A Geometric Uncertainty Principle and Pleijel's estimate

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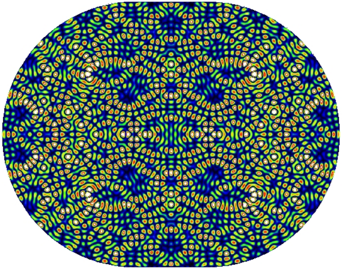


Given a domain $\Omega \subset \mathbb{R}^2$ and

$$\begin{aligned} -\Delta \phi_n(x) &= \lambda_n \phi_n(x) \quad \text{on } \Omega \\ \phi_n(x) &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}$$

what bounds can be proven on on the number of connected components of

$$\Omega \setminus \{x \in \Omega : \phi_n(x) = 0\}?$$

Courant's nodal line theorem (1924)

The number $N(n)$ of connected components of

$$\Omega \setminus \{x \in \Omega : \phi_n(x) = 0\}$$

satisfies

$$N(n) \leq n.$$

Pleijel's estimate (1956)

The number N of connected components of

$$\Omega \setminus \{x \in \Omega : \phi_n(x) = 0\}$$

satisfies

$$\limsup_{n \to \infty} \frac{N(n)}{n} \leq \left(\frac{2}{j}\right)^2 \sim 0.69\ldots$$

Proof of Pleijel's estimate

Denote the nodal domains via

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$

Then

$$\frac{4\pi n}{|\Omega|} \sim \lambda_n(\Omega) \geq \lambda_1(\Omega_i) \geq \frac{\pi j^2}{|\Omega_i|}.$$

Therefore

$$\frac{|\Omega_i|}{|\Omega|} \geq \left(\frac{j}{2}\right)^2 \frac{1}{n}.$$

Polterovich, 2009

Remark 2.5. It is clear from the proof of (1.1) that this estimate is not sharp for both Dirichlet and Neumann boundary conditions. Indeed, the Faber-Krahn inequality is an equality only for the disk, and nodal domains of an eigenfunction can not be all disks at the same time. Therefore, a natural question is to find an optimal constant in (1.1). Motivated by the results of (Blum, Gnutzmann & Smilansky) we suggest that for any regular bounded planar domain with either Dirichlet or Neumann

$$\limsup_{k \to \infty} \frac{n_k}{k} \leq \frac{2}{\pi} \sim 0.63$$

If true, this estimate (which is quite close to Pleijel's bound) is sharp and attained for the basis of separable eigenfunctions on a rectangle.

Bourgain (2013).

The number N of connected components of

$$\Omega \setminus \{x \in \Omega : \phi_n(x) = 0\}$$

satisfies

$$\limsup_{n \to \infty} \frac{N(n)}{n} \leq \left(\frac{2}{j}\right)^2 - 3 \cdot 10^{-9}$$

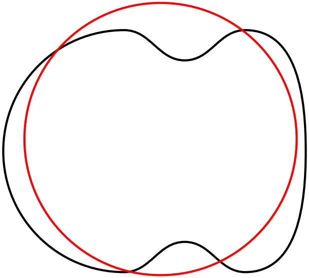
Replace nodal domains by their inradius. If domains deviate a lot from their inradius, they have to be big in measure (Hansen-Nadirashvili stability estimate). Combine this with a geometric insight

Theorem (Blind). *A collection of disks with radii $r_1, r_2, \dots$ such that*

$$\inf_{i,j} \frac{r_i}{r_j} \geq \frac{3}{4}$$

has packing density bounded from above by $\pi/\sqrt{12}$.

Fraenkel asymmetry



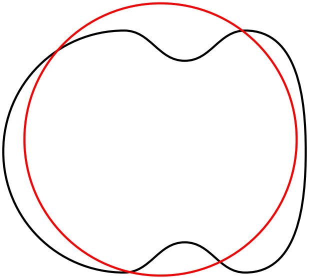
Given a domain Ω, we define the Fraenkel asymmetry via

$$\mathcal{A}(\Omega) = \inf_{B} \frac{|\Omega \triangle B|}{|\Omega|},$$

where the infimum ranges over all disks $B \subset \mathbb{R}^2$ with $|B| = |\Omega|$ and $\triangle$ is the symmetric difference

$$\Omega \triangle B = (\Omega \setminus B) \cup (B \setminus \Omega).$$

Fraenkel asymmetry



$$\mathcal{A}(\Omega) = \inf_{B} \frac{|\Omega \triangle B|}{|\Omega|}$$

Scale-invariant quantity with

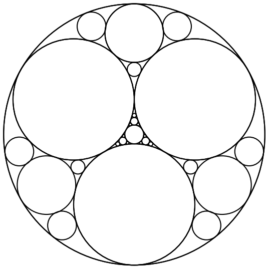
$$0 \leq \mathcal{A}(\Omega) \leq 2.$$

Stability estimate

Very recently, Brasco, De Philippis & Velichkov (improving an earlier result of Fusco, Maggi & Pratelli) have shown that

$$\frac{\lambda_1(\Omega) - \lambda_1(\Omega_0)}{\lambda_1(\Omega_0)} \gtrsim \mathcal{A}(\Omega)^2.$$

Then, however,



Faber-Krahn does not allow small domains!

Deviation measure

Given a decomposition

$$\Omega = \bigcup_{i=1}^{N} \Omega_i,$$

we define

$$D(\Omega_i) = \frac{|\Omega_i| - \min_{1 \leq j \leq N} |\Omega_j|}{|\Omega_i|}.$$

D is scale invariant and satisfies

$$0 \leq D(\Omega_i) \leq 1.$$

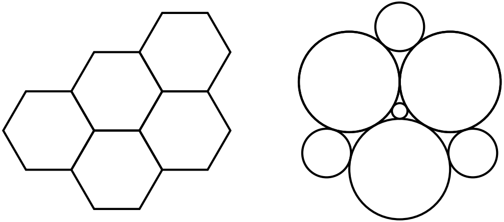
Geometric uncertainty principle

If for N sufficiently large (depending on Ω)

$$\Omega = \bigcup_{i=1}^{N} \Omega_i, \quad \mathcal{A}(\Omega) = \inf_{B} \frac{|\Omega \triangle B|}{|\Omega|}, \quad D(\Omega_i) = \frac{|\Omega_i| - \min_{1 \leq j \leq N} |\Omega_j|}{|\Omega_i|},$$

then

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq \frac{1}{60000}.$$



Extensions

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i)\right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i)\right) \geq \frac{1}{60000}.$$

The statement remains true in higher dimensions with some optimal constant $c_n > 0$.

Conjecture. We have

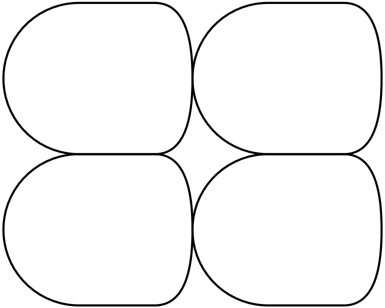
$$0.07 \leq c_2 < c_3 < c_4 < \cdots \leq 2,$$

where $c_2 \sim 0.07$ comes from assuming that the hexagonal packing is extremal. Is it? What can be said about extremizers?

Extensions

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i)\right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i)\right) \geq c_K > 0.$$

The statement also remains true if we replace the ball B in the definition of Fraenkel asymmetry $\mathcal{A}$ by any strictly convex body K.



Back to Pleijel estimates

$$\left(\sum_{i=1}^{N}\frac{|\Omega_i|}{|\Omega|}\mathcal{A}(\Omega_i)\right)+\left(\sum_{i=1}^{N}\frac{|\Omega_i|}{|\Omega|}D(\Omega_i)\right)\geq\frac{1}{60000}.$$

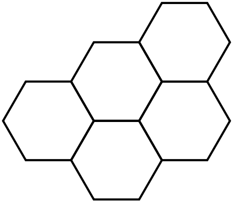
One of the two terms is 'large'. If it is the first one, then an average nodal domain is not a disk and the stability estimate implies the result. If it is the second one, then an average nodal domain is a constant factor bigger than what the Faber-Krahn inequality predicts and we are done.

Being extremely optimistic with regards to unknown constants yields

$$\limsup_{n\to\infty}\frac{N(n)}{n}\leq\left(\frac{2}{j}\right)^2-10^{-6}.$$

What can we hope for?

Spectral partition problem (Helffer & Hoffmann-Ostenhof, Lin & Caffarelli): hexagonal extremizers.

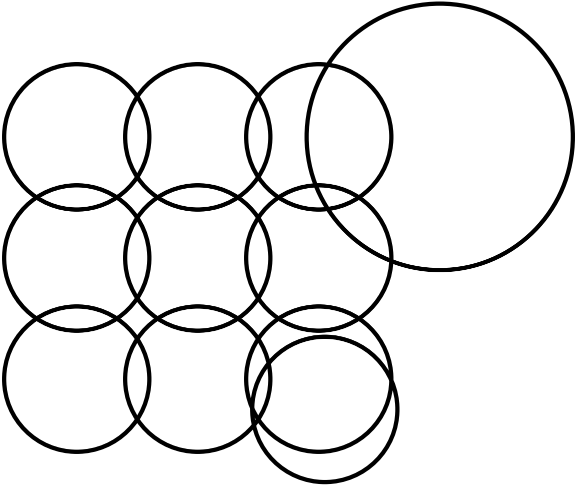


'Elliptic techniques assuming a given partition' cannot prove

$$\limsup_{n \to \infty} \frac{N(n)}{n} \leq 0.67.$$

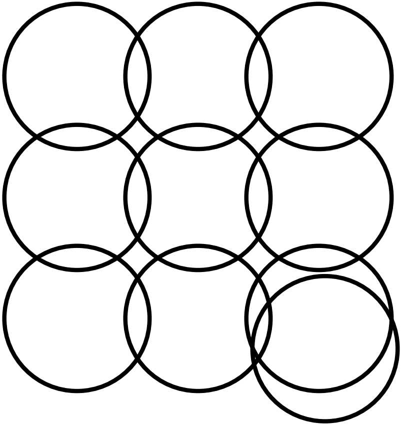
Sketch of the global proof I

Replace all partition elements by Fraenkel balls.

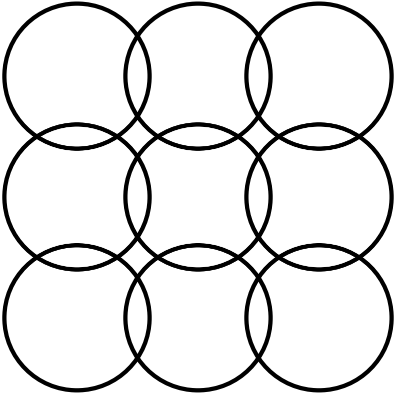


Sketch of the global proof

Remove all the big balls. There are few of them anyway.

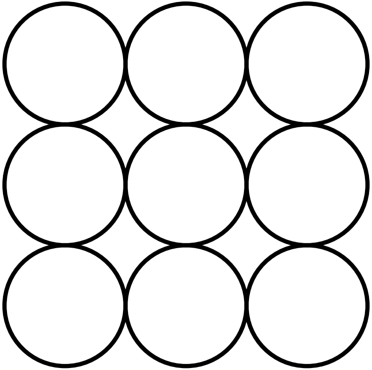


Remove all the balls intersecting another ball too strongly. There are few of them anyway.



Sketch of the global proof

The remaining balls are roughly equal sized and keep a certain distance from each other. Shrink them a tiny bit to make them disjoint.



Use Blind's result on the packing density of disks.

A more subtle question

Define for *any* set $S \subset \mathbb{R}^2$

$$\mathcal{A}_S(\Omega) = \inf_S \frac{|\Omega \triangle S|}{|\Omega|},$$

where the infimum is taking over all rescaled and rotated translates of S.

Define the evil set $\mathcal{E}$ containing all sets in the Euclidean plane such that they admit a tiling of the plane by just rotation and translation. Let $K \subset \mathbb{R}^2$ be some set and assume

$$\inf_{E \in \mathcal{E}} \mathcal{A}_E(K) > \varepsilon.$$

Does this imply a geometric uncertainty principle

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i) \right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq c(\varepsilon)?$$