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On the spectrum of the Hodge Laplacian and the John ellipsoid

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We give upper and lower bounds for the first eigenvalue of the Hodge Laplacian acting on p-forms of a compact, convex Euclidean domain Ω (smooth boundary, absolute boundary conditions). We denote this eigenvalue by the symbol

$$\lambda_1^{[p]}(\Omega).$$

Perhaps the main scope is to stress the geometric meaning of this eigenvalue, and to relate it with a classical object in convex geometry: the John ellipsoid of the domain.

Known estimates on functions: the Dirichlet problem.
Ω = compact, *convex* domain in $\mathbf{R}^n$ with smooth boundary.

- Classical Dirichlet eigenvalue problem:

$$\begin{cases} \Delta f = \lambda f \quad \text{on } \Omega, \\ f = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Let

$$\lambda_1^D(\Omega)$$

be its first eigenvalue.
Classical bounds:

$$\frac{\pi^2}{4R(\Omega)^2} \leq \lambda_1^D(\Omega) \leq \frac{c_n}{R(\Omega)^2}$$

where

$$R(\Omega) = \text{inner radius of } \Omega.$$

Lower bound: Hersch, Li and Yau.
Upper bound: domain monotonicity.

• Given geometric functionals $\Gamma_1(\Omega), \Gamma_2(\Omega)$ we say that $\Gamma_1(\Omega)$ *is comparable to* $\Gamma_2(\Omega)$ if there exist constants a, b not depending on Ω such that

$$a\Gamma_1(\Omega) \leq \Gamma_2(\Omega) \leq b\Gamma_1(\Omega).$$

and we will write:

$$\Gamma_1(\Omega) \sim \Gamma_2(\Omega).$$

Theorem. *For any convex domain Ω one has $\lambda_1^D(\Omega) \sim \dfrac{1}{R(\Omega)^2}$. In other words:*

$$\frac{1}{\sqrt{\lambda_1^D(\Omega)}} \sim R(\Omega).$$

That is, the fundamental wavelength *for the Dirichlet problem is comparable to the inner radius (large drums produce a low tone).*

Remark. The above fact does not hold in other spaces: for example, in hyperbolic space $\mathbf{H}^n$ one has the Mc Kean inequality:

$$\lambda_1^D(\Omega) \geq \frac{(n-1)^2}{4}$$

for *any* compact domain Ω (not necessarily convex).

Known estimates on functions: the Neumann problem.

- The Neumann eigenvalue problem:

$$\begin{cases} \Delta f = \lambda f & \text{on } \Omega, \\ \dfrac{\partial f}{\partial N} = 0 & \text{on } \partial\Omega, \end{cases}$$

where N is the unit normal vector. Let

$$\lambda_1^N(\Omega)$$

be its first *positive* eigenvalue. One knows (Polya):

$$\lambda_1^N(\Omega) < \lambda_1^D(\Omega)$$

and that:

$$\frac{\pi^2}{\mathrm{diam}(\Omega)^2} \leq \lambda_1^N(\Omega) \leq \frac{n\pi^2}{\mathrm{diam}(\Omega)^2}.$$

Lower bound: Payne and Weinberger.
Upper bound: particular case of an estimate of Cheng.

Theorem. *For any convex domain Ω one has $\lambda_1^N(\Omega) \sim \frac{1}{\text{diam}(\Omega)^2}$. In other words:*

$$\frac{1}{\sqrt{\lambda_1^N(\Omega)}} \sim \text{diam}(\Omega).$$

Hence: the fundamental wavelength for the Neumann problem is comparable to the diameter.

- Conclusion: given the fundamental tones for the Dirichlet and Neumann problems:

$$\lambda_1^D(\Omega), \lambda_1^N(\Omega)$$

one can *roughly hear* both the inner radius and the diameter of the domain.

The John ellipsoid. The shape of a convex domain Ω can be roughly described by a suitable ellipsoid.

Theorem (F. John, 1948) *Given any convex domain Ω in $\mathbf{R}^n$ there exists a unique ellipsoid of maximal volume included in Ω, denoted by $\mathcal{E}_\Omega$. Moreover one has:*

$$\mathcal{E}_\Omega \subseteq \Omega \subseteq n \cdot \mathcal{E}_\Omega,$$

(origin in the center of $\mathcal{E}_\Omega$).

Uniqueness apparently due to Löwner.

- $\mathcal{E}_\Omega$ is called the *John ellipsoid* of Ω. Set:

  $D_p(\mathcal{E}_\Omega) = p-$th longest principal axis of $\mathcal{E}_\Omega$.

Ordering:

$$D_1(\mathcal{E}_\Omega) \geq D_2(\mathcal{E}_\Omega) \geq \cdots \geq D_n(\mathcal{E}_\Omega).$$

Observe that

$$D_1(\mathcal{E}_\Omega) \sim \mathrm{diam}(\Omega), \quad D_n(\mathcal{E}_\Omega) \sim 2R(\Omega) \sim R(\Omega).$$

(clear for true ellipsoids; in general apply the inclusions in John theorem). The classical estimates give:

Theorem. *Let Ω be a convex domain in $\mathbf{R}^n$ and $\mathcal{E}_\Omega$ its John ellipsoid. Then:*

$$\frac{1}{\sqrt{\lambda_1^N(\Omega)}} \sim D_1(\mathcal{E}_\Omega) \quad \textit{and} \quad \frac{1}{\sqrt{\lambda_1^D(\Omega)}} \sim D_n(\mathcal{E}_\Omega).$$

In particular, if Ω is a convex plane domain then the two fundamental tones determine (up to constants) the two principal axis of the John ellipse of Ω.

- Satisfactory in dimension 2, but incomplete in dimensions $n \geq 3$.

- Do the other principal axes of the John ellipsoid have a similar spectral interpretation?

The Hodge Laplacian. Laplacian acting on differential p-forms:

$$\Delta = d\delta + \delta d$$

where $\delta = d^\star$.

- Eigenvalue problem for the *absolute boundary conditions* :

$$\begin{cases} \Delta\omega = \lambda\omega \quad \text{on } \Omega, \\ i_N\omega = 0 \quad \text{on } \partial\Omega, \\ i_N d\omega = 0 \quad \text{on } \partial\Omega. \end{cases}$$

N = inner unit normal vector
i_N = interior multiplication by N.

The spectrum is discrete:

$$\lambda_1^{[p]} \le \lambda_2^{[p]} \le \cdots \le \lambda_k^{[p]} \le \cdots$$

(the degree is in the superscript).

- Variational characterization.

$$\lambda_1^{[p]}(\Omega) = \inf\left\{ \frac{\int_\Omega |d\omega|^2 + |\delta\omega|^2}{\int_\Omega |\omega|^2} : \omega \in \Lambda^p(\Omega), i_N\omega = 0 \text{ on } \partial\Omega \right\}$$

• Identify 1-forms and vector fields via the metric. In 3-space:

$$\lambda_1^{[1]}(\Omega) = \inf\left\{\frac{\int_\Omega |\mathrm{div}X|^2 + |\mathrm{curl}X|^2}{\int_\Omega |X|^2} : \langle X, N\rangle = 0\right\},$$

that is, the infimum is taken over all vector fields which are tangent to the boundary.

Motivation for the boundary conditions:

The space of harmonic p-forms satisfying the absolute conditions is isomorphic with the absolute de Rham cohomology of Ω in degree p.

If Ω is convex one has $\lambda_1^{[p]} > 0$ for all $p \geq 1$.

• $0-$forms are functions: absolute boundary conditions .... Neumann conditions. Then:

$$\lambda_1^{[0]} = \lambda_1^N.$$

• $n-$forms are identified with functions (through the $\star$ operator). Dual conditions ... Dirichlet. Then

$$\lambda_1^{[n]} = \lambda_1^D.$$

The Hodge $\star$ operator transforms absolute boundary conditions into *relative* boundary conditions ... corresponding dual eigenvalue problem:

$$\begin{cases} \Delta\omega = \mu\omega \quad \text{on } \Omega, \\ J^\star\omega = 0 \quad \text{on } \partial\Omega, \\ J^\star\delta\omega = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $J^\star$ denotes restriction of forms to the boundary.

We will call, for $p = 0, \ldots, n$:

- $\lambda_1^{[p]}$: *fundamental tone in degree p*
- $\dfrac{1}{\sqrt{\lambda_1^{[p]}}}$: *fundamental wavelength in degree p*

Problem: estimate $\lambda_1^{[p]}$ for all degrees p.

Eigenvalue estimates for the Hodge Laplacian.

- Estimating the first eigenvalue for p-forms is, generally speaking, more difficult than for functions.

- Main tool: Bochner formula, giving estimates involving pointwise lower bounds of the principal curvatures of the boundary (joint works with P. Guerini and S. Raulot).

- For a convex domain it is desirable to have lower bounds depending on global invariants, rather than local ones.

Theorem (Guerini-S. 2004) *For any convex domain in $\mathbf{R}^n$ one has:*

$$\lambda_1^{[0]} = \lambda_1^{[1]} \leq \lambda_1^{[2]} \leq \cdots \leq \lambda_1^{[n]}.$$

That is, the fundamental tones of the Hodge Laplacian form an increasing sequence (with respect to the degree).

- All fundamental tones of the Hodge Laplacian belong to the interval $[\lambda_1^N, \lambda_1^D]$.

Sketch of proof.

1. Let ω be an eigenform associated to $\lambda_1^{[p]}$ and V a parallel vector field in $\mathbf{R}^n$ (of unit length). Then

$$i_V\omega$$

is a test-form for the eigenvalue $\lambda_1^{[p-1]}$ (because $i_N i_V = -i_V i_N$).

2. min-max principle:

$$\lambda_1^{[p-1]}\int_\Omega |i_V\omega|^2 \leq \int_\Omega |di_V\omega|^2 + |\delta i_V\omega|^2.$$

3. Identify the set of parallel vector fields of unit length with $\mathbf{S}^{n-1}$ and integrate both sides with respect to $V \in \mathbf{S}^{n-1}$. After some work and the Bochner formula, get:

$$\lambda_1^{[p-1]} \leq \lambda_1^{[p]}.$$

In particular, $\lambda_1^{[0]} \leq \lambda_1^{[1]}$.

Note: convexity is needed!

4. $\lambda_1^{[0]} \geq \lambda_1^{[1]}$ is always true (by differentiating Neumann eigenfunctions).

Hence:

$$\lambda_1^{[0]} = \lambda_1^{[1]}$$

and equality holds at the first step. •

Monotonicity property: $\lambda_1^N \leq \lambda_1^{[p]} \leq \lambda_1^D$. Hence, for all p:

$$\frac{\pi^2}{\mathrm{diam}(\Omega)^2} \leq \lambda_1^{[p]} \leq \frac{c_n}{R(\Omega)^2}.$$

But we can do much better than that.

The main estimate. From the previous results we have only n significant fundamental tones: these can be estimated in terms of the John ellipsoid of the domain.

Theorem (S. 2011) *Let Ω be a convex body and $\mathcal{E}_\Omega$ its John ellipsoid. Order the principal axes of $\mathcal{E}_\Omega$ from longest to shortest:*

$$D_1(\mathcal{E}_\Omega) \geq D_2(\mathcal{E}_\Omega) \geq \cdots \geq D_n(\mathcal{E}_\Omega).$$

Then, for all $p = 1, \dots, n$ one has:

$$\frac{a_{n,p}}{D_p(\mathcal{E}_\Omega)^2} \leq \lambda_1^{[p]}(\Omega) \leq \frac{a'_{n,p}}{D_p(\mathcal{E}_\Omega)^2},$$

where $a_{n,p}$ and $a'_{n,p}$ are explicit constants. Precisely:

$$a_{n,p} = \frac{4}{n^2 \cdot \binom{n}{p-1}}, \quad a'_{n,p} = 4p(n+2)n^n.$$

Remark. The constants are not sharp.

- Main result is that the fundamental wavelength in degree p is comparable with the p-th longest principal axis of its John ellipsoid:

$$\frac{1}{\sqrt{\lambda_1^{[p]}(\Omega)}} \sim D_p(\mathcal{E}_\Omega)$$

for all $p = 1, \ldots, n$.

- Philosophy: knowing all fundamental tones:

$$\lambda_1^{[1]}, \lambda_1^{[2]}, \ldots, \lambda_1^{[n]}$$

one can *roughly hear* the John ellipsoid (hence the shape) of the domain.

- What is the physical interpretation of $\lambda_1^{[p]}(\Omega)$?

Spectrum and volume of cross-sections. Set, for $p = 1, \ldots, n$:

$$\mathrm{vol}^{[p]}(\Omega) = \sup\{\mathrm{vol}(\Sigma) : \Sigma = \Omega \cap \pi_p,\ \pi_p \text{ is a } p\text{-dimensional plane}\}.$$

$\mathrm{vol}^{[p]}(\Omega)$ is the maximal volume of a p-dimensional cross-section of Ω.

- Note:

$$\begin{cases} \mathrm{vol}^{[1]}(\Omega) = \mathrm{diam}(\Omega) \\ \mathrm{vol}^{[n]}(\Omega) = \mathrm{vol}(\Omega) \end{cases}$$

- The functional $\mathrm{vol}^{[p]}$ is monotone increasing with respect to inclusion. Recall John's theorem:

$$\mathcal{E}_\Omega \subseteq \Omega \subseteq n \cdot \mathcal{E}_\Omega.$$

Hence:

$$\mathrm{vol}^{[p]}(\mathcal{E}_\Omega) \leq \mathrm{vol}^{[p]}(\Omega) \leq n^p \mathrm{vol}^{[p]}(\mathcal{E}_\Omega),$$

and

$$\begin{aligned}\mathrm{vol}^{[p]}(\Omega) &\sim \mathrm{vol}^{[p]}(\mathcal{E}_\Omega)\\ &\sim D_1(\mathcal{E}_\Omega)\cdots D_p(\mathcal{E}_\Omega)\\ &\sim \frac{1}{\sqrt{\lambda_1^{[1]}\cdots\lambda_1^{[p]}}}\end{aligned}$$

Therefore we get a spectral estimate involving cross-sections:

Corollary. *For every $p = 1, \ldots, n$ one has:*

$$\mathrm{vol}^{[p]}(\Omega) \sim \frac{1}{\sqrt{\lambda_1^{[1]}\cdots\lambda_1^{[p]}}}$$

This means of course:

$$\frac{c_{n,p}}{\sqrt{\lambda_1^{[1]}\cdots\lambda_1^{[p]}}} \leq \mathrm{vol}^{[p]}(\Omega) \leq \frac{c'_{n,p}}{\sqrt{\lambda_1^{[1]}\cdots\lambda_1^{[p]}}}.$$

for explicit (but not sharp) constants.

An inequality for the volume. Taking $p = n$:

$$\mathrm{vol}(\Omega) \sim \frac{1}{\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[n]}}}.$$

That is, the volume is comparable to the product of all fundamental wavelengths.

Remark. Is something like this true in a more general situation? (for example, closed manifolds with some curvature assumptions?)

- Consequence: (weak) Faber-Krahn inequality.

In fact, from monotonicity: $\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[n]}} \leq \left(\lambda_1^{[n]}\right)^{n/2}$ hence:

$$\lambda_1^{[n]} \geq \frac{c_n}{\mathrm{vol}(\Omega)^{2/n}},$$

and we know that $\lambda_1^{[n]} = \lambda_1^D$. That is:

$$\lambda_1^D \geq \frac{c_n}{\mathrm{vol}(\Omega)^{2/n}}.$$

(of course, c_n can't be sharp).

A Faber-Krahn type inequality for $\lambda_1^{[p]}$.

Again from monotonicity: $\sqrt{\lambda_1^{[1]} \cdots \lambda_1^{[p]}} \leq \left(\lambda_1^{[p]}\right)^{p/2}$.

Corollary. *For all $p = 1 \ldots, n$:*

$$\lambda_1^{[p]}(\Omega) \geq \frac{c_{n,p}}{\left(\mathrm{vol}^{[p]}(\Omega)\right)^{2/p}}.$$

- Case $p = 1$. We have

$$\mathrm{vol}^{[1]}(\Omega) = \mathrm{diam}(\Omega) \quad \text{and} \quad \lambda_1^{[1]} = \lambda_1^N$$

hence

$$\lambda_1^N(\Omega) \geq \frac{c_n}{\mathrm{diam}(\Omega)^2},$$

... Payne-Weinberger inequality for the first Neumann eigenvalue.

- Case $p = n$: Faber-Krahn inequality for the first Dirichlet eigenvalue. Then:

- the bound in the Corollary is an isoperimetric inequality for forms connecting these two classical inequalities on functions.

- Problem: find the optimal constant for all p.

Conjecture. *Let Ω be convex in $\mathbf{R}^n$ and $p = 1, \ldots, n$. Let $\bar{\Omega}_p$ be the p-th dimensional ball such that*

$$\mathrm{vol}(\bar{\Omega}_p) = \mathrm{vol}^{[p]}(\Omega).$$

Then

$$\lambda_1^{[p]}(\Omega) \geq \lambda_1^{[p]}(\bar{\Omega}_p) = \lambda_1^D(\bar{\Omega}_p)$$

For $p = 1$ this is the Payne-Weinberger inequality (with the optimal constant). In fact:

$$\mathrm{vol}^{[1]}(\Omega) = \mathrm{diam}(\Omega), \quad \bar{\Omega}_1 = [0, \mathrm{diam}(\Omega)]$$

hence

$$\lambda_1^{[1]}(\bar{\Omega}_1) = \lambda_1^D(\bar{\Omega}_1) = \frac{\pi^2}{\mathrm{diam}(\Omega)^2}$$

As $\lambda_1^{[1]} = \lambda_1^N$ the above reads: $\lambda_1^N \geq \dfrac{\pi^2}{\mathrm{diam}(\Omega)^2}$.

- Equivalent form of the conjecture:

$$\lambda_1^{[p]}(\Omega) \geq \frac{c_p}{\Big(\mathrm{vol}^{[p]}(\Omega)\Big)^{2/p}}$$

where

$$c_p = \lambda_1^D(B_p) \cdot \mathrm{vol}(B_p)^{2/p}$$

Scheme of the proof.

- Recall the statement: $\lambda_1^{[p]}(\Omega) \sim 1/D_p(\mathcal{E}_\Omega)^2$.

- The upper bound is given in terms of any ellipsoid $\mathcal{E}_-$ contained in Ω (no convexity needed).

Theorem 1. *Let Ω be an arbitrary domain in $\mathbf{R}^n$ and let $\mathcal{E}_-$ be an ellipsoid contained in Ω, with principal axes $D_1(\mathcal{E}_-) \geq D_2(\mathcal{E}_-) \geq \cdots \geq D_n(\mathcal{E}_-)$. Then:*

$$\lambda_1^{[p]}(\Omega) \leq 4p(n+2) \cdot \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\mathcal{E}_-)} \cdot \frac{1}{D_p(\mathcal{E}_-)^2}$$

If Ω is convex ... take $\mathcal{E}_- = \mathcal{E}_\Omega$, then

$$\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\mathcal{E}_\Omega)} \leq n^n$$

because $\Omega \subseteq n\mathcal{E}_\Omega$. Get

$$\lambda_1^{[p]}(\Omega) \leq \frac{c_{n,p}}{D_p(\mathcal{E}_\Omega)^2}$$

where $c_{n,p} = 4p(n+2)n^n$.

- Main tool: Hodge decomposition for manifolds with boundary.

- Test-form. Fix coordinates so that $\mathcal{E}_-$ has equation:
$$\frac{x_1^2}{D_1^2}+\cdots+\frac{x_n^2}{D_n^2}\leq 4,$$
where $D_k = D_k(\mathcal{E}_-)$. Let $\omega = dx_1\wedge\cdots\wedge dx_{p+1}$. The test form will be the canonical primitive of ω restricted to $\mathcal{E}_-$ (this is explicitly computable).

- The *canonical primitive* of ω is the unique co-exact $(p-1)$-form θ such that $d\theta=\omega$ and $i_N\theta=0$ on $\partial\Omega$.
It minimizes the L^2-norm among all primitives of ω.

Lower bound.

Lower bound is given in terms of *any* ellipsoid $\mathcal{E}_+$ containing Ω (convexity is needed!).

Theorem 2. *Let Ω be a convex body in $\mathbf{R}^n$ and $\mathcal{E}_+$ an ellipsoid containing Ω, with principal axes $D_1(\mathcal{E}_+) \geq D_2(\mathcal{E}_+) \geq \cdots \geq D_n(\mathcal{E}_+)$. Then, for all $p \geq 2$:*

$$\lambda_1^{[p]}(\Omega) \geq 4 \binom{n}{p-1}^{-1} \cdot \frac{1}{D_p(\mathcal{E}_+)^2}.$$

Now take $\mathcal{E}_+ = n\mathcal{E}_\Omega$. Get

$$\lambda_1^{[p]}(\Omega) \geq \frac{4}{n^2 \binom{n}{p-1}} \cdot \frac{1}{D_p(\mathcal{E}_\Omega)^2}.$$

Thus, John's theorem is used to relate the upper and lower bounds.

Main steps.

- First step: reduce the problem to a lower bound of the energy.

Let ω be a p-eigenform. Bochner formula:

$$\langle \Delta\omega, \omega\rangle = |\nabla\omega|^2 + \frac{1}{2}\Delta|\omega|^2.$$

Integrating on Ω:

$$\lambda_1^{[p]} \int_\Omega |\omega|^2 = \int_\Omega |\nabla\omega|^2 + \frac{1}{2}\int_\Omega \Delta|\omega|^2.$$

Now:

$$\begin{aligned}
\frac{1}{2}\int_\Omega \Delta|\omega|^2 &= \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial N}|\omega|^2 \\
&= \int_{\partial\Omega} \langle \nabla_N\omega, \omega\rangle \\
&= \int_{\partial\Omega} \langle S^{[p]}\omega, \omega\rangle \\
&\geq 0
\end{aligned}$$

where $S^{[p]}$ = self-adjoint extension of the shape operator S to $\Lambda^{[p]}(\partial\Omega)$ (by convexity, one has $S \geq 0$ hence also $S^{[p]} \geq 0$).

- Hence for any p-eigenform:

$$\lambda_1^{[p]} \geq \frac{\int_\Omega |\nabla \omega|^2}{\int_\Omega |\omega|^2}.$$

- Second step: estimate from below the energy of co-closed, tangential forms.

Theorem 3. *Let ω be a co-closed $(p-1)$-form on Ω such that $i_N\omega = 0$ on $\partial\Omega$. Let $\mathcal{E}_+$ be an ellipsoid containing Ω, with principal axes $D_1(\mathcal{E}_+) \geq D_2(\mathcal{E}_+) \geq \cdots \geq D_n(\mathcal{E}_+)$. Then:*

$$\frac{\int_\Omega |\nabla \omega|^2}{\int_\Omega |\omega|^2} \geq 4 \binom{n}{p-1}^{-1} \cdot \frac{1}{D_p(\mathcal{E}_+)^2}.$$

- Use the Payne-Weinberger lower bound on suitable cross-sections of Ω to obtain a lower bound for the energy of the components of ω.

Proof of the upper bound. As usual, to produce upper bounds we need suitable test-forms. Recall the variational property of the first Hodge-eigenvalue:

$$\lambda_1^{[p]}(\Omega) = \inf\left\{\frac{\int_\Omega |d\omega|^2 + |\delta\omega|^2}{\int_\Omega |\omega|^2} : \omega \in \Lambda^p(\Omega), i_N\omega = 0 \text{ on } \partial\Omega.\right\}$$

As Δ commutes with both d and δ, it preserves the space of exact (resp. co-exact) forms. Hence:

$$\lambda_1^{[p]} = \min\{\lambda_1^{[p]'}, \lambda_1^{[p]''}\}$$

where $\lambda_1^{[p]'}$ (resp. $\lambda_1^{[p]''}$) is the first eigenvalue of Δ when restricted to exact (resp. co-exact) p-forms. By differentiating eigenforms one sees that:

$$\lambda_1^{[p]'} = \lambda_1^{[p-1]''}.$$

From the Hodge decomposition theorem for manifolds with boundary (Hodge-Morrey decomposition), one sees that, if ω is an exact p-form on Ω, then there exists a unique $(p-1)$-form $\theta = \theta_{\omega,\Omega}$ such that:

$$\begin{cases} \omega = d\theta, \\ \theta \text{ is co-exact and } i_N\theta = 0 \text{ on } \partial\Omega. \end{cases}$$

The form θ above is called the *canonical primitive of* ω. It has the following important property:

• the canonical primitive is the primitive with the least L^2-norm.

Now $\theta = \theta_{\omega,\Omega}$ is a test-form for the eigenvalue $\lambda_1^{[p-1]''}$. Hence:

$$\begin{aligned}\lambda_1^{[p]} &\leq \lambda_1^{[p]'} \\ &= \lambda_1^{[p-1]''} \\ &\leq \frac{\int_\Omega |d\theta|^2}{\int_\Omega |\theta|^2} = \frac{\int_\Omega |\omega|^2}{\int_\Omega |\theta_{\omega,\Omega}|^2}\end{aligned}$$

Now if $\mathcal{E}_- \subseteq \Omega$ we see that, for *any* exact p-form ω one has:

$$\lambda_1^{[p]}(\Omega) \leq \frac{\int_\Omega |\omega|^2}{\int_{\mathcal{E}_-} |\theta_{\omega,\mathcal{E}_-}|^2}$$

where $\theta_{\omega,\mathcal{E}_-}$ is the canonical primitive of ω on $\mathcal{E}_-$. Let us choose ω so that everything will be computable. Fix coordinates so that $\mathcal{E}_-$ has equation:

$$\frac{x_1^2}{D_1^2} + \cdots + \frac{x_n^2}{D_n^2} \leq 4,$$

and take

$$\omega = dx_1 \wedge \cdots \wedge dx_p.$$

Then $|\omega|^2 = 1$ and its canonical primitive on the ellipsoid $\mathcal{E}_-$ is explicitly computable. One ends-up with the desired upper bound:

$$\lambda_1^{[p]}(\Omega) \leq 4p(n+2) \cdot \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\mathcal{E}_-)} \cdot \frac{1}{D_p(\mathcal{E}_-)^2}.$$