Recent improvements of Berezin - Li & Yau type inequalities

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A survey talk

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Spectrum of Dirichlet Laplacians.

Let $\Omega \subset \mathbb{R}^d$ be an open domain of finite Lebesgue measure. Consider in $L^2(\Omega)$ the Dirichlet boundary value problem:

$$-\Delta^{\mathcal{D}} u = \lambda u, \qquad u\Big|_{\partial\Omega} = 0.$$

Denote by $\{\lambda_k\}_{k=1}^{\infty}$ the eigenvalues of $-\Delta^{\mathcal{D}}$ and let

$$L^{cl}_{\gamma,d} = (2\pi)^{-d} \int (1 - |\xi|^2)_+^{\gamma} \, d\xi.$$

Weyl asymptotics:

$$N(\lambda, -\Delta^{\mathcal{D}}) = \#\{k : \lambda_k < \lambda\} = (2\pi)^{-d} \int_{|\xi|^2<\lambda} \int_{\Omega} dx \, d\xi$$
$$= L^{cl}_{0,d} \, |\Omega| \, \lambda^{d/2} + o(\lambda^{d/2}).$$

When proving Weyl's asymptotic for domains with "bad" boundaries one usually needs the uniform estimate

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq C\,|\Omega|\,\lambda^{d/2}. \qquad (*)$$

Birman-Solomyak and Ciesielski obtained this result for bounded domains.

Rosenblum, Lieb and Metivier obtained it for domains Ω of finite measure.

The best constant in $(*)$ is unknown.

Pólya conjecture:

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq L_{0,d}^{cl}\, |\Omega|\, \lambda^{d/2}.$$

This conjecture was proved by Pólya for tiling domains.

It was also justified for domains with the structure

$$\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \qquad d_1 \geq 2,$$

where Ω_1 satisfies Pólya conjecture (for example tiling) and Ω_2 is arbitrary.

Open problem: Prove Pólya conjecture for $\Omega = \{x \in \mathbb{R}^d : |x| < 1\}$.

Weyl formula for Riesz means:

$$\sum_k (\lambda - \lambda_k)_+^\gamma = (2\pi)^{-d} \int (\lambda - |\xi|^2)_+^\gamma \, d\xi \int_\Omega dx$$

$$= L_{\gamma,d}^{cl} |\Omega| \, \lambda^{d/2+\gamma} + o(\lambda^{d/2+\gamma}).$$

This formula is supported by the Berezin inequality:

If $\gamma \geq 1$, then

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq L_{\gamma,d}^{cl} |\Omega| \, \lambda^{d/2+\gamma}.$$

Proof for $\gamma = 1$.

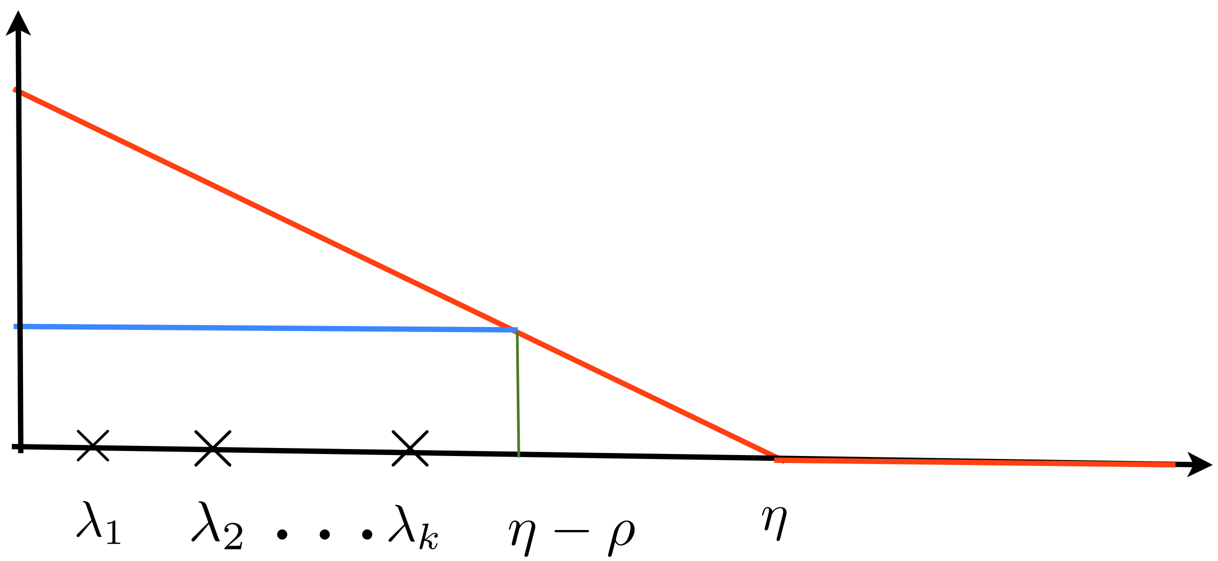
Let $\{\varphi_k\}$ be orth.n. eigenfunctions of $-\Delta^{\mathcal{D}}$. Then

$$
\begin{aligned}
\sum_k (\lambda - \lambda_k)_+ &= \sum_k \big(< (\lambda - \lambda_k)\varphi_k, \varphi_k > \big)_+ \\
&= \sum_k \big(< (\lambda + \Delta^{\mathcal{D}})\varphi_k, \varphi_k > \big)_+ \\
&= (2\pi)^{-d} \sum_k \Big(\int (\lambda - |\xi|^2)\, |\hat{\varphi}_k(\xi)|^2 \, d\xi \Big)_+ \\
&\leq (2\pi)^{-d} \sum_k \int (\lambda - |\xi|^2)_+ \, |\hat{\varphi}_k(\xi)|^2 \, d\xi \\
&= (2\pi)^{-d} \int (\lambda - |\xi|^2)_+ \sum_k \Big| \int \varphi_k(x)\, e^{-ix\xi} \, dx \Big|^2 d\xi \\
&= (2\pi)^{-d} \int (\lambda - |\xi|^2)_+ \, d\xi \; |\Omega|.
\end{aligned}
$$

Corollary.

$$N(\lambda, -\Delta^{\mathcal{D}}) \le \frac{d}{2}\left(1 + \frac{2}{d}\right)^{1+d/2} L^{cl}_{1,d}\, |\Omega|\, \lambda^{d/2}.$$

Indeed:



$$N(\eta - \rho, -\Delta^{\mathcal{D}}) \le \frac{1}{\rho} \sum_k (\eta - \lambda_k)_+ \le \frac{\eta^{1+d/2}}{\rho} L_{1,d}\, |\Omega|.$$

Letting $\eta = (1+\tau)\lambda$, $\rho = \tau\lambda$ and $\tau = 2/d$ we find

$$N(\lambda, -\Delta^{\mathcal{D}}) \le \frac{(1+\tau)^{1+d/2}}{\tau} L_{1,d}\, |\Omega|\, \lambda^{d/2} = \frac{d}{2}\left(1 + \frac{2}{d}\right)^{1+d/2} L^{cl}_{1,d}\, |\Omega|\, \lambda^{d/2}.$$

Remarks.

1. If $d = 2$ then the latter inequality could be rewritten as

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq 2\, L_{0,2}^{cl}\, |\Omega|\, \lambda^{d/2}.$$

2. Using the Legendre transform we obtain Li & Yau inequality from Berezin's inequality

$$\sum_{k=1}^{n} \lambda_k \geq \frac{d}{d+2} (L_{0,d}^{cl}\, |\Omega|)^{-2/d}\, n^{\frac{d+2}{d}}.$$

3. A simple argument allows us to obtain from Berezin's inequality sharp inequalities for all $\gamma \geq 1$

$$\sum_{k} (\lambda - \lambda_k)_+^{\gamma} \leq L_{\gamma,d}^{cl}\, |\Omega|\, \lambda^{\gamma + d/2}.$$

Neumann Problem:

$$-\Delta^{\mathcal{N}} v = \mu\, v, \qquad \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0.$$

Theorem.

$$\sum_k (\mu - \mu_k)_+ \geq (2\pi)^{-d}\, |\Omega| \int (\mu - |\xi|^2)_+ \, d\xi.$$

Proof. Let $\{\psi_k\}$ be orth.n. eigenfunctions of $-\Delta^{\mathcal{N}}$. Then

$$\sum_k (\mu - \mu_k)_+ = \sum_k (\mu - \mu_k)_+ \|\psi_k\|^2 = (2\pi)^{-d} \sum_k (\mu - \mu_k)_+ \int |\hat{\psi}_k(\xi)|^2 \, d\xi.$$

Note that

$$|\Omega|^{-1}\sum_k |\hat{\psi}_k(\xi)|^2 = |\Omega|^{-1}\sum_k | < \psi_k, e^{i\cdot\xi} > |^2 = |\Omega|^{-1}\,\|e^{-i\cdot\xi}\|_{L^2(\Omega)} = 1.$$

Therefore, by using Jensen's inequality we find

$$\begin{aligned}\sum_k(\mu-\mu_k)_+ &= (2\pi)^{-d}\,|\Omega|\int |\Omega|^{-1}\sum_k(\mu-\mu_k)_+|\hat{\psi}_k(\xi)|^2\,d\xi \\ &\geq (2\pi)^{-d}\,|\Omega|\int \Big(|\Omega|^{-1}\sum_k(\mu-\mu_k)|\hat{\psi}_k(\xi)|^2\Big)_+\,d\xi \\ &= (2\pi)^{-d}\,|\Omega|\int \Big(\mu-|\Omega|^{-1}\sum_k \mu_k| < \psi_k, e^{i\cdot\xi} >_{L^2(\Omega)} |^2\Big)_+\,d\xi \\ &= (2\pi)^{-d}\,|\Omega|\int \Big(\mu-|\Omega|^{-1}\int_\Omega |\nabla_x e^{-ix\xi}|^2\,dx\Big)_+\,d\xi \\ &= (2\pi)^{-d}\,|\Omega|\int \Big(\mu-|\Omega|^{-1}\,|\xi|^2\|e^{i\cdot\xi}\|^2_{L_2(\Omega)}\Big)_+\,d\xi = (2\pi)^{-d}\,|\Omega|\int(\mu-|\xi|^2)_+\,d\xi.\end{aligned}$$

Remarks.

1. Using Legendre's transform we obtain Kröger's inequality:

$$\sum_{k=1}^{n} \mu_k \leq \frac{d}{d+2} (L_{0,d}^{cl} |\Omega|)^{-2/d} n^{(d+2)/d}.$$

2. Similar proofs could be given for both Dirichlet and Neumann spectrum of arbitrary differential operators (systems of operators) with constant coefficients and finite phase volume.

Open Problem:

Obtain an inequality for the Riesz means of the eigenvalues of the Hodge Laplacian.

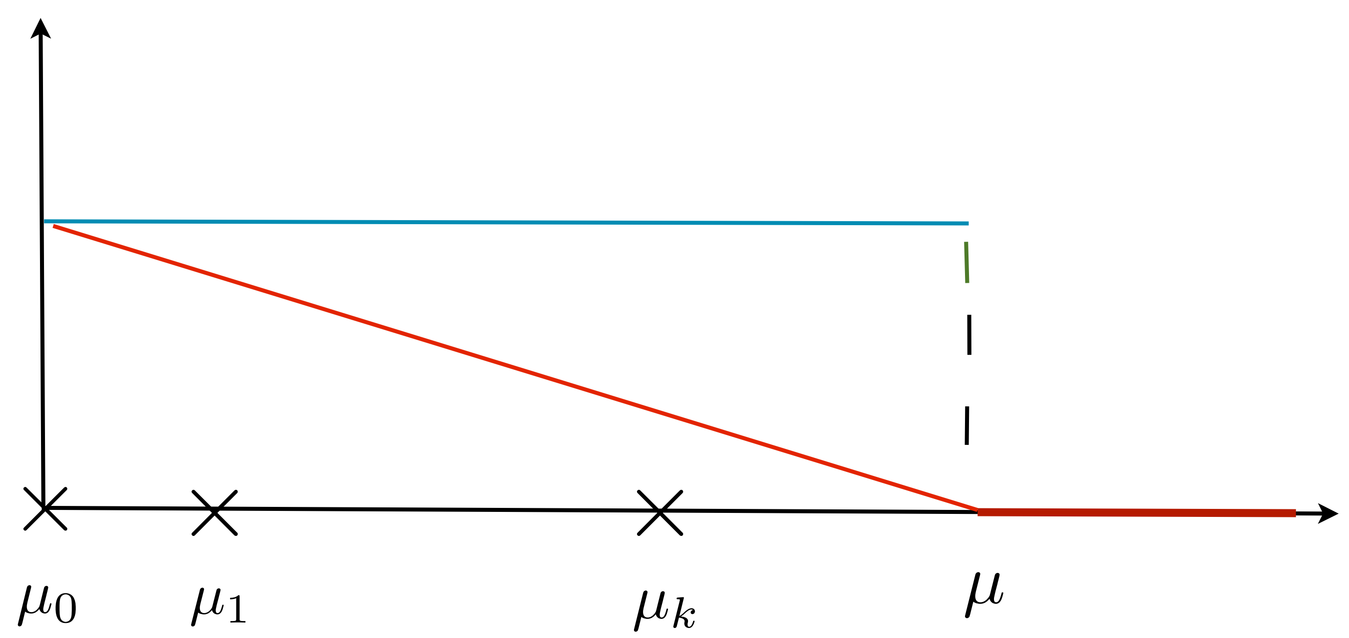
Counting function for the Neumann Laplacian.

Let $N(\mu, -\Delta^{\mathcal{N}}) = \#\{k : \mu_k < \mu\}$. Then

$$N(\mu, -\Delta^{\mathcal{N}}) \geq \frac{1}{\mu} \sum_k (\mu - \mu_k)_+ \geq \frac{1}{\mu} \mu^{1+d/2} |\Omega| L^{cl}_{1,d} = \mu^{d/2} |\Omega| L^{cl}_{1,d}.$$

Note, that if $d = 2$, then

$$N(\mu, -\Delta^{\mathcal{N}}) \geq \mu^{d/2} |\Omega| \frac{1}{2} L^{cl}_{0,d}.$$



Two terms Weyl asymptitics for the Dirichlet Laplacian:

$$\sum_k (\lambda - \lambda_k)_+^\gamma = L_{1,d}^{cl} \, |\Omega| \, \lambda^{\gamma+d/2} - \frac{1}{4} \, L_{\gamma,d-1}^{cl} |\partial\Omega| \, \lambda^{\gamma+(d-1)/2} + o(\lambda^{\gamma+(d-1)/2}).$$

The proof of Weyl's conjecture for $\gamma = 0$ was obtained by V.Ivrii '80.

It suggests that there is a possibility of improving Berezin-Li & Yau inequalities.

Indeed, recently, several improvements of the Berezin-Li & Yau inequality have been found. The first result is due to Melas '03.

$$\sum_n (\lambda - \lambda_n)_+^\gamma \le L_{\gamma,d}^{cl} \, |\Omega| \Big(\lambda - M_d \frac{|\Omega|}{I(\Omega)} \Big)_+^{\gamma + d/2} , \quad \lambda > 0 \, , \quad \gamma \ge 1 \, ,$$

where $M_d \ge \frac{1}{24(d+2)}$ and $I(\Omega)$ denotes the "moment of inertia" of Ω, that is,

$$I(\Omega) = \min_{a \in R^d} \int_\Omega |x - a|^2 \, dx$$

- the second moment of Ω.

Remark.
Note that this correction does not capture the correct order in λ from the second term of Weyl's asymptotics.

Another result.

Let $u \in \mathbb{S}^{d-1}$, $x \in \Omega$ and let

$$\theta(x,u) = \inf\{t > 0 \,:\, x + tu \notin \Omega\}\ , \quad d(x,u) = \inf\{\theta(x,u), \theta(x,-u)\},$$

Theorem (Geisinger, AL, Weidl).
Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathbb{S}^{d-1}$. If $\gamma \geq 3/2$, then for the eigenvalues $\{\lambda_k\}$ of the Dirichlet Laplacian in Ω we have

$$\sum_k (\lambda - \lambda_n)_+^\gamma \leq L_{\gamma,d}^{cl} \int_\Omega \left(\lambda - \frac{1}{4d^2(x,u)}\right)_+^{\gamma + d/2} dx.$$

"Proof". The proof is based on the following 1D inequality:

Let $-d^2/dt^2$ is defined on the interval $(0,l)$ with Dirichlet boundary conditions. Obviously the eigenvalues of this operator are equal to

$$k^2\,\frac{\pi^2}{l^2}, \qquad k = 1, 2, 3, \ldots .$$

Let $\gamma \geq 1$. Then

$$\sum_{k=1}^{\infty} \left(\lambda - \frac{k^2\pi^2}{l^2}\right)_+^{\gamma} \leq L_{\gamma,1}^{cl} \int_0^l \left(\lambda - \frac{1}{4\delta^2(t)}\right)^{\gamma+1/2} dt,$$

where $\delta(t) = \min\{t, l-t\}$.

Remark. This inequality is sharp and involves the term $1/4\delta^2$ which usually appears in Hardy's inequalities.

Let $\delta(x) = \inf\{|x-y| : y \notin \Omega\}$.

Corollary.(GLW)

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex domain with smooth boundary and assume that the curvature of $\partial\Omega$ is bounded from above by $1/R$. Then for $\gamma \geq 3/2$ and all $\lambda > 0$ we have

$$\sum (\lambda - \lambda_n)_+^\gamma \leq L_{\gamma,d}^{cl} |\Omega| \lambda^{\gamma + d/2} - L_{\gamma,d}^{cl} 2^{-d-2} |\partial\Omega| \lambda^{\gamma+(d-1)/2} \int_0^1 \Big(1 - \frac{d-1}{4R\sqrt{\lambda}} s\Big)_+ ds .$$

"Proof". The proof uses operator-valued Lieb-Thirring inequalities and some properties of convex domains.
Namely, let $\Omega_t = \{x \in \Omega : \delta(x) > t\}$ be the inner parallel set of Ω. Then we use Steiner's theorem

$$|\partial\Omega_t| \geq \Big(1 - \frac{d-1}{R} t\Big)_+ |\partial\Omega| .$$

Remark.

It would be interesting to know if for bounded convex domains in $\mathbb{R}^2$ without extra smoothness of the boundary

$$|\partial\Omega_t| \geq \left(1 - \frac{3t}{l_0}\right)_+ |\partial\Omega|, \qquad (*)$$

where

$$l_0 = \inf_{u\in\mathbb{S}} \sup_{x\in\Omega} l(x,u) = \inf_{u\in\mathbb{S}} \sup_{x\in\Omega} \left(\theta(x,u) + \theta(x,-u)\right).$$

This is true for a large class of convex domains, including circles, regular polygons and arbitrary triangles.

In particular, this would allow us to state the previous result without any conditions of the curvature of the boundary.

Open problem. Prove or disprove that $(*)$ holds true for all bounded convex domains in $\mathbb{R}^2$.

Theorem (Kovarik and Weidl)

Let $\Omega \subset \mathbb{R}^d$ be convex. Then

$$\sum_{k=1}^{n} \lambda_k \geq \frac{d}{d+2}(L_{0,d}^{cl}\,|\Omega|)^{-2/d}\, n^{\frac{d+2}{d}} + \frac{n}{64R_i^2},$$

where R_{in} is in-radius of Ω.

Remark.
This estimate is better than the one by Melas if $d \geq 3$ and also definitely better for thin domains.
Note that Melas' result could be rewritten in the Li & Yau form as

$$\sum_{k=1}^{n} \lambda_k \geq \frac{d}{d+2}(L_{0,d}^{cl}\,|\Omega|)^{-2/d}\, n^{\frac{d+2}{d}} + M_d\,\frac{|\Omega|}{I(\Omega)}\, n,$$

where $M_d \geq \frac{1}{24(d+2)}$.

Inequalities for operators with magnetic fields.

Consider the Dirichlet boundary value problem for Laplacians with magnetic fields in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$

$$-\Delta_A^{\mathcal{D}} u = (-i\nabla - A)^2 u = \lambda, \qquad u\Big|_{\partial\Omega} = 0.$$

It is known that if $\gamma \geq 3/2$ then

$$\sum_k (\lambda - \lambda_k)^\gamma \leq L_{\gamma,d}^{cl} \lambda^{\gamma + d/2} |\Omega|. \qquad (*)$$

Theorem. (Erdös, Loss and Vugalter)

If $A = \frac{B}{2}(-x_2, x_1)$, then $(*)$ holds even for $\gamma \geq 1$.

A more precise result was obtained in the paper of Frank, Loss and Weidl '09.

Theorem.

Let $\Omega \subset \mathbb{R}^2$ be an open domain of finite measure and let $\gamma \geq 0$. Then for the Dirichlet eigenvalues of the magnetic Laplacian $-\Delta_A^{\mathcal{D}}$ with constant magnetic field

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq R_\gamma L_{\gamma,2}^{cl} \, |\Omega| \, \lambda^{\gamma + d/2},$$

where

$$R_0 = 2, \qquad R_\gamma = 2 \left(\frac{\gamma}{\gamma + 1} \right)^\gamma.$$

This inequality is sharp and cannot be improved even for tiling domains.

Theorem. (Kovarik and Weidl)

Let $\Omega \subset \mathbb{R}^2$ be bounded and convex. Then for the Dirichlet eigenvalues of the magnetic Laplacian $-\Delta_A^{\mathcal{D}}$ with constant magnetic field

$$\sum_{k=1}^{n} \lambda_k \geq \frac{2\pi\, n^2}{|\Omega|} + \frac{n}{64 R_{in}},$$

where R_{in} is the in-radius of Ω.

Spectral inequalities for Heisenberg Laplacians.

Let

$$X_1 = \partial_{x_1} + \frac{1}{2}\, x_2\, \partial_{x_3}, \qquad X_2 = \partial_{x_2} - \frac{1}{2}\, x_1\, \partial_{x_3}$$

and condider in $L^2(\Omega)$ the problem

$$H\, u = (X_1^* X_1 + X_2^* X_2)\, u = \lambda u, \qquad u\Big|_{\partial\Omega} = 0.$$

Theorem. (Hansson, AL)
Let $|\Omega| < \infty$. Then

$$\sum_k (\lambda - \lambda_k)_+ \le \frac{1}{96}\, |\Omega|\, \lambda^3. \qquad (*)$$

Remark.

The operator H is not elliptic. The corresponding phase volume for its symbol is infinite. Therefore in $(*)$ we have λ^3 instead of $\lambda^{1+3/2}$,

Open problem: Prove the opposite inequality for the Neumann problem.

Inequalities on the continuous spectrum.

Theorem.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and let $\mathbb{R}^d \setminus \Omega$ is of finite Lebesgue measure. Then for any $\gamma \geq 1$

$$\operatorname{Tr}\left((-\Delta_{\mathbb{R}^d} - \lambda)_+^\gamma - (-\Delta_\Omega^{\mathcal{D}} - \lambda)_+^\gamma\right) \geq L_{\gamma,d}^{cl} |\mathbb{R}^d \setminus \Omega| \, \lambda^{\gamma+d/2}.$$

Open problem:

Show a similar inequality for the Neumann problem.

Pólya conjecture for the unit disk in $\mathbb{R}^2$.

Let $D = \{x \in \mathbb{R}^d : \, |x| < 1\}$ and consider the spectral problem for the Dirichlet Laplacian in $L^2(D)$

$$-\Delta^{\mathcal{D}} u(x) = \lambda \, u(x), \qquad u\Big|_{\partial D} = 0.$$

Open problem: Find the best constant C in

$$N(\lambda, -\Delta^{\mathcal{D}}) = \#\{k : \, \lambda_k < \lambda\} \le C \, |B|^2 \, \lambda^{d/2}.$$

The quadratic form of the operator $-\Delta^{\mathcal{D}}$ in polar coordinates $x = (r, \theta)$ equals

$$\int_D |\nabla_x u|^2 \, dx = \int_0^\infty \int_{\mathbb{S}} \left(|\partial_r u|^2 + \frac{|\partial_\theta u|^2}{r^2} \right) r \, d\theta \, dr.$$

The standard exponential change of variables $r = e^{-t}$ implies

$$\int_D |\nabla_x u|^2 \, dx = \int_0^\infty \int_{\mathbb{S}} \left(|\partial_t u|^2 + |\partial_\theta u|^2 \right) d\theta \, dt.$$

Note that we also have

$$\lambda \int_D |u|^2 \, dx = \lambda \int_0^\infty \int_{\mathbb{S}} |u|^2 \, e^{-2t} \, d\theta dt.$$

By using Glazman's lemma we obtain that the value $N(\lambda, -\Delta^{\mathcal{D}})$ coinsides with the dimension of the subspace of functions satisfying the inequality

$$\int_0^\infty \int_{\mathbb{S}} \left(|\partial_t u|^2 + |\partial_\theta u|^2 - \lambda\, e^{-2t} |u|^2 \right) d\theta\, dt < 0.$$

The latter is equivalent to the study of the number $N(\lambda, \tilde{H})$ of the negative eigenvalues of the Schrödinger operator on the cylinder $(0, \infty) \times S$ with Dirichlet boundary conditions at $t = 0$, where

$$\tilde{H} u = -\partial_t^2 u - \partial_\theta^2 u - \lambda\, e^{-2t}\, u, \qquad u|_{t=0} = 0.$$

Considering λ as a parameter and using the inequality

$$e^{-2t} \leq (\cosh t)^{-2}$$

we find that $N(\lambda, \tilde{H}) \leq N(\lambda, H)$, where

$$Hu = -\partial_t^2 u - \partial_\theta^2 u - \lambda (\cosh t)^{-2} u, \qquad u|_{t=0} = 0.$$

The spectrum of this operator could be easily computed and we finally obtain

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq N(\lambda, H) \leq 1 + \frac{1}{2} [\sqrt{\lambda} - 2]^2,$$

whereas the Pólya conjecture states that

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq \frac{1}{4} \lambda.$$

Thank you