex;*
- *There are no closed minimal surfaces in $\bar{\Omega}$.*

If $(\Omega, \Sigma, K, \mathcal{L})$ is a minimal disk closure, then each leaf L of $\mathcal{L}$ is either a disk, an annulus or a Möbius band.

Sharpness

The preceding theorem is sharp in the following sense:

- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so that one leaf of $\mathcal{L}$ is a Möbius band. The lamination $\mathcal{L}$ cannot occur as the lamination of a minimal disk sequence.
- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so Ω contains a minimal torus which is a leaf of $\mathcal{L}$.

Sharpness

The preceding theorem is sharp in the following sense:

- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so that one leaf of $\mathcal{L}$ is a Möbius band. The lamination $\mathcal{L}$ cannot occur as the lamination of a minimal disk sequence.
- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so Ω contains a minimal torus which is a leaf of $\mathcal{L}$.

Sharpness

The preceding theorem is sharp in the following sense:

- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so that one leaf of $\mathcal{L}$ is a Möbius band. The lamination $\mathcal{L}$ cannot occur as the lamination of a minimal disk sequence.
- There is a minimal disk closure, $(\Omega, \Sigma, \emptyset, \mathcal{L})$, so Ω contains a minimal torus which is a leaf of $\mathcal{L}$.

Further Questions

Some further questions:

- Is the theorem for minimal disk sequences sharp?
- To what extent are both theorems true even for regions which contain closed minimal surfaces? For instance, can one rule out leaves with non-abelian fundamental group even if the three-manifold admits closed minimal surfaces?

Further Questions

Some further questions:

- Is the theorem for minimal disk sequences sharp?
- To what extent are both theorems true even for regions which contain closed minimal surfaces? For instance, can one rule out leaves with non-abelian fundamental group even if the three-manifold admits closed minimal surfaces?